Dynamic Contracting with Flexible Monitoring*

Liang Dai † Yenan Wang ‡ Ming Yang §

Shanghai Jiao Tong University and Duke University

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Abstract

We analyze a continuous-time moral hazard problem in which the principal can flexibly combine "carrots" and "sticks" to incentivize the agent. That is, he can flexibly allocate his limited attention between seeking confirmatory evidence and seeking contradictory evidence about the agent's effort as the basis for rewards and punishment. Such flexibility calls for a joint design of monitoring and compensation schemes novel in the literature. When the agent's continuation value is low, the principal only resorts to carrots with a constant rewards; When the agent's continuation value reaches a threshold, the principal switches to stick-dominant mode and the level of rewards decreases in the continuation value. Moreover, the agent's effort could be perpetuated if and only if both the flexibility in monitoring and the synergy relative to the agent's private benefit from shirking are sufficiently large.

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†Antai College of Economics and Management, Shanghai Jiao Tong University, 1954 Huashan Road, Shanghai, 200030, China, liangdai@sjtu.edu.cn

‡The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, yenan.wang@duke.edu

§The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, ming.yang@duke.edu.

1 Introduction

In a long-term contractual relation, the principal usually has the flexibility in assigning weights to different performance indicators of the agent as the basis for the incentive scheme. The weight assigned to a particular indicator affects the optimal reward or punishment associated with it, and vice versa. How does such flexibility affect the design of incentive schemes of agents at different stages of career? Would the principal ever choose to perpetuate the agent's effort? If so, when? These are prevalent issues in real life, such as in human resource management, bureaucratic systems and educational practices. However, existing literature either assumes a single exogenous performance indicator, or focuses on how much attention should be devoted to a *given* indicator. In this paper, we develop a dynamic framework that incorporates the *joint* determination of both the optimal combination of *multiple* performance indicators and the reward or punishment scheme associated with it. We show that such flexibility makes a qualitative difference in shaping the optimal contract.

To fix ideas, consider a continuous-time setup, in which the principal ("he") has a project that needs the agent ("she") to operate. The agent is less patient than the principal, and can work or shirk at each instant. From the perspectives of both the principal and social welfare, it is optimal for the agent to work, but the agent enjoys a private benefit if she shirks. To incentivize the agent, at each instant, the principal chooses a combination of "carrots" and "sticks". That is, he can allocate his fixed amount of attention to seek the following two types of evidence, and determine how much to reward or punish the agent upon its arrival. The C-evidence (C for "carrots") could arrive only if the agent has worked, while the S-evidence (S for "sticks") could arrive only if the agent has shirked. The principal can also terminate the project at any time, which is socially inefficient.

In addition to the standard incentive versus interest tradeoff in (DeMarzo and Sannikov 2006) and (Sannikov 2008), the principal faces the tradeoff between carrots and sticks as means to incentivize the agent. On one hand, carrots generate greater variation than sticks in the agent's continuation value, and are thus less favorable to the principal, who is effectively risk averse in the relevant range of the agent's continuation value. This is because, given that the agent indeed works, S-evidence would not arrive and thus requires no adjustment to the agent's continuation value; while C-evidence does arrive in equilibrium, which necessarily involves the reward upon its arrival and the downward adjustment to the agent's continuation value in the absence of C-evidence. On the other hand, sufficiently high continuation value is required as the agent's skin in the game for sticks to be an effective incentive device on its own, whereas the effectiveness of carrots does not depend on the agent's continuation value. Moreover, even if sticks can work alone, a high continuation value of the agent has to be maintained, which involves interest expenditure to the principal, making sticks less favorable than carrots. This tradeoff between carrots and sticks, together with the incentive versus interest tradeoff, shapes the optimal incentive scheme.

When the agent's continuation value is low, the principal allocates all his attention to seeking C-evidence and thus completely relies on carrots. Instead of paying the agent immediately when C-evidence arrives, the principal adds the whole reward to the agent's continuation value, so as to build a buffer against inefficient termination and make sticks effective in the future. In addition, since the arrival rate of C-evidence is set to its maximum, the reward upon the arrival of each piece of C-evidence should be just enough to deter shirking.

When the agent's continuation value is accumulated to the level enough for sticks to be effective, but not enough for sticks alone to deter shirking, the optimal incentive scheme features a "phase change". That is, instead of the carrot-only mode, the principal now mainly relies on sticks, and sets the penalty upon observing S-evidence to its maximum — to confiscate the whole stake promised to the agent, resulting in the termination of the project. Carrots are still used to make up for the sticks, but the reward upon observing C-evidence is set larger to minimize the reliance on carrots. Such reward decreases as the agent's continuation value grows further.

When the agent's continuation value further grows beyond the payout boundary, the conflict of interest between the principal and the agent is so little that there is little need to further incentivize the agent. But the interest accrued from deferred payment is large. Thus, it is optimal for the principal to make payment at once, so as to reduce the continuation value of the agent back to the payout boundary.

Concerning the perpetuation of the agent's effort, the flexibility in combining carrots and sticks offers the principal the option of first building up the agent's stake in the game (i.e., her continuation value) with carrots, and then perpetuating the agent's effort mainly with sticks, which avoids the inefficient termination. We show that such option is optimal if and only if the synergy relative to the agent's private benefit from shirking *and* the principal's flexibility in allocating weights between carrots and sticks are *both* sufficiently large. In other words, with insufficient flexibility as such, which is the central piece of this paper, the agent's effort should not be perpetuated even if the synergy is large relative to frictions. This contrasts with models without such flexibility in adjusting weights on multiple performance indicators, and provides new insight on human resource management and the design of bureaucratic systems. Moreover, the value function may be convex in the vicinity of the payout boundary if public randomization is not allowed. This reflects the fact that the higher is the agent's continuation value, the more likely that it reaches the absorbing payout boundary and the contract becomes completely immune to inefficient termination.

Our model also yields empirically plausible predictions. First, junior employees of a firm are incentivized mainly based on confirmatory evidence of their contribution to their employer, while senior employees are incentivized mainly based on contradictory evidence of their contribution. Second, concerning the compensation scheme, the reward for each piece of confirmatory evidence of contribution to the employer varies little among junior employees, but decreases with seniority for senior employees, and features an upward jump when a junior employee becomes senior. The penalty for each piece of contradictory evidence of contribution to the employer increases with seniority for both junior and senior employees. Third, except for those hired permanently, all employees are more prone to unemployment absent the arrival of confirmatory evidence of their contribution, and the more so if the employees are less senior. Lastly, employers offer permanent positions if and only if both their flexibility in adjusting monitoring schemes and the potential synergy created by employees (relative to frictions) are sufficiently large.

1.1 Literature Review

Our work is mainly related to the continuous-time dynamic contracting literature, pioneered by (DeMarzo and Sannikov 2006) and (Sannikov 2008). Early work on dynamic moral hazard models also includes (Biais, Mariotti, Rochet, and Villeneuve 2010). Like our model, (Biais, Mariotti, Rochet, and Villeneuve 2010) use a Poisson process instead of Brownian motions to model discrete losses in continuous time, whose arrival rate depends on the hidden action of the agent. (Myerson 2015) considers a similar problem under a political economics framework where a political leader uses randomized punishment to motivate governors to work. Opposite to the discrete losses in (Biais, Mariotti, Rochet, and Villeneuve 2010), (Sun and Tian 2017) use

Poisson processes to model arrivals of discrete revenue. Similarly, (He 2012) considers a risk-averse agent who can save privately and whose hidden effort affects the arrival rate of discrete revenue. In those models, monitoring technologies are exogenous. In other words, the output processes, which are functions of hidden actions and other random factors, are exogenously assumed, and play dual roles as both direct physical payoff determinants and bases for monitoring and contracting. The essence of our model is to separate these two roles, so as to study the interaction between the design of monitoring technology and that of contracts.

Recent work also endogenizes monitoring scheme in dynamic moral hazard models. On top of the framework of (DeMarzo and Sannikov 2006), (Piskorski and Westerfield 2016) allow the principal to monitor the agent at a cost increasing with his monitoring intensity. Based on a framework similar to that of (Biais, Mariotti, Rochet, and Villeneuve 2010), (Chen, Sun, and Xiao 2017) consider the timing decision of monitoring, where monitoring is modeled as paying a fixed cost for a credible guarantee of the agent taking the desired action. (Varas, Marinovic, and Skrzypacz 2019) consider a problem where monitoring serves as an incentive device and also provides information to the principal. In (Orlov 2018), the principal can change his monitoring intensity. In (Georgiadis and Szentes 2019), the principal can observe a diffusion process with the drift being the agent's effort at a cost proportional to the time at which the principal stops observing this process, and the principal is to determine the optimal stopping time. While these papers probe into *how much* attention should be devoted to a *given* monitoring technology and the optimal timing, our focus is on the principal's *optimal allocation* of attention into *multiple* information sources, as the basis for both his monitoring activities and his design of incentive scheme.

In a static setup, (Li and Yang 2019) and (Georgiadis and Szentes 2019) also study the impact of the principal's flexibility in designing his monitoring scheme. Based instead on a dynamic setup, we are able to explore when it is optimal for the principal to perpetuate the agent's effort. In addition, our notion of flexibility is different from that in (Georgiadis and Szentes 2019). The information source in (Georgiadis and Szentes 2019) is a *single* exogenous (conditional on the agent's effort) linear diffusion process, and the flexibility that they consider refer to the freedom of the principal to stop observing that process earlier if existing observations already suffice to prove the deviation of the agent from the desired action. The notion of flexibility in our paper instead refers to the freedom of the principal to allocate different levels of attention on *various* processes (interpreted as different performance indicators) contingent on the whole history summarized by the agent's continuation value.

Our work is also related to (Che and Mierendorff 2019). In a dynamic setting, (Che and Mierendorff 2019) study an individual's decision among immediate action, confirmatory learning (i.e., seeking evidence that would confirm the state he finds relatively more likely) and contradictory learning (i.e., seeking evidence that would confirm the state he finds relatively less likely), before taking actions that affect his state-contingent payoff. While carrots and sticks in our model have informational similarity to the two types of learning in theirs, the problem that we study is fundamentally different from theirs. (Che and Mierendorff 2019) study an individual's decision problem, in which the fact to learn (i.e., the state of nature) is exogenous, while we are studying strategic situation featuring moral hazard, in which the fact to learn (i.e., whether the agent is shirking) is endogenous to the choice of monitoring technologies and in turn the design of the incentive scheme by the principal.

2 The Model

2.1 Setup

Time is continuous and infinite. There is a principal ("he") and an agent ("she").[1] Both of them are risk neutral. The principal has a discount rate $r > 0$ and unlimited access to capital. The agent has a discount rate $\rho > r$ and is protected by the limited liability constraint. The principal owns a project that needs the agent to operate, which involves an action $a_t \in [0, 1]$ taken by the agent. The action can be understood as the level of shirking. If action a_t is taken at instant t, in the period $[t, t + dt]$, the agent enjoys a private benefit of $\lambda \cdot a_t dt$, while the benefit to the principal is $z \cdot (1 - a_t)\, dt > 0$. Here we interpret z as the latent progress of a project or reputation of an entity that is lost absent the agent's due diligence and is not discernible immediately. Therefore, contracts cannot be made contingent on whether z is accrued. We refer to z as the "synergy" (between the principal and the agent) hereafter. The principal can terminate the project at any time, and the project generates a payoff of zero for both players since then.

To model the carrot-or-stick decision, we assume that at each instant the principal can choose how to allocate his μ units of attention to seek one of two types of evidence. C-evidence reveals whether the agent has good performance, while S-evidence reveals whether she has bad performance. Specifically, if the principal allocates a fraction $\alpha_t \in [0, \bar{\alpha}]$ of his μ units of attention to seeking S-evidence and the remaining $1 - \alpha_t$ fraction of his attention for the C-evidence, he receives S-evidence with arrival rate $\mu \cdot \alpha_t \cdot a_t$, and C-evidence with arrival rate $\mu \cdot (1 - \alpha_t) \cdot (1 - a_t)$. Hence, the agent's chance of being caught shirking is proportional to a_t, the level of shirking, and $\mu \cdot \alpha_t$, the attention allocated to monitoring shirking. Intuitively, if the agent does not

[1] We do not intentionally associate the players with particular genders.

shirk, no evidence of shirking exists in the first place and the principal cannot find S-evidence no matter how much attention is allocated to seeking such evidence; if the principal pays no attention to monitor shirking, no S-evidence arrives regardless of the agent's action. The arrival rate of C-evidence can be interpreted similarly. More specifically, the cumulative number of arrivals of S-evidence, Y_1, and that of C-evidence, Y_0, satisfy

$$dY_{1,t} = \begin{cases} 1, & \text{with probability } \mu\alpha_t a_t dt \\ 0, & \text{otherwise} \end{cases},$$

and

$$dY_{0,t} = \begin{cases} 1, & \text{with probability } \mu\left(1-\alpha_t\right)\left(1-a_t\right)dt \\ 0, & \text{otherwise} \end{cases},$$

respectively. To save the notation, we write $Y = (Y_0, Y_1)$.

It is worth noting that upper bound $\bar{\alpha}$ measures the flexibility of the principal in allocating his attention across different performance indicators. To highlight the role of this flexibility, we assume $\bar{\alpha}$ to be close to 1. However, if $\bar{\alpha} = 1$, we show in the Appendix that the optimal reward to the agent upon the arrival of C-evidence would be infinity with positive probability, and thus we preclude this case in the text.[2] Formally, we assume that

Assumption 1 $\bar{\alpha} \in [1 - \frac{\rho}{\mu}, 1)$.

In addition, we assume that the principal is more patient than the agent, and that the principal has enough attention to discipline the agent in this contractual relation.

Assumption 2 $r < \rho < \mu$.

[2] See the discussion after Equation (14) for details.

Moreover, we assume that z is large enough so that action 1 is inefficient even taking into account the agent's private benefit. Then it is without loss of generality to focus only on contracts implementing $a_t = 0$ for all t.

Assumption 3 $z > \lambda > 0$.

A contract X specifies the action a taken by the agent, the monitoring scheme α, the cumulative payment I to the agent and the time of termination τ as functions of the history of past evidence. As mentioned before, we focus on contracts that implement $a_t = 0$ for all t, so that we suppress a and write $X = (\alpha, I, \tau)$.

Given the contract X and an action process a, the expected discounted utility of the agent is

$$E^a \left[\int_0^\tau e^{-\rho t} \left(dI_t + \lambda a_t dt \right) \right],$$

and that of the principal is

$$E^a \left[\int_0^\tau e^{-rt} \left(z \left(1 - a_t\right) dt - dI_t \right) \right]. \tag{1}$$

For notational convenience, we suppress all time subscripts hereafter when no confusion can be caused.

While contracts involving randomization are of theoretical interest, they are typically not practical in reality. Therefore, we postpone the discussion of public randomization to Section 5, and only consider deterministic contracts for the rest of this paper unless mentioned otherwise.

2.2 Incentive Compatibility and Limited Liability

To characterize the incentive compatibility condition, we rely on martingale techniques similar to those introduced by (Sannikov 2008). When choosing her action at time t, the agent considers how it will affect her continuation value, defined as

$$w_t(X,a) = E^a\left[\int_t^\tau e^{-\rho u}\left(dI_u + \lambda a_u du\right)\Big|\, \mathcal{F}_t\right] 1_{\{t<\tau\}},$$

where $\{\mathcal{F}_t\}$ is the filtration generated by Y. Martingale representation theorem yields the following lemma.

Lemma 2.1 *For any contract X that implements $a_t = 0$ for all $t \leq \tau$, there exist predictable processes (β_0, β_1) such that w_t evolves before termination $(t \leq \tau)$ as*

$$dw_t = \rho w_t dt - dI_t + \beta_{0,t}\left[dY_{0,t} - \mu\left(1-\alpha_t\right)dt\right] - \beta_{1,t} dY_{1,t}\ . \tag{2}$$

The contract is incentive compatible if and only if

$$\mu\alpha_t\beta_{1,t} + \mu(1-\alpha_t)\beta_{0,t} \geq \lambda\ . \tag{IC}$$

And the contract satisfies the limited liability constraint of the agent if and only if

$$\beta_{1,t} \leq w_t \tag{3}$$

and

$$\beta_{0,t} + w_t \geq 0\ . \tag{4}$$

The proofs of this lemma and all the other lemmas and propositions are relegated to the Appendix unless otherwise specified. Intuitively, β_0 refers to the reward to the agent upon the arrival of C-evidence, and β_1 refers to the punishment to her upon the arrival of S-evidence. Inequality (IC) highlights the key feature of our model. Its left-hand side consists of the two means for the principal to incentivize the agent —

carrots and sticks, which have to sum up to at least λ, the agent's private benefit from shirking. The principal can choose not only the allocation of his attention α, but also β_0 and β_1, the magnitudes of reward of carrots and punishment of sticks. We require β_0 and β_1 to be deterministic functions of the agent's continuation value w.

Two limited liability constraints in Lemma 2.1 restrict the magnitudes of reward and punishment. Inequality (3) requires that the punishment of sticks should be no more than the whole stake promised to the agent. The other constraint (4) says that the reward for carrots plus the stake already promised to the agent has to be non-negative, which will be shown slack.

3 Basic Properties of the Optimal Contract

To build intuition, this section provides a heuristic derivation of some basic properties of the optimal contract. In Theorem 3.1 at the end of this section, we verify that this contract is indeed optimal.

Let $B(w)$ denote the principal's value function. We have the Hamilton–Jacobi–Bellman (HJB) equation in the continuation region ($t < \tau$)

$$rB\left(w\right) = \max_{\alpha\in[0,\bar{\alpha}],\beta_0,\beta_1} z + \left(1-\alpha\right)\mu\left[B\left(w+\beta_0\right) - B\left(w\right)\right] + \left[\rho w - \beta_0\mu\left(1-\alpha\right)\right]B^{'}\left(w\right)\ , \tag{5}$$

subject to

$$\mu\alpha\beta_1 + \mu(1-\alpha)\beta_0 \geq \lambda\ ; \tag{IC}$$

$$\beta_1 \leq w\ ; \tag{6}$$

$$\beta_0 + w \geq 0\ ; \tag{7}$$

and

$$\alpha \in [0, \bar{\alpha}] \ . \tag{8}$$

The left-hand side of Equation (5) is the principal's expected flow of value at instant t. The first term on the right-hand side, z, is the flow of synergy. The second term is due to the reward β_0 he gives to the agent if C-evidence arrives, which happens with probability $(1-\alpha)\mu dt$ conditional on $a = 0$ being implemented from t to $t + dt$. The third term arises due to the drift of w, where ρw is the rate at which the interests accrue, and $-\beta_0\mu(1-\alpha)$ is the flip side of carrots due to promise keeping — if C-evidence does not arrive, the principal reduces the agent's continuation value at this rate to balance against the reward for carrots, so that the continuation value w net of ρw is a martingale and thus the contract does deliver w in expectation to the agent.

Note that there is no term in Equation (5) that corresponds to sticks (i.e., no term containing β_1), because S-evidence never arrives if the agent does follow the contract and take $a = 0$ at each instant. In this sense, sticks serve only as an off-equilibrium threat. Therefore, the limited liability constraint (6) must be binding — If S-evidence were observed, the principal would maximize the penalty by terminating the project and confiscating the whole stake w promised to the agent.

Notationally, superscript * denotes objects of the optimal contract hereafter.

Property 1 $\beta_1^*(w) = w$.

Instead of $B(w)$, it is equivalent but more convenient to continue our analysis based on $V(w) = B(w) + w$, the sum of the principal's value function and the agent's continuation value, or their joint surplus. Equation (5) then becomes

$$rV(w) = \max_{\alpha \in [0,\bar{\alpha}], \beta_0} z + [\rho w - \beta_0\mu(1-\alpha)]V'(w) + (1-\alpha)\mu[V(w+\beta_0) - V(w)] - (\rho - r)w \ . \tag{9}$$

Next, since $r < \rho$, we guess and later verify that there is a payout boundary $\bar{w}$ as standard in existing dynamic contracting models, e.g., (DeMarzo and Sannikov 2006) and (Biais, Mariotti, Rochet, and Villeneuve 2010) — If $w > \bar{w}$, the principal will simply pay $dI = w - \bar{w}$ immediately and reduce the continuation value to $\bar{w}$; Otherwise, the principal will use backloading; i.e., waiting for the agent's continuation value w to increase instead of paying her immediately (i.e., $dI = 0$). By construction, $V(\bar{w} + \beta_0) = V(\bar{w})$, so that when $w = \bar{w}$, the third term in equation (9) equals zero, and $V'(\bar{w}) = 0$ if it exists. If $V'(\bar{w})$ does not exist, i.e., the left and the right derivatives are not equal, (9) is not defined at $w = \bar{w}$ and the coefficient in front of $V'(w)$ must be zero at $\bar{w}$. Notice that this coefficient is the drift of the continuation value. Hence, when $V'(\bar{w})$ does not exist, $\bar{w}$ is an absorbing payout boundary. As a result, no matter whether the payout boundary $\bar{w}$ is absorbing or not, the second term in equation (9) must also equal zero when $w = \bar{w}$, so that

$$V(\bar{w}) = \frac{z}{r} - (\rho - r)\frac{\bar{w}}{r} \tag{10}$$

and

$$B(\bar{w}) = \frac{z}{r} - \frac{\rho}{r}\bar{w} \; . \tag{11}$$

Moreover, we must have $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. Assume otherwise, once the continuation value reaches $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, the principal could always incentivize the agent with the following contract: paying out $\bar{w} - \frac{\lambda}{\rho+\mu\bar{\alpha}}$ immediately to reduce the agent's continuation value to $\frac{\lambda}{\rho+\mu\bar{\alpha}}$; setting $\alpha = \bar{\alpha}$, $\beta_1 = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ and $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$, so that (IC) is binding, and that $\beta_0\mu(1-\bar{\alpha}) = \rho\frac{\lambda}{\rho+\mu\bar{\alpha}}$, i.e., the drift of the agent's continuation value is zero with payment $I = 0$ and thus $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is an absorbing state. Then the principal's payoff becomes

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho + \mu\bar{\alpha}} - (\bar{w} - \frac{\lambda}{\rho + \mu\bar{\alpha}}) > \frac{z}{r} - \frac{\rho}{r}\bar{w} = B(\bar{w}) \ ,$$

where the inequality follows $\bar{w} > \frac{\lambda}{\rho+\mu\bar{\alpha}}$, contradicting the optimality of $B(\bar{w})$.

Property 2 *There exists a* $\bar{w} \in (0, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$ *such that i)* $dI^* = (w - \bar{w})^+$ *; ii)* V *is increasing in* $[0, \bar{w}]$*; iii) if* $w \geq \bar{w}$*,*

$$V(w) = z/r - (\rho - r)\bar{w}/r \ ; \tag{12}$$

and iv) either $V'(\bar{w}) = 0$*, or* $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w})$*, the drift at* $w = \bar{w}$*, is* 0*.*

Together with Assumption 2 and Property 1, we have $\beta_1^*(w) = w < \bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}} < \lambda/\mu$ for $w < \bar{w}$; i.e., by the (IC) constraint, sticks alone are not sufficient to incentivize the agent to work. Moreover, (IC) and Property 1 imply that $w\alpha^* + \beta_0^*(1 - \alpha^*) \geq \lambda/\mu$, and thus $\beta_0^*(w) \geq \lambda/\mu > \frac{\lambda}{\rho+\mu\bar{\alpha}}$ if $w < \bar{w}$. This, together with Property 2, implies

Property 3 $w + \beta_0^* \geq \bar{w}$ *for all* $w < \bar{w}$*.*

That is, a single piece of C-evidence suffices to make the continuation value w jump to the payout region $[\bar{w}, +\infty)$, so that $V(w + \beta_0^*) = V(\bar{w})$; i.e., β_0^*, the optimal reward for carrots, only raises their joint surplus from $V(w)$ to $V(\bar{w})$, and the remaining reward, $\beta_0^* - (\bar{w} - w)$, is an immediate transfer from the principal to the agent and has no impact on their joint surplus. Also, the limited liability constraint (7) slacks as conjectured.

Property 3 plays a crucial role in the derivation of the optimal contract, given that the value function V may not always be concave.[3] To see this, by Property 3,

[3]This possibility is discussed in Section 4.2.

Equation (9) becomes

$$rV(w) = max_{\beta_0,\alpha} z + [\rho w - \beta_0 \mu(1-\alpha)]V^{'}(w) + (1-\alpha)\mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \quad (13)$$

whose right-hand side is always decreasing in β_0. This has two important implications. First, it indicates an advantage of using sticks relative to using carrots, regardless of the concavity of V. In equilibrium, S-evidence never arrives, and thus sticks incentivize the agent without causing variation in her continuation value w. But if carrots are used (i.e., $\beta_0 > 0$), C-evidence does arrive in equilibrium and generate variation in w. Property 3 implies that effectively, the upward jump in w when C-evidence arrives is always $\bar{w} - w$ (after the bonus payment), which is independent of α and β_0. But the magnitude of the downward drift of w absent the arrival of C-evidence, $\beta_0 \mu(1-\alpha)$, is increasing in both the attention allocated to carrots, $\mu(1-\alpha)$, and the associated reward, β_0. Therefore, the more the principal resorts to carrots, the more adverse variation in w is generated, making it disadvantageous relative to sticks.

Second, the fact that the right-hand side of Equation (13) is decreasing in β_0 implies a binding incentive compatibility constraint (IC) in the no-payment region $[0, \bar{w}]$, i.e.,

Property 4 $\mu\,[\alpha^* w + (1-\alpha^*)\beta_0^*] = \lambda.$

The incentive compatibility constraint (IC) plays a central role in this model. Property 4 establishes that the combination of carrots and sticks should be just enough to overcome the agent's private benefit of shirking.

Note that the principal still has two degrees of freedom to set the sensitivities of the agent's continuation value to news reflecting her action. As mentioned in the literature review, this contrasts with the counterpart in models without the choice among multiple performance indicators, e.g., in (Sannikov 2008) and (Biais, Mariotti, Rochet, and Villeneuve 2010), where there is no such degree of freedom.

Now we are ready to derive the central piece of the model — the optimal allocation of attention, α, and the optimal reward for carrots, β_0, in the no-payment region $[0, \bar{w}]$. By Properties 1 and 4,

$$\beta_0^* = \frac{\lambda - \mu\alpha^* w}{\mu(1-\alpha^*)}. \tag{14}$$

Equation (14) highlights the substitution between the attention allocated to carrots, $1-\alpha$, and pecuniary reward β_0 for them, which is peculiar to our setup with flexibility in monitoring practice. The more attention is allocated to carrots, the higher is the probability of the arrival of C-evidence confirming the agent's effort, and thus the less reward is needed to incentivize the agent. Conversely, a higher pecuniary reward for carrots provides stronger incentive for the agent, and thus reduces the reliance of the principal on the arrival of C-evidence, enabling him to utilize sticks. Note that $\beta_0^* \to \infty$ as $\alpha^* \to 1$, and thus we assume $\bar{\alpha} < 1$ in Assumption 1 to preclude this situation.

In the no-payment region, we have $dI = 0$ by definition and $V(w + \beta_0) = V(\bar{w})$ by Property 2. Thanks to Equation (14) in addition, the HJB equation (9) becomes

$$rV(w) = \max_{\alpha \in [0,\bar{\alpha}]} z - (\rho - r)w + (1-\alpha)\mu[V(\bar{w}) - V(w)] + (\rho w - \lambda + \mu\alpha w)V'(w). \tag{15}$$

Notice that α affects the right-hand side of Equation (15) through the last two terms. As explained before, the third term reflects its impact through the reward of carrots; i.e., raising α reduces the arrival rate of C-evidence and that of the contingent increment $V(\bar{w}) - V(w)$ in their joint surplus. This in turn reduces the expected instantaneous joint surplus $(1-\alpha)\mu[V(\bar{w}) - V(w)]$. The impact is linear in α, and the marginal impact is $-\mu[V(\bar{w}) - V(w)]$, whose absolute value monotonically decreases with w.

The last term of the right-hand side of Equation (15) reflects the impact of α through the flip side of carrots; i.e., a lower arrival rate of C-evidence also reduces the downward drift of the agent's continuation value w due to promise keeping.[4] This increases the expected instantaneous joint surplus $(\rho w - \lambda + \mu\alpha w)V^{'}(w)$. This effect is also linear in α, with a marginal impact $\mu w V^{'}(w)$, which could be non-monotonic in w. Since the total impact of α is linear, with marginal impact

$$\mu\left[wV^{'}(w) + V(w) - V(\bar{w})\right], \tag{16}$$

we have the following corner solution.

Property 5 *If* $wV^{'}(w) + V(w) < V(\bar{w})$*, then* $\alpha^* = 0$ *and* $\beta_0^* = \lambda/\mu$*;*

If $wV^{'}(w) + V(w) = V(\bar{w})$*, then* $\alpha^* \in [0, \bar{\alpha}]$ *and* $\beta_0^* = \frac{\lambda - \mu\alpha^* w}{\mu(1-\alpha^*)}$*;*

If $wV^{'}(w) + V(w) > V(\bar{w})$*, then* $\alpha^* = \bar{\alpha}$ *and* $\beta_0^* = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$*.*

The following Theorem verifies that the contract that we derive is indeed optimal.

Theorem 3.1 *Under Assumption 3, the solution V to the HJB equation* (9) *is the principal's value function. Moreover, the optimal contract is characterized by Property 5.*

4 The Role of Flexible Monitoring

This section highlights the critical role of flexible monitoring, which is the central piece of this paper. Section 4.1 shows that such flexibility is indeed utilized by and

[4]Note that $\rho w - \lambda + \mu\alpha w \le 0$ since $\bar{w} \le \frac{\lambda}{\rho + \mu\bar{\alpha}}$. Raising α thus reduces the magnitude of the downward drift.

thus valuable to the principal. Section 4.2 further articulates that such flexibility enables the principal to perpetuate the agent's effort with positive probability when the synergy z is large relative to the agent's private benefit of shirking, λ. Section 4.3 summarizes these results with a graphic illustration using the narratives of career path and provides a few empirically plausible predictions.

4.1 Flexibility in Monitoring is Utilized

Property 5 establishes that other than knife-edge cases, the optimal attention allocated to sticks, α^*, is either 0 or $\bar{\alpha}$. This subsection further establishes that an optimal contract necessarily involves both possibilities. Specifically, Proposition 4.1 establishes that $\alpha^*(w) = 0$ when the agent's continuation value w is close to 0, and $\alpha^*(w) = \bar{\alpha}$ when w is close to the payout boundary $\bar{w}$. This indicates that the flexibility in allocating attention between different performance indicators allows the principal to incentivize the agent differently at different stages of her career, and is thus valuable to the principal.

Proposition 4.1 *There exists a $\hat{w}_0 \in (0, \bar{w})$ and a $\hat{w}_{\bar{\alpha}} \in [\hat{w}_0, \bar{w})$, such that $\alpha^*(w) = 0$ and $\beta_0^*(w) = \lambda/\mu$ if $w \in (0, \hat{w}_0)$, and that $\alpha^*(w) = \bar{\alpha}$ and $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$ if $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$.*

By Property 5, the optimal contract only involves $\alpha = 0$ and $\alpha = \bar{\alpha}$ except for the knife-edge case featuring indifference, from Equation (15) we know that for each $w \in (0, \bar{w})$, either $\alpha = 0$ and

$$rV(w) = z + [\rho w - \lambda]V'(w) + \mu[V(\bar{w}) - V(w)] - (\rho - r)w \ , \tag{17}$$

or $\alpha = \bar{\alpha}$ and

$$rV(w) = z + (1-\bar{\alpha})\mu[V(\bar{w}) - V(w)] + [\rho w - \lambda + \mu\bar{\alpha}w]V^{'}(w) - (\rho - r)w \ . \quad (18)$$

Both equations can be solved in closed form, and interested readers are referred to the Appendix. It can be verified that $V'(0)$ is finite. This implies $0 \cdot V^{'}(0) + V(0) = 0 < V(\bar{w})$, and by continuity, there is a neighborhood of $w = 0$ such that $wV^{'}(w) + V(w) < V(\bar{w})$. Thus, the principal relies completely on carrots when the agent's continuation value w is low. The statement for $(\hat{w}_{\bar{\alpha}}, \bar{w})$ can also be proved with the closed-form solutions.

Intuitively, when the agent's continuation value w is low, the principal should not rely on sticks at all, because the agent has little to lose even if she is confirmed to have shirked. Relying on carrots, on the other hand, also maximizes the chance that C-evidence (i.e., hard evidence confirming the effort of the agent) arrives. This helps the principal quickly accumulate the agent's "skin in the game", which makes sticks (which is costless to the principal) more effective in the future, and pushes the project away from termination (which is socially inefficient). When the agent's continuation value w is higher, the principal is able to impose large penalty if S-evidence arrives. Since such penalty is just an off-equilibrium threat, making sticks less costly than carrots, the principal should rely on sticks as much as possible.

The flexibility in combining carrots and sticks allows the principal to exploit their respective advantages. On one hand, carrots generate greater variation than sticks in the agent's continuation value, and are thus less favorable to the principal. This is because, given that the agent indeed works, S-evidence would not arrive and thus requires no adjustment to the agent's continuation value; while C-evidence does arrive in equilibrium, which necessarily involves the reward upon its arrival and the downward adjustment to the agent's continuation value in the absence of C-evidence. On the other hand, sufficiently high continuation value is required as the agent's skin in the game for sticks to be an effective incentive device on its own, whereas the effectiveness of carrots does not depend on the agent's continuation value. Moreover, even if sticks could work alone, a high continuation value of the agent has to be maintained, which involves interest expenditure to the principal, making sticks less favorable than carrots. This tradeoff between carrots and sticks makes the principal rely only on carrots when w is low, and on sticks as much as possible when w is high.

Concerning β_0, the reward for carrots, recall that the right-hand side of Equation (13) is decreasing in β_0, since an increase in β_0 makes the drift of the agent's continuation value, $\rho w - \beta_0 \mu(1-\alpha)$, more negative due to promise keeping, and thus makes the project more prone to termination. Hence, given the optimal attention allocation α^*, β_0^* should be set as low as possible — such that the incentive compatibility constraint (IC) is binding. Thus, for agents facing $\alpha^* = 0$, including those with $w \in (0, \hat{w}_0)$, we have $\beta_0^*(w) = \lambda/\mu$, and the resulting drift of w is $\rho w - \lambda < 0$; For agents facing $\alpha^* = \bar{\alpha}$, including those with $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$, we have $\beta_0^*(w) = \frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$, and the resulting drift of w is $\rho w - \lambda + \mu\bar{\alpha}w \leq 0$.[5]

Note first that $\beta_0^*(w)$ is constant in the region with $\alpha^*(w) = 0$, but is decreasing in the region with $\alpha^* = \bar{\alpha}$. This is because in the latter case, the penalty for sticks increases with w, partially substituting the reward for carrots that is required by the incentive compatibility constraint (IC). Second, $\beta_0^*(w)$ features an upward jump when α^* switches from 0 to $\bar{\alpha}$. To see this, notice the fact that any switching point $w < \frac{\lambda}{\rho + \mu\bar{\alpha}} \leq \frac{\lambda}{\mu}$ implies that the size of the upward jump is $\frac{\lambda - \mu\bar{\alpha}w}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} > \frac{\lambda - \mu\bar{\alpha}\cdot\frac{\lambda}{\mu}}{\mu(1-\bar{\alpha})} - \frac{\lambda}{\mu} = 0$. Third, the drift of w increases (i.e., becomes less negative) with w, due to the interest accrued (i.e., due to the term ρw) and the increasing reliance on sticks in lieu of

[5]This is because $w \leq \bar{w}$, and $\bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$ by Property 2.

carrots (i.e., due to the term $\mu\bar{\alpha}w$). Lastly, the drift of w is negative, which moves w towards 0, the termination boundary, unless w reaches the payout boundary $\bar{w}$ and $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, where the drift is zero; i.e., the project and the agent's effort are perpetuated. Section 4.2 characterizes when such perpetuation is optimal.

4.2 Possibility of Perpetuation of the Agent's Effort

This subsection discusses whether the optimal contract involves the perpetuation of the agent's effort with positive probability. Mathematically, this refers to whether the payout boundary $\bar{w}$ is an absorbing state. We show that this is related to the (local) convexity of the value function, which is in turn determined by the flexibility of the principal's attention allocation as captured by $\bar{\alpha}$, and by the ratio of the synergy z to that of the agent's private benefit of shirking, λ. Specifically, 1) $\bar{w}$ is absorbing if and only if $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$; 2) $\bar{w}$ is absorbing if and only if the value function V is not universally concave[6]. More precisely, $\bar{w}$ is absorbing if and only if V is convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$ given by Proposition 4.1; and 3) $\bar{w}$ is absorbing if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and $z/\lambda \geq \theta$, where the threshold θ is a function of $\bar{\alpha}$, r, ρ, μ and λ.

Again, the role of the flexibility in attention allocation worths a highlight. We show that without such flexibility, perpetuation of the agent's effort is impossible.

Recall from Property 2 that $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. We have in addition

Lemma 4.1 *$\bar{w}$ is absorbing if and only if $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$.*

Proof. First consider the "if" statement. If $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, we show that the following strategy is feasible and optimal, and makes $\bar{w}$ absorbing: $\alpha = \bar{\alpha}$, $\beta_0 = \frac{\rho\lambda}{\mu(1-\bar{\alpha})(\rho+\mu\bar{\alpha})}$ and $\beta_1 = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$. Feasibility results from the binding (IC) constraint. To see

[6] Recall from Section 2 that we will discuss public randomization in Section 5 and preclude it in the rest of the paper unless otherwise mentioned.

why $\bar{w}$ is absorbing, when $w = \bar{w}$, the positive component of the drift of the agent's continuation value due to accrued interest is $\rho\bar{w}dt = \frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, and the negative component due to Carrots is $\mu(1-\bar{\alpha})\beta_0 dt$, which also equals $\frac{\rho\lambda}{\rho+\mu\bar{\alpha}}dt$, so that w remains constant when no C-evidence arrives, and the whole reward β_0^* upon its arrival is paid out immediately so that w is also unchanged.

To see the optimality of this strategy, observe that the principal's expected payoff at $w = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is $E(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t})$. Since $Y_{0,t} - \mu(1-\bar{\alpha})t$ is a martingale,

$$E(\int_0^{+\infty} ze^{-rt}dt - \beta_0 \int_0^{+\infty} e^{-rt}dY_{0,t}) = \frac{z}{r} - \frac{\beta_0\mu(1-\bar{\alpha})}{r} = \frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} \ .$$

Thus, the expected joint surplus is

$$\frac{z}{r} - \frac{\rho}{r} \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} + \bar{w} = \frac{z}{r} - \frac{\rho - r}{r} \cdot \bar{w} \ .$$

From equation (10), this strategy achieves the optimal joint surplus at the payout boundary.

To show the "only if" statement, by Property 2, it suffices to show that any $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ cannot be absorbing. Any contract respecting the (IC) constraint satisfies

$$\beta_0\mu(1-\bar{\alpha}) \geq \lambda - \bar{w}\mu\bar{\alpha} > \rho \cdot \frac{\lambda}{\rho+\mu\bar{\alpha}} > \rho\bar{w} \ .$$

Thus, when no C-evidence arrives, the agent's continuation value always has a downward drift term $\rho\bar{w} - \beta_0\mu(1-\bar{\alpha}) < 0$. This implies that the payout boundary is reflective. ■

Notice the role of the flexibility in attention allocation here. If $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, the way for the principal to perpetuate the agent's effort there is to set $\alpha = \bar{\alpha}$; i.e., to rely on sticks as much as possible. But if $w_0 < \hat{w}_0$, such monitoring scheme is not viable since the agent has too little to lose if caught shirking. To avoid inefficient termination, the principal has to first rely on carrots to build up the agent's skin in the game while keeping her working, and then switch to stick-dominant mode when the continuation value is high enough. This approach is infeasible without the flexibility in attention allocation.

As the main proposition of this subsection, Proposition 4.2 further establishes the connection between the possibility of the perpetuation of the agent's effort, the (local) convexity of the value function, and the conditions on exogenous parameters.

Proposition 4.2 *Let $\hat{w}_0$ and $\hat{w}_{\bar{\alpha}}$ be given by Proposition 4.1. Then, $\bar{w}$ is absorbing if and only if V is convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$, which holds if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and $z/\lambda \geq \theta$, where the threshold θ is a function of exogenous parameters r, ρ, μ and $\bar{\alpha}$. Moreover, if $\bar{w}$ is absorbing, then $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$.*

That is, the agent's effort could be perpetuated if and only if *both* the principal enjoys sufficient flexibility in attention allocation *and* the synergy relative to the friction of the contractual relation is large enough. The Appendix gives the exact closed-form formula for the threshold θ. In other words, no matter how large is the synergy relative to the friction, the contractual relation would terminate in probability one as long as the principal does not have sufficient flexibility of attention allocation. This again stresses the importance of such flexibility in shaping the optimal contract.

Proposition 4.2 also establishes the equivalence relation between $\bar{w}$ being absorbing and the local convexity of the value function V. If V is globally concave, then the payout boundary $\bar{w}$ is reflective as in (Biais, Mariotti, Rochet, and Villeneuve 2010) and (DeMarzo and Sannikov 2006). That is, the agent receives a lumpy bonus of

$\beta_0 - (\bar{w} - w)$ and a jump of $\bar{w} - w$ in continuation value upon each arrival of C-evidence, but her continuation value will then still drift downward from $\bar{w}$ until the next arrival, and will eventually reach zero in probability one, resulting in termination of the project.

Moreover, if the agent's effort could be perpetuated, there is only one switching point in $(0, \bar{w})$ for attention allocation; i.e., the optimal $\alpha = 0$ in $(0, \hat{w}_{\bar{\alpha}})$, and $\alpha = \bar{\alpha}$ in $(\hat{w}_{\bar{\alpha}}, \bar{w})$.

The proof of Proposition 4.2 is based on the closed-form solutions to Equations (17) and (18). Interested readers are referred to the Appendix. Here we provide an intuitive explanation for the role of the flexibility based on these two equations.

In the range of w in which the optimal $\alpha = 0$, which at least includes $(0, \hat{w}_0)$ by Proposition 4.1, the value function V satisfies Equation (17), which yields

$$V''(w) = (\rho w - \lambda)^{-1} \left[(\rho - r) + (r + \mu - \rho) V'(w) \right] .^{7} \tag{19}$$

By Property 2, $\bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}} < \lambda/\rho$, so the drift of w, $\rho w - \lambda$, must be negative in this range. In the square brackets, $\rho - r$ is due to the interest accrued, $(\rho - r)w$; $rV'(w)$ is due to the interest $rV(w)$ earned by the principal; $\mu V'(w)$ is due to the expected jump in the principal's value, $\mu[V(\bar{w}) - V(w)]$, which falls by $\mu V'(w)\,dw$ for an increase in w by dw; $-\rho V'(w)$ is due to the drift of w, which rises by ρdw and costs the principal $\rho V'(w)\,dw$ for an increase in w by dw. Since $V'(w) > 0$ for $w < \bar{w}$, and both $\rho - r$ and $r + \mu - \rho$ are positive by Assumption 2, we have $V''(w) < 0$ for all w in this range; i.e., the value function is concave. This reflects the tradeoff between the constant marginal cost of backloading due to interest accrued and the decreasing marginal benefit of accumulating the cushion against termination, which happens when w hits zero for the first time. In addition, since the drift of w

is negative, if only carrots are accessible to the principal; i.e., if the principal cannot flexibly allocate attention between carrots and sticks, the payout boundary $\bar{w}$ must be non-absorbing and satisfy $V'(\bar{w}) = 0$. This echoes the results in existing models with only one-sided control, such as (DeMarzo and Sannikov 2006) and (Biais, Mariotti, Rochet, and Villeneuve 2010).

Now consider the range of w in which the optimal $\alpha = \bar{\alpha}$, which at least includes $(\hat{w}_{\bar{\alpha}}, \bar{w})$ by Proposition 4.1. The value function V satisfies Equation (18) there, which yields

$$V''(w) = (\rho w - \lambda + \mu w \bar{\alpha})^{-1} \{(\rho - r) + [r + (1 - \bar{\alpha})\mu - \rho - \mu\bar{\alpha}] V'(w)\}. \qquad (20)$$

Notice the difference between Equations (19) and (20) due to the attention $\bar{\alpha}$ allocated to sticks. First, it increases the drift of w by $\mu w \bar{\alpha}$. To see why, recall that in equilibrium, C-evidence does arrive and rewards for carrots do incur, which requires a downward drift to balance it due to promise keeping. However, S-evidence does not arrive in equilibrium and thus does not require a drift term to balance the penalty. Hence, when attention $\bar{\alpha}\mu$ is reallocated from carrots to sticks, it reduces the downward drift balancing the reward for carrots by $\mu w \bar{\alpha}$. By Lemma 4.1, the drift is still negative for $w < \frac{\lambda}{\rho + \mu\bar{\alpha}}$. This also interprets the new term $-\mu\bar{\alpha}V'(w)$ in the braces. Second, it reduces the expected jump in the principal's value from $\mu[V(\bar{w}) - V(w)]$ to $(1 - \bar{\alpha})\mu[V(\bar{w}) - V(w)]$, which accounts for the difference between $\mu V'(w)$ in Equation (19) and $(1 - \bar{\alpha})\mu V'(w)$ in Equation (20).

With those differences, if the terms in the square brackets, $r + (1 - \bar{\alpha})\mu - \rho - \mu\bar{\alpha}$ is negative; i.e., if $\bar{\alpha} > \frac{r - \rho + \mu}{2\mu}$ as in Proposition 4.2, V'' may be positive for some $w \in (\hat{w}_{\bar{\alpha}}, \bar{w})$; in other words, V may be convex in this range. In the Appendix, based on the closed-form solution to Equations (17) and (18), we show that this is the case if and only if $\bar{w}$ is absorbing. This reflects the fact that the higher is the agent's continuation value, the more likely that it reaches the absorbing payout boundary and the contract becomes completely immune to termination. This is impossible if the principal does not have the flexibility to reallocate attention from carrots to sticks.

In the proof, we further establish that besides $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$, the ratio of the synergy to the agent's private benefit of shirking, z/λ, has to be no less than θ, as the sufficient and necessary condition for $\bar{w}$ to be absorbing. Moreover, in this case, $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$, so that attention μ is reallocated only once as the agent's continuation value w rises from 0 to $\bar{w}$.

4.3 A Career Path Narrative

Using the narrative of career path, this subsection summarizes with a graphic illustration the role of flexibility in monitoring practice studied in this section, which is the core of this paper, and provides a few empirically plausible predictions.

Proposition 4.1 establishes that the optimal monitoring and compensation schemes for junior employees (i.e., agents with continuation value $w \in (0, \hat{w}_0)$) are qualitatively different from those for senior employees (agents with continuation value $w \in (\hat{w}_{\bar{\alpha}}, \bar{w})$). Concerning monitoring schemes, junior employees are incentivized in carrot-only mode (i.e., $\alpha = 0$), since they need to accumulate cushion against unemployment (i.e., termination) and have little to lose even caught shirking; Senior employees are instead incentivized in stick-dominant mode (i.e., $\alpha = \bar{\alpha}$), since they have enough skin in the game, and sticks are based on off-equilibrium penalties, which are less costly than on-equilibrium reward for carrots. Thus, our model predicts that

Prediction 1 *Junior employees are incentivized mainly based on confirmatory evidence of their contribution to their employer, while senior employees are incentivized* *mainly based on contradictory evidence of their contribution.*

Concerning compensation schemes, as an off-equilibrium threat, the penalty for sticks is always the whole continuation value w and thus increases in seniority of employees. The reward for carrots for junior employees is set to the minimum level required for inducing effort, and is constantly λ/μ, while that for senior employees, $\frac{\lambda-\mu\bar{\alpha}w}{\mu(1-\bar{\alpha})}$, decreases with their seniority, since the reward for carrots is more substitued by the penalty for sticks at higher continuation values. For super-senior employees, i.e., agents with continuation value $w > \bar{w}$, their promised stakes are so large that a payment $w - \bar{w}$ to reduce interest accrued is so urgent as to dominate their incentive problems. Hence,

Prediction 2 *The reward for each piece of confirmatory evidence of contribution to the employer varies little among junior employees, but decreases with seniority for senior employees. The penalty for each piece of contradictory evidence of contribution to the employer increases with seniority for both junior and senior employees.*

Concerning the possibility of being fired (i.e., termination), absent the arrival of C-evidence, the drift of the continuation value of junior employees is $\rho w - \lambda$, and that of senior employees is $\rho w - \lambda + \mu\bar{\alpha}w$. They are both negative unless $w = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, i.e., unless the (senior) employee is permanently hired. They are less negative the higher the continuation value w for two reasons. First, larger stakes in the game carry larger interest income; Second, larger stakes also allow for larger penalty for sticks and less reliance on carrots, and thus less downward drift in continuation value to balance the in-equilibrium reward for carrots. Therefore,

Prediction 3 *Except for those hired permanently, absent the arrival of confirmatory evidence of their contribution, an employee is more prone to unemployment than before, and the more so if the employees are less senior.*

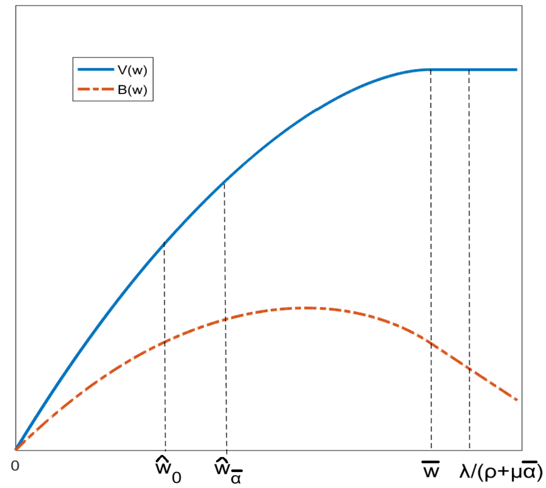

Figure 1: Reflective Payout Boundary $\bar{w}$

Then, when is it possible for employees to be hired permanently? Proposition 4.2 shows that it is the case if and only if both the flexibility in adjusting monitoring schemes and the potential synergy created by employees (relative to frictions) are sufficiently large.

First consider the case $\bar{\alpha} \leq \frac{r-\rho+\mu}{2\mu}$ as illustrated in Figure 1; i.e., the flexibility in monitoring is not large enough. The blue solid line corresponds to the value function V (in terms of the joint surplus), and the red dash-dotted line corresponds to the principal's value function $B(w) = V(w) - w$. The value function V is strictly concave in $(0, \hat{w}_0)$, reflecting the standard incentive versus interest tradeoff; V is also strictly concave in $(\hat{w}_{\bar{\alpha}}, \bar{w})$, where the fact that the reward for carrots decreases with the agent's stake in the game makes V less concave; However, since $\bar{\alpha}$ is low, the principal does not have enough flexibility to rely on sticks to the extent he wants, so that $\bar{w}$ is reflective and thus given by $V'(\bar{w}) = 0$. That is, senior employees who just receive a reward for carrots still face a downward drift in their promised stakes and thus the risk of being fired. This is the case no matter how large is the synergy z relative to λ, the agent's private benefit of shirking.

Now fix an $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$, so that the principal does have enough flexibility in adjusting monitoring scheme. If $z/\lambda < \theta$, the synergy is too low to justify perpetuation of the agent's effort, so the optimal contract is qualitatively similar to that of the case $\bar{\alpha} \leq \frac{r-\rho+\mu}{2\mu}$; Once z/λ grows beyond the threshold θ, the optimal contract changes fundamentally as shown in Figure 2. — Now the principal has both the flexibility and the desire to perpetuate the agent's effort, so now the payout boundary $\bar{w}$ becomes absorbing. That is, the agent is "tenured" once her effort is confirmed by an arrival of C-evidence. In addition, the possibility of complete avoidance of termination creates new marginal benefit of increasing continuation value, and thus makes the value function V convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$. Moreover, $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$, so that α, the attention allocated to sticks, only switches once as the agent's continuation value rises from 0 to $\bar{w}$. Thus, we have

Prediction 4 *Employers offer permanent positions if and only if both their flexibility in adjusting monitoring schemes and the potential synergy created by employees (relative to frictions) are sufficiently large.*

5 Public Randomization

So far we have been focusing on deterministic contracts, on the basis that random contracts are of little practical relevance in reality. This is without loss of generality theoretically as well if the resulting value function is globally concave as in the case illustrated in Figure 1 and as in most existing models in the literature as well. However, as established in Proposition 4.2, our value function is convex in the vicinity of the payout boundary $\bar{w}$ if it is absorbing (as illustrated in Figure 2). For this situation, this section discusses the extension in which public randomization of the following

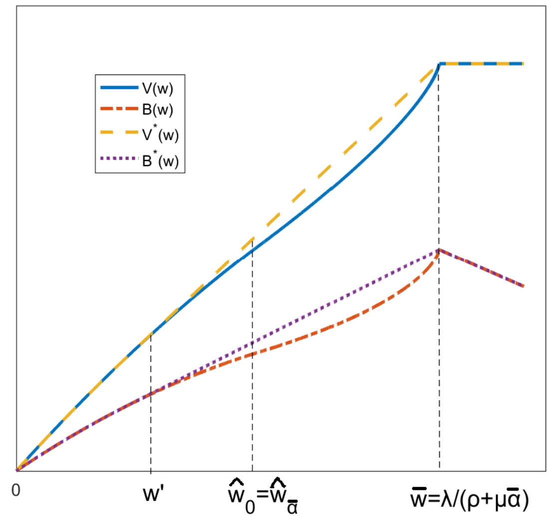

Figure 2: Absorbing Payout Boundary $\bar{w}$

form is allowed. At time 0, in addition to starting the contractual relation with a deterministic continuation value w_0, the principal can choose a mean-preserving spread of w_0 as the basis for random contracts, but no further randomization is allowed for $t > 0$. Since $B = V - w$, and the linear term has no effects on the concavification operation, we can work with the joint surplus function V without loss of generality.

Proposition 5.1 *With public randomization, the principal's value function $B^* = V^* - w$, where V^* is the concavification of V.*

Proof. Proposition 4.2 establishes that when V is not globally concave, we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $V(\bar{w})$ is uniquely determined by Property 2. In addition, V is concave in $(0, \hat{w})$ and convex in $(\hat{w}, \bar{w})$, where $\hat{w} \equiv \hat{w}_0 = \hat{w}_{\bar{\alpha}}$. Therefore, the concavification of V must be over $\bar{w}$ and some $w' \in (0, \hat{w})$ as shown with the yellow broken line in Figure 2.[8]

We check that the values of non-randomized states are not changed. First, $V(\bar{w})$ does not change because $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ is absorbing and its value does not depend on the

[8]The purple dotted line in Figure 2 illustrates the corresponding concavification B^* of the principal's value function B.

values of other states. For $w \in (0, w')$, notice that the continuation value may only drift downward or jump upward over $\bar{w}$. Since $V(\bar{w})$ remains the same and $V^* = V$ for $w \in (0, w')$, the values of these states satisfy the same HJB equation and thus remain the same. ■

6 Conclusion

This paper studies a continuous-time moral hazard problem in which the principal can flexibly combine "carrots" with "sticks" to incentivize the agent. That is, he can flexibly allocate his limited attention between confirmatory and contradictory evidence about the agent's effort as the basis for rewards and punishment. We find that such flexibility generates rich dynamics, which differs qualitatively from the situation where only one of the two means is feasible. When the agent has little skin in the game, the principal only resorts to carrots; When the agent has large skin in the game, the principal instead assigns the highest possible weights on sticks. Moreover, only with such flexibility can the agent's effort be perpetuated with positive probability when the agent is less patient than the principal.

References

Biais, B., T. Mariotti, J.-C. Rochet, and S. Villeneuve (2010). Large risks, limited liability, and dynamic moral hazard. *Econometrica 78*(1), 73–118.

Che, Y.-K. and K. Mierendorff (2019). Optimal dynamic allocation of attention. *American Economic Review 109*(8), 2993–3029.

Chen, M., P. Sun, and Y. Xiao (2017). Optimal monitoring schedule in dynamic contracts. *Available at SSRN 3038034*.

DeMarzo, P. M. and Y. Sannikov (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance 61*(6), 2681–2724.

Georgiadis, G. and B. Szentes (2019). Optimal monitoring design. Technical report.

He, Z. (2012). Dynamic compensation contracts with private savings. *The Review of Financial Studies 25*(5), 1494–1549.

Li, A. and M. Yang (2019). Optimal incentive contract with endogenous monitoring technology. *Theoretical Economics forthcoming.*

Myerson, R. B. (2015). Moral hazard in high office and the dynamics of aristocracy. *Econometrica 83*(6), 2083–2126.

Orlov, D. (2018). Frequent monitoring in dynamic contracts. Technical report, working paper.

Piskorski, T. and M. M. Westerfield (2016). Optimal dynamic contracts with moral hazard and costly monitoring. *Journal of Economic Theory 166*, 242–281.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *The Review of Economic Studies 75*(3), 957–984.

Sun, P. and F. Tian (2017). Optimal contract to induce continued effort. *Management Science 64*(9), 4193–4217.

Varas, F., I. Marinovic, and A. Skrzypacz (2019). Random inspections and periodic reviews: Optimal dynamic monitoring.

7 Appendix

7.1 Proofs in Section 2

7.1.1 Proof of Lemma 2.1

Proof. The proof is a standard application of the martingale representation theorem. For any given contract $X = (\alpha, I, \tau)$ and effort process a, define

$$M_t^{1,a} = Y_t^1 - \mu \int_0^t \alpha_s a_s ds$$

and

$$M_t^{0,a} = Y_t^0 - \mu \int_0^t (1 - \alpha_s)(1 - a_s) ds \ .$$

If the agent follows the effort process a, her lifetime expected payoff conditional on information at time t is

$$U_t = \int_0^{t \wedge \tau} e^{-\rho s}(dI_s + \lambda a_s ds) + e^{-\rho t} W_t \ .$$

Let $\tilde{a}$ be an arbitrary effort process. Let $\tilde{U}_t$ denote the agent's lifetime expected payoff conditional on information at t if she follows a till time t and then reverts to $\tilde{a}$. Then by the martingale representation theorem, U_t can be written as

$$U_t = U_0 - \int_0^{t \wedge \tau} e^{-\rho s} \beta_{1,s} dM_t^{1,a} + \int_0^{t \wedge \tau} e^{-\rho s} \beta_{0,s} dM_t^{0,a}$$

For each $t \geq 0$,

$$
\begin{aligned}
\tilde{U}_t =& U_t + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s}dM_t^{1,a} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s}dM_t^{0,a} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
=& U_0 - \int_0^{t\wedge\tau} e^{-\rho s}\beta_{1,s}dM_t^{1,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\beta_{0,s}dM_t^{0,\tilde{a}} + \int_0^{t\wedge\tau} e^{-\rho s}\lambda(\tilde{a}_s - a_s)ds \\
& - \int_0^{t\wedge\tau} e^{-\rho s}\mu\alpha_s\beta_{1,s}(\tilde{a}_s - a_s)ds - \int_0^{t\wedge\tau} e^{-\rho s}\mu(1-\alpha_s)\beta_{0,s}(\tilde{a}_s - a_s)ds
\end{aligned}
$$

Hence, $a_t = 0$ for all t is incentive compatible if and only if the drift term of the above expression is non-positive for any effort process $\tilde{a} \neq 0$, i.e.,

$$\lambda \leq \mu\alpha_t\beta_{1,t} + \mu(1-\alpha_t)\beta_{0,t}$$

for all t before termination. ■

7.2 Proofs in Section 3

We provide the proofs for Property 2 and Theorem 3.1 here, and those of all the other properties are straightforward from the text and thus omitted.

7.2.1 Proof of Property 2

Proof. Note that the joint value function V must be nondecreasing in continuation value w. This is because in any region where V is strictly decreasing in w, the principal can benefit from paying out to the agent, contradicting to the optimality of V. Let $A \subset \mathbb{R}_+$ denote the region of continuation values in which V is strictly increasing. Then the principal does not make any payment when $w \in A$ and $\mathbb{R}_+ \backslash A$ is the payout region. Since $\rho > r$, deferring payment becomes infinitely costly as $w \to +\infty$. Thus the payout region $\mathbb{R}_+ \backslash A$ is nonempty and there exists a $\bar{w} = \inf(\mathbb{R}_+ \backslash A)$.

By construction, V is strictly increasing for $w \in [0, \bar{w}]$ and is constant in a right neighborhood of $\bar{w}$, $(\bar{w}, \bar{w} + \Delta)$. Then, if $V'(\bar{w})$ exists, it must be zero. If $V'(\bar{w})$ does not exist, i.e., the left and the right derivatives are not equal, (9) is not defined at $w = \bar{w}$ and the coefficient in front of $V'(w)$ must be zero at $\bar{w}$. Notice that this coefficient is the drift of the continuation value. Hence, when $V'(\bar{w})$ does not exist, $\bar{w}$ is an absorbing payout boundary. As a result, no matter whether $V'(\bar{w})$ exists or not, the second and the third terms in the right hand side of equation (9) must be zero when $w = \bar{w}$, leading to $V(\bar{w}) = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$.

By definition, the payout region is a subset of $(\bar{w}, +\infty)$. Actually, the payout region is $(\bar{w}, +\infty)$. Otherwise, there exists an interval $(\bar{w}' - \Delta, \bar{w}') \subset (\bar{w}, +\infty)$ such that V is strictly increasing on $[\bar{w}' - \Delta, \bar{w}']$ and is constant in a right neighborhood of $\bar{w}'$. It must be the case that $\bar{w}' < \infty$, since $\rho > r$ and deferring payment is infinitely costly as $w \to +\infty$. Then similar argument regarding the existence of $V'(\bar{w})$ also applies here – no matter whether $V'(\bar{w}')$ exists or not, the second and the third terms in the right hand side of equation (9) must be zero when $w = \bar{w}'$, and thus $V(\bar{w}') = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}' < \frac{z}{r} - \frac{\rho - r}{r}\bar{w} = V(\bar{w})$, a contradiction to the non-decreasing property of V. Hence, the above defined $\bar{w}$ is the payout boundary and the payout region is $(\bar{w}, +\infty)$. As an immediate implication, the optimal payment is $dI^* = (w - \bar{w})^+$ and for $w \in [\bar{w}, +\infty)$, $V(w) = V(\bar{w})$.

The above proof has already shown that either $V'(\bar{w}) = 0$, or $V'(\bar{w})$ does not exist and $\rho\bar{w} - \mu(1 - \alpha^*(\bar{w}))\beta_0^*(\bar{w})$, the drift at $w = \bar{w}$, is 0.

The proof for $\bar{w} \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$ is straightforward from the text. ■

7.2.2 Proof of Theorem 3.1

Lemma 7.1 *For any* $\bar{w} \in (0, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$*, let* $\bar{V} \equiv \frac{z}{r} - \frac{\rho-r}{r}\bar{w}$ *and* α *be determined by Property 5, then the ODE*

$$rV(w) = z - (\rho - r)w + \rho w V^{'}(w) + (1-\alpha)\mu[\bar{V} - V(w)] - (\lambda - \mu\alpha w)V^{'}(w) \quad (21)$$

with boundary condition $V(0) = 0$ *has a unique solution on* $[0, \bar{w}]$*.*

Proof. For any $w < \frac{\lambda}{\rho+\bar{\alpha}\mu}$, since $\lambda - \mu\alpha w - \rho w > 0$, we can rearrange Equation (21) to obtain

$$V^{'} = \frac{z - (\rho - r)w + (1-\alpha)\mu[\bar{V} - V] - rV}{\lambda - \mu\alpha w - \rho w} \ .$$

Let

$$F(w, V) = \frac{z - (\rho - r)w + (1-\alpha)\mu[\bar{V} - V] - rV}{\lambda - \mu\alpha w - \rho w} \ .$$

For any fixed $\epsilon > 0$, for any $(w_1, V_1), (w_2, V_2) \in [0, \frac{\lambda}{\rho+\bar{\alpha}\mu} - \epsilon] \times [0, \bar{V}]$, there exists an M such that $|F(w_1, V_1) - F(w_2, V_2)| \leq M|V_1 - V_2|$. Then, by the Cauchy-Lipschitz theorem, the initial value problem has a unique solution over $[0, \frac{\lambda}{\rho+\bar{\alpha}\mu} - \epsilon]$. Further notice that V is increasing and upper bounded, thus V does not explode as $w \to \bar{w}$. Then the maximum interval of existence reaches the boundary $\bar{w}$ for all $\bar{w} \leq \frac{\lambda}{\rho+\bar{\alpha}\mu}$. When $\bar{w} = \frac{\lambda}{\rho+\bar{\alpha}\mu}$, taking $\epsilon \to 0$, we can extend the solution over $\left[0, \frac{\lambda}{\rho+\bar{\alpha}\mu}\right]$. ■

Proposition 7.1 *Consider two ODEs*

$$rV_1 = max_{\alpha\in[0,\bar{\alpha}]} z - (\rho - r)w + \rho w V_1^{'} + (1-\alpha)\mu[\bar{V}_1 - V_1] - (\lambda - \mu\alpha w)V_1^{'}$$

and

$$rV_2 = max_{\alpha\in[0,\bar{\alpha}]} z - (\rho - r)w + \rho w V_2^{'} + (1-\alpha)\mu[\bar{V}_2 - V_2] - (\lambda - \mu\alpha w)V_2^{'} \ ,$$

where $\bar{V}_1 = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}_1$, $\bar{V}_2 = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}_2$, $\bar{w}_1 < \bar{w}_2 \leq \frac{\lambda}{\rho + \bar{\alpha}\mu}$; *and* $V_1(0) = V_2(0) = 0$. *Then* $V_1 > V_2$ *for* $w \in (0, \bar{w}_1)$.

Proof. Suppose the contrary holds. Note that $V_1^{'}(0) > V_2^{'}(0)$. Then, there exists a $w \in (0, \bar{w}_1)$ such that $V_1(w) = V_2(w)$. Define $\tilde{w} = \inf\{w \in (0, \bar{w}_1) : V_1(w) = V_2(w)\}$. By the continuity of V_1 and V_2, we have $V_1(\tilde{w}) = V_2(\tilde{w})$. Let α_2 be the α that solves the maximization problem for V_2 at $\tilde{w}$. Taking the difference between the two ODEs at $w = \tilde{w}$, we obtain

$$(\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda) \cdot (V_1 - V_2)^{'} + (1 - \alpha_2)\mu(\bar{V}_1 - \bar{V}_2) \leq 0 \ .$$

Since $\alpha_2 < 1$ and $\bar{V}_1 - \bar{V}_2 > 0$,

$$(\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda) \cdot (V_1 - V_2)^{'} < 0 \ .$$

Since $\bar{w}_1 < \frac{\lambda}{\rho + \bar{\alpha}\mu}$, $\rho\tilde{w} + \mu\alpha_2\tilde{w} - \lambda < 0$. Thus, $V_1^{'}(\tilde{w}) - V_2^{'}(\tilde{w}) > 0$. Note that this inequality holds whenever $V_1 = V_2$. Since $V_1(w) - V_2(w)$ is continuous and the inequality is strict, it also holds for w close to $\tilde{w}$, i.e., $V_1^{'}(w) - V_2^{'}(w) > 0$ in $(\tilde{w} - \delta, \tilde{w})$ for some $\delta > 0$. By the defintion of $\tilde{w}$, $V_1(w) - V_2(w) > 0$ for $w \in (\tilde{w} - \delta, \tilde{w})$. Then, it is impossible to have $V_1(\tilde{w}) = V_2(\tilde{w})$, a contradiction. ■

According to the above results, the candidate of the optimal payout boundary is the smallest $\bar{w} \in (0, \frac{\lambda}{\rho + \bar{\alpha}\mu}]$ such that the solution of ODE (15) satisfies $V(\bar{w}) = \frac{z}{r} - \frac{\rho - r}{r}\bar{w}$. The existence of such $\bar{w}$ is guaranteed by the continuity of V. Now we are ready to prove Theorem 3.1.

Proof. Let τ denote the first time that w_t hits zero. We first verify that the principal's value function can be induced by the proposed control processes in Property 5 and the proposed payment process $dI_t = (\beta_0 + w - \bar{w})^+ dY_t^0$. Note that by Property 3,

$\beta_0 + w > \bar{w}$, so that $dI_t = (\beta_0 + w - \bar{w})dY_t^0$. By Ito's Formula for jump processes,

$$e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) = B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - rB(w_s)]ds + \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 \ .$$

Under the optimal control processes, the HJB equation becomes

$$rB(w) = z + (\rho w - \beta_0\mu(1-\alpha))B^{'}(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w + \beta_0 - \bar{w})] \ .$$

Thus,

$$B(w_0) = \int_0^{t\wedge\tau} e^{-rs}[z + (1-\alpha_s)\mu(B(\bar{w}) - B(w_s) - (w_s + \beta_{0,s} - \bar{w}))]ds - \int_0^{t\wedge\tau} e^{-rs}[B(\bar{w}) - B(w_s)]dY_s^0 - e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \ .$$

Due to the fact that $Y_s^0 - (1-\alpha_s)\mu s$ is a martingale and $w_\tau = 0$, letting $t \to \infty$ and taking expectation on the right hand side of the above equation, we obtain

$$B(w_0) = E(\int_0^{\tau} e^{-rs}[zds - (w_s + \beta_{0,s} - \bar{w})dY_s^0]) \ ,$$

which verifies that the principal's expected payoff given by (1) is indeed achieved at the proposed control and payment processes.

We then verify the proposed contract is optimal. Since the cumulative payment process is increasing in time, without loss of generality, write a general payment process as

$$I_t = I_t^c + I_t^d \ ,$$

where I_t^c is a continuous increasing process and I_t^d includes discrete upward jumps.

By Ito's Formula for jump processes,

$$
\begin{aligned}
e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) =& B(w_0) + \int_0^{t\wedge\tau} e^{-rs}[(\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - rB(w_s)]ds \\
& - \int_0^{t\wedge\tau} e^{-rs}B^{'}(w_s)dI_s^c + \int_0^{t\wedge\tau} e^{-rs}[B(w_s+\beta_{0,s}) - B(w_s)]dY_s^0 \\
& + \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s+\beta_{0,s}\Delta Y_s^0 - \Delta I_s^d) - B(w_s+\beta_{0,s}\Delta Y_s^0)] \ ,
\end{aligned}
$$

where $\Delta Y_s^0 \equiv Y_s^0 - Y_{s-}^0$. We then rearrange the terms to get

$$
\begin{aligned}
B(w_0) =& e^{-r(t\wedge\tau)}B(w_{t\wedge\tau}) \\
& + \int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - (1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds \\
& + \int_0^{t\wedge\tau} B^{'}(w_s)e^{-rs}dI_s^c + \int_0^{t\wedge\tau}[B(w+\beta_{0,s}) - B(w)][(1-\alpha_s)\mu ds - dY_s^0] \\
& - \sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s+\beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s+\beta_{0,s}\Delta Y_s^0)] \ .
\end{aligned}
$$

Taking expectation on both sides and using the fact that $Y_t^0 - \int_0^s(1-\alpha_s)\mu ds$ is a martingale, we obtain

$$
\begin{aligned}
B(w_0) =& E(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) \\
& + E(\int_0^{t\wedge\tau} e^{-rs}\{rB(w_s) - (\rho w_s - \beta_{0,s}\mu(1-\alpha_s))B^{'}(w_s) - (1-\alpha_s)\mu[B(w+\beta_{0,s}) - B(w)]\}ds) \\
& + E(\int_0^{t\wedge\tau} B^{'}(w_s)e^{-rs}dI_s^c) \\
& - E(\sum_{s\in[0,t\wedge\tau]} e^{-rs}[B(w_s+\beta_{0,s}\Delta Y_s^0 - \Delta I_t^d) - B(w_s+\beta_{0,s}\Delta Y_s^0)]) \ .
\end{aligned}
$$

Notice that

$$
rB(w) \geq z + (\rho w - \beta_0\mu(1-\alpha))B^{'}(w) + (1-\alpha)\mu[B(\bar{w}) - B(w) - (w+\beta_0-\bar{w})]
$$

and for any incentive compatible contract,

$$B(w+\beta_{0,s}) = B(\bar{w}) - (w+\beta_{0,s}-\bar{w}) \ .$$

Moreover, since $B^{'}(w) \geq -1$,

$$B(w_0) \geq E(e^{-r(t\wedge\tau)}B(w_{t\wedge\tau})) + E(\int_0^{t\wedge\tau} ze^{-rs}ds - \int_0^{t\wedge\tau} e^{-rs}dI_s^c) - E(\sum_{s\in[0,t\wedge\tau]} e^{-rs}\Delta I_t^d) \ .$$

Letting $t \to \infty$ and using the fact that $B(w)$ is bounded, we obtain

$$B(w_0) \geq E(\int_0^{\tau} e^{-rs}(zds - dI_s)) \ .$$

Therefore, any function satisfying all these conjectured properties is indeed the value function for the principal. ■

7.3 Proofs in Section 4

7.3.1 Proof of Proposition 4.1

Proof. Recall from Property 2 that $\bar{w} \leq \frac{\lambda}{\rho+\mu\bar{\alpha}}$. If $w = \bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, Equation (18) is exactly Equation (10), so $\alpha^*(w) = \bar{\alpha}$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, Equation (15) is equivalent to

$$V'(w) = \max_{\alpha\in[0,\bar{\alpha}]} \frac{z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]-rV(w)}{\lambda-\rho w-\mu\alpha w}. \tag{22}$$

Let $G(\alpha;w) = \frac{z-(\rho-r)w+(1-\alpha)\mu[V(\bar{w})-V(w)]-rV(w)}{\lambda-\rho w-\mu\alpha w}$, which is obviously continuous in both α and w. Property 5 establishes that the maximizer of the right-hand side (RHS) of Equation (22) must be 0 or $\bar{\alpha}$. So to figure out $\alpha^*(w)$, it suffices to compare $G(0;w)$

with $G(\bar{\alpha}; w)$, taking as given $V(0) = 0$ and $V(\bar{w})$.

For $w \in \left(0, \frac{\lambda}{\rho+\mu\bar{\alpha}}\right)$, $G(\bar{\alpha}; w) \geq G(0; w)$ is equivalent to

$$w\left[z - (\rho - r) w - rV\right] \geq \left[\lambda - (\mu + \rho) w\right] \left(V(\bar{w}) - V(w)\right) . \tag{23}$$

Let $\hat{w}_0 = \min\left\{\bar{w}, \frac{\lambda}{\rho+\mu}\right\}$. For any $w \in (0, \hat{w}_0)$, the left-hand side of Inequality (23) is negative, but its right-hand side is positive. So it fails, establishing the optimality of $\alpha(w) = 0$ in this range.

Now we establish the optimality of $\alpha(w) = \bar{\alpha}$ for w in the vicinity of $\bar{w}$. Note that by Equation (10), (23) is equivalent to

$$w(\rho - r)(\bar{w} - w) \geq \left[\lambda - (\mu + \rho + r) w\right] \left(V(\bar{w}) - V(w)\right), \tag{24}$$

which holds for all $w \geq \frac{\lambda}{\rho+\mu+r}$. So if $\bar{w} \in (\frac{\lambda}{\rho+\mu+r}, \frac{\lambda}{\rho+\mu\bar{\alpha}}]$, $\alpha^*(w) = \bar{\alpha}$ for all $w \in (\frac{\lambda}{\rho+\mu+r}, \bar{w}]$.

Note that Inequality (24) is equivalent to $\frac{\bar{w}-w}{V(\bar{w})-V(w)} \geq \frac{\lambda-(\mu+\rho+r)w}{(\rho-r)w}$. If $\bar{w} \leq \frac{\lambda}{\rho+\mu+r} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, by Lemma 4.1 (whose proof does not require Proposition 4.1), $\bar{w}$ is reflective so that $V'(\bar{w}) = 0$. Then by L'Hospital's rule, $\lim_{w\to\bar{w}^-} \frac{\bar{w}-w}{V(\bar{w})-V(w)} = \lim_{w\to\bar{w}^-} \frac{1}{V'(w)} = +\infty$, while $\lim_{w\to\bar{w}^-} \frac{\lambda-(\mu+\rho+r)w}{w(\rho-r)} = \frac{\lambda-(\mu+\rho+r)\bar{w}}{(\rho-r)\bar{w}} < +\infty$. Hence, there also exists a $\hat{w}_{\bar{\alpha}} < \bar{w}$, such that $\alpha(w) = \bar{\alpha}$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$.

$\beta_0^*(w)$ for $w \in (0, \hat{w}_0) \cup (\hat{w}_{\bar{\alpha}}, \bar{w}]$ results from Equation (14). ■

Here we provide the closed-form solutions to Equations (17) and 18). As a first-order linear ODE, Equation (17) has general solutions

$$V(w) = \frac{\rho - r}{r + \mu - \rho}(\frac{\lambda}{\rho} - w) + \frac{\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho}}{r + \mu} + K(\frac{\lambda}{\rho} - w)^{\frac{r+\mu}{\rho}} , \tag{25}$$

which are all strictly concave in $(0, \bar{w})$. From $V(0) = 0$, we can pin down for $w \in (0, \hat{w}_0)$ that $K = -\frac{\rho(\rho-r)}{(r+\mu)(r+\mu-\rho)} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu-\rho}{\rho}} - \frac{\mu V(\bar{w})+z}{r+\mu} \cdot \frac{\lambda}{\rho}^{-\frac{r+\mu}{\rho}}$.

Also as a first-order linear ODE, Equation (18) has general solutions

$$V(w) = \frac{\rho - r}{r + (1-\bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)}(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho + \bar{\alpha}\mu}}{r + \mu(1-\bar{\alpha})} + K(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r+(1-\bar{\alpha}}{\rho+\mu\bar{\alpha}}} \tag{26}$$

if $r + (1-\bar{\alpha})\mu \neq \rho + \bar{\alpha}\mu$ and

$$V(w) = -\frac{\rho - r}{\rho + \mu\bar{\alpha}}(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) ln(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) + \frac{(1-\bar{\alpha})\mu V(\bar{w}) + z - (\rho - r)\frac{\lambda}{\rho + \bar{\alpha}\mu}}{r + \mu(1-\bar{\alpha})} + K(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) \tag{27}$$

if $r + (1-\bar{\alpha})\mu = \rho + \bar{\alpha}\mu$. It is shown later in the proof of Proposition 4.2 that the solutions that are increasing in $(0, \bar{w})$ are strictly convex in $(0, \bar{w})$ if $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K < 0$, linear if $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K = 0$, and strictly concave in $(0, \bar{w})$ otherwise.

With the closed-form solutions and their concavity properties discussed above, we show the following proposition:

Proposition 7.2 *If $\bar{w} \geq \frac{\lambda}{\rho+\mu+r}$, then $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$.*

To prove Proposition 7.2, we first prove Lemma 7.2, which articulates that the optimal α takes values in $\{0, \overline{\alpha}\}$ almost surely.

Lemma 7.2 *There does not exist an interval (w_1, w_2) such that $w \cdot V'(w) = V(\bar{w}) - V(w)$ for all $w \in (w_1, w_2)$.*

Proof. Suppose the contrary. Then $w \cdot V'(w) = V(\bar{w}) - V(w)$ implies

$$V(w) = \frac{c}{w} + V(\bar{w}) \tag{28}$$

in (w_1, w_2) for some constant c. Plugging $w \cdot V'(w) = V(\bar{w}) - V(w)$ into the HJB equation (9) we obtain

$$V(w) = \frac{z - (\rho - r)\, w + (\rho + \mu - \lambda/w)\, V(\bar{w})}{r + \rho + \mu - \lambda/w} \ . \tag{29}$$

It is straightforward to verify that Equations (28) and (29) cannot be both satisfied in any interval. ■

Lemma 7.3 shows that the convexity of V in an interval below the payout boundary $\bar{w}$ is "contagion" up to $\bar{w}$.

Lemma 7.3 *If there exists an interval* $[w_1, w_2) \subset (0, \bar{w})$ *such that* $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$ *and* V *is convex in* (w_1, w_2)*, then* $\alpha^*(w) = \bar{\alpha}$ *for all* $w \in (w_1, \bar{w}]$ *and* V *is convex in* $[w_1, \bar{w}]$*.*

Proof. If V is convex in (w_1, w_2), since V is continuously differentiable in $(0, \bar{w})$, $w \cdot V'(w) + V(w)$ is strictly increasing in $[w_1, w_2)$. Given that $w_1 \cdot V'(w_1) \geq V(\bar{w}) - V(w_1)$, we have $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_2)$. So there exists $w_3 \in (w_2, \bar{w})$ such that $w \cdot V'(w) > V(\bar{w}) - V(w)$ for all $w \in (w_1, w_3)$. Iteration of this argument yields $w \cdot V'(w) > V(\bar{w}) - V(w)$ and thus $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w})$. By Proposition 4.1, $\alpha^*(\bar{w}) = \bar{\alpha}$ as well.

Given that $\alpha^*(w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, the specific solution to Equation (18) that matches the value function V in $[w_1, w_2)$ must also matches V in $[w_1, \bar{w}]$. Since V is convex in $[w_1, w_2)$, that specific solution must be given by Equation (26) with $K \leq 0$ and $r + (1 - \bar{\alpha})\mu < \rho + \bar{\alpha}\mu$. This proves the convexity of V in $[w_1, \bar{w}]$. ■

With Lemmas 7.2 and 7.3, we can now prove Proposition 7.2.

Proof. Let $\hat{W} \equiv \left\{ w \in (0, \bar{w}) : w \cdot V'(w) = V(\bar{w}) - V(w) \right\}$. We are to show that $\hat{W}$ is a singleton if $\bar{w} \geq \frac{\lambda}{\rho + \mu + r}$. By Proposition 4.1, $\hat{W}$ is non-empty and has a maximum.

Without loss of generality, assume $\hat{w}_{\bar{\alpha}} = \max \hat{W}$. Then V must be strictly concave in $(0, \hat{w}_{\bar{\alpha}}]$. To see this, Lemma 7.2 and the properties of the general solutions to Equations (17) and 18) imply that V must be piecewise concave or convex in $(0, \hat{w}_{\bar{\alpha}}]$. If there is an interval $(w_1, w_2) \subset (0, \hat{w}_{\bar{\alpha}}]$ such that V is convex in it, then by Lemma 7.3, $\alpha^* (w) = \bar{\alpha}$ for all $w \in (w_1, \bar{w}]$, contradicting the fact that $\hat{w}_{\bar{\alpha}} = \max \hat{W}$.

Note that Equation (29) holds for $w = \hat{w}_{\bar{\alpha}}$. Plug it into $V^{'} (\hat{w}_{\bar{\alpha}}) = \frac{V(\bar{w}) - V(\hat{w}_{\bar{\alpha}})}{\hat{w}_{\bar{\alpha}}}$, we have $V^{'} (\hat{w}_{\bar{\alpha}}) = \frac{\rho - r}{r + \rho + \mu} (1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}_{\bar{\alpha}}})$. Similarly, if there exists $\hat{w}' \in \hat{W}$ such that $\hat{w}' < \hat{w}_{\bar{\alpha}}$, then $V^{'} (\hat{w}') = \frac{\rho - r}{r + \rho + \mu} (1 - \frac{\frac{\lambda}{r+\mu+\rho} - \bar{w}}{\frac{\lambda}{r+\mu+\rho} - \hat{w}'})$. If $\bar{w} \geq \frac{\lambda}{\rho + \mu + r}$, then we have $V' (\hat{w}') \leq V' (\hat{w}_{\bar{\alpha}})$, contradicting the concavity of V in $(0, \hat{w}_{\bar{\alpha}}]$. ■

7.3.2 Proof of Proposition 4.2

Proof. By Proposition 4.1, $\alpha^* (w) = \bar{\alpha}$ if $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$, so here we focus on the solutions to Equation (18) when studying the property of the payout boundary $\bar{w}$. Let V_K be the solution with constant K in Equation (26) or (27).

Case 1: If $r + (1 - \bar{\alpha})\mu > \rho + \bar{\alpha}\mu$, we must have $\bar{w} < \frac{\lambda}{\rho + \mu\bar{\alpha}}$, and thus $\bar{w}$ is reflective by Lemma 4.1. To see this, Equation (26) yields

$$V_K^{'} = -\frac{\rho - r}{r + (1 - \bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)} - K \frac{r + (1 - \bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} (\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r + (1 - \bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} - 1}. \quad (30)$$

Since the first term of the right-hand side of Equation (30) is negative, K must be negative, otherwise $V_K^{'} < 0$ for all $w \leq \frac{\lambda}{\rho + \mu\bar{\alpha}}$, contradicting the optimality of $\alpha^* (w) = \bar{\alpha}$ for $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. Since $\frac{r + (1 - \bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} - 1 > 0$, V_K is concave. Moreover, as $w \to \frac{\lambda}{\rho + \mu\bar{\alpha}}$, $V_K^{'} \to -\frac{\rho - r}{r + (1 - \bar{\alpha})\mu - (\rho + \bar{\alpha}\mu)} < 0$, contradicting the optimality of $\alpha^* (w) = \bar{\alpha}$ for $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$ if $\bar{w} = \frac{\lambda}{\rho + \mu\bar{\alpha}}$.

Case 2: If $r + (1-\bar{\alpha})\mu = \rho + \bar{\alpha}\mu$, we have

$$V_K^{'} = -K + \frac{\rho - r}{\rho + \mu\bar{\alpha}} + \frac{\rho - r}{\rho + \mu\bar{\alpha}} ln(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w) \ .$$

Regardless of the value of K, V_K is concave and as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_K^{'} \to -\infty$. Thus, analogous to the previous case, it must be that $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$ and $\bar{w}$ is reflective.

Case 3: If $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$, since $-\frac{\rho - r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$, K in Equation (26) can be either positive or negative. Equation (30) yields

$$V_K^{''} = K\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}}(\frac{r + (1-\bar{\alpha})\mu}{\rho + \mu\bar{\alpha}} - 1)(\frac{\lambda}{\rho + \mu\bar{\alpha}} - w)^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 2} \ .$$

If $K > 0$, since $\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}} - 1 < 0$, $V_K^{''} < 0$ so that V_K is concave. Moreover, as $w \to \frac{\lambda}{\rho+\mu\bar{\alpha}}$, $V_{\bar{\alpha}}^{'} \to -\infty$. Again, it must be that $\bar{w} < \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is reflective.

If $K = 0$, then $V_K^{'}(w) = -\frac{\rho - r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)} > 0$ for all $w \in (\hat{w}_{\bar{\alpha}}, \bar{w}]$. Thus we must have $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$ as an absorbing state.

If $K < 0$, $V_K^{''} > 0$ so that V_K is strictly convex. Thus, the value function V satisfies $V^{'} > 0$ for all $w < \frac{\lambda}{\rho+\mu\bar{\alpha}}$. This implies that $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$, and $\bar{w}$ is absorbing by Lemma 4.1.

To summarize all the cases above, we have $\bar{w}$ is absorbing (i.e., $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}}$) if and only if $r + (1-\bar{\alpha})\mu < \rho + \bar{\alpha}\mu$ and $K \leq 0$; i.e., if and only if V is (weakly) convex in $(\hat{w}_{\bar{\alpha}}, \bar{w})$.

If $\bar{w}$ is absorbing, $\bar{w} = \frac{\lambda}{\rho+\mu\bar{\alpha}} > \frac{\lambda}{\rho+\mu+r}$, so we have $\hat{w}_0 = \hat{w}_{\bar{\alpha}}$ by Proposition 7.2. Let $\hat{w} = \hat{w}_0 = \hat{w}_{\bar{\alpha}}$ in this case.

Now we prove the second "if and only if" claim; i.e., $\bar{w}$ is absorbing if and only if $\bar{\alpha} > \frac{r-\rho+\mu}{2\mu}$ and $z/\lambda \geq \theta$.

From Equation (25), we have

$$V^{'}(\hat{w})=\frac{\rho-r}{r+\mu-\rho}(1-\frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1}+\frac{\mu V\left(\bar{w}\right)+z}{\lambda}(1-\frac{\rho}{\lambda}\hat{w})^{\frac{r+\mu}{\rho}-1}-\frac{\rho-r}{r+\mu-\rho}\ . \quad (31)$$

On the other hand, by Property 5, we have $V'\left(\hat{w}\right)=\frac{V(\bar{w})-V}{\hat{w}}$. Plug this into Equation (17), we have

$$V^{'}(\hat{w})=\frac{\rho-r}{r+\rho+\mu}(1-\frac{\frac{\lambda}{r+\mu+\rho}-\bar{w}}{\frac{\lambda}{r+\mu+\rho}-\hat{w}})\ . \quad (32)$$

As $\hat{w}$ increases from 0 to $\frac{\lambda}{r+\mu+\rho}$, the right-hand side of Equation (31) is decreasing from $\frac{\mu V(\bar{w})+z}{\lambda}$, and that of Equation (32) is increasing from $\frac{z-V(\bar{w})}{\lambda}$ to $+\infty$. Thus, there exists a unique $\hat{w}\in(0,\frac{\lambda}{r+\mu+\rho})$ such that both equations hold simultaneously.

Next, we show that V_K is convex if and only if $\hat{w}\geq\frac{\lambda}{2(\rho+\mu\bar{\alpha})}$. Observe that $V(\hat{w})$ should also satisfy Equation (26), and thus

$$V^{'}(\hat{w})=-\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)}-K\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}(\frac{\lambda}{\rho+\mu\bar{\alpha}}-\hat{w})^{\frac{r+(1-\bar{\alpha})\mu}{\rho+\mu\bar{\alpha}}-1}\ . \quad (33)$$

We have shown that V_K is convex if and only if $r+(1-\bar{\alpha})\mu<\rho+\bar{\alpha}\mu$ and $K\leq 0$. From Equations (32) and (33),

$$K\leq 0\Leftrightarrow\frac{\rho-r}{r+\rho+\mu}(1-\frac{\frac{\lambda}{r+\mu+\rho}-\bar{w}}{\frac{\lambda}{r+\mu+\rho}-\hat{w}})\geq-\frac{\rho-r}{r+(1-\bar{\alpha})\mu-(\rho+\bar{\alpha}\mu)},$$

which reduces to $\hat{w}\geq\frac{\lambda}{2(\rho+\mu\bar{\alpha})}$.Notice that if $r+(1-\bar{\alpha})\mu<\rho+\bar{\alpha}\mu$, $\frac{\lambda}{2(\rho+\mu\bar{\alpha})}<\frac{\lambda}{r+\mu+\rho}$.

Therefore, V_K is convex if and only if $r+(1-\bar{\alpha})\mu<\rho+\bar{\alpha}\mu$ (i.e., $\bar{\alpha}>\frac{r-\rho+\mu}{2\mu}$) and the right-hand side of Equation (31) and that of Equation (32) intersect at some $\hat{w}\in[\frac{\lambda}{2(\rho+\mu\bar{\alpha})},\frac{\lambda}{r+\mu+\rho})$. The second condition holds if and only if

$$\left(\frac{\rho-r}{r+\mu-\rho}+\frac{\mu V\left(\bar{w}\right)+z}{\lambda}\right)(1-\frac{\rho}{\lambda}\frac{\lambda}{2(\rho+\mu\bar{\alpha})})^{\frac{r+\mu}{\rho}-1}-\frac{\rho-r}{r+\mu-\rho}\geq\frac{\rho-r}{r+\rho+\mu}(1-\frac{\frac{\lambda}{r+\mu+\rho}-\frac{\lambda}{\rho+\mu\bar{\alpha}}}{\frac{\lambda}{r+\mu+\rho}-\frac{\lambda}{2(\rho+\mu\bar{\alpha})}}),$$

which is equivalent to

$$\frac{z}{\lambda} \geq \frac{r(\rho - r)}{\mu + r} \left\{ \frac{2(\rho + \mu\bar{\alpha})}{\rho + 2\mu\bar{\alpha}} \frac{2\mu\bar{\alpha}}{(r + \mu - \rho)[(\rho + \mu\bar{\alpha}) - (r + \mu(1 - \bar{\alpha}))]} - \frac{1}{r + \mu - \rho} + \frac{\mu}{\rho + \mu\bar{\alpha}} \right\} \equiv \theta \ .$$

■