A Theory of Participation in OTC and Centralized Markets*

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September 3, 2021

Abstract

Should regulators encourage the migration of trade from over-the-counter (OTC) to centralized markets? To address this question, we study a model in which banks make costly decisions to participate in an OTC market, a centralized market, or both markets at the same time. Banks differ in their ability to take large positions, what we call their trading capacity. In equilibrium, intermediate-capacity banks find it optimal to participate in the centralized market. In contrast, low- and high-capacity banks find it optimal to participate in the OTC market, due to an endogenous complementarity. Namely, low capacity banks receive worse terms of trade than in the centralized market but better risk sharing, thanks to the intermediation services offered by high-capacity banks. High-capacity banks receive worse risk sharing than in the centralized market, but profit from the provision of intermediation services to low-capacity banks. While the social optimum has qualitatively similar participation patterns, it prescribes that more customers migrate to the centralized market, and that more dealers enter the OTC market.

*An earlier version of this paper was circulated under the title "Platform Trading with an OTC Market Fringe." We'd like to thank, for fruitful comments and suggestions, the Editor Veronica Guerrieri and the four referees, as well as Ana Babus, David Cimon, Hugues Dastarac, Darrell Duffie, Selman Erol, Valentin Haddad, Peter Hoffmann, Gregor Jarosch, Peter Kondor, Jordi Mondria, Carlos Ramirez, Marzena Rostek, Norman Schürhoff, Zhaogang Song, and Jing Zeng, as well as seminar participants at UCLA, University of Luxembourg, Johns Hopkins University, University of Colorado Boulder, the Federal Reserve Bank of St. Louis, University of Toronto, UC Irvine, Texas A&M University, Universidad Diego Portales, the Bank of Canada Workshop on Money, Banking, Payments, and Finance, the 2nd LAEF OTC Markets and Securities Workshop at UCSB, the 2018 ASSA Meetings in Philadelphia, the Spring 2018 Midwest Macroeconomics Meetings at UW-Madison, the 2018 Annual Meeting of the Society for Economic Dynamics in Mexico, the CEPR 2018 European Summer Symposium in Financial Markets in Gerzensee, the European Finance Association 2018 Meetings in Warsaw, the Northern Finance Association 2018 Meetings in Charlevoix, the 20th Members Meeting of the Finance Theory Group at CMU, the Spring 2019 Southwest Search and Matching Workshop, the Banque de France-ESSEC 2nd Workshop on OTC Markets: Recent Advances in Research, the 2nd Asset Pricing Conference by LTI, the 15th Annual Central Bank Conference on the Microstructure of Financial Markets. Minghao Pan provided expert research assistance.

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1 Introduction

Over-the-counter (OTC) markets have a decentralized structure: trade is bilateral, opaque, and generates substantial price dispersion. A common policy concern is that, in OTC markets, dealers make substantial profits at the expense of customers, who pay high prices for low-quality intermediation services. In response, regulators have made proposals and taken measures to increase investors' participation in centralized markets.[1] But it is not obvious that such policies are welfare improving, since market participation decisions are endogenous: if customers were not content with the intermediation services provided by dealers, private parties could have successfully offered them to participate in centralized markets. In fact, when there is a centralized market option, volume often concentrates in the OTC market.[2] Therefore, to make a case for these policies, one must answer the following question: can it be socially optimal for investors to participate in a centralized market, when they find it privately optimal to participate as customers in an OTC market?

To address this question, we study an equilibrium in which investors, called "banks," make costly decisions to participate in an OTC market, a centralized market, or both markets at the same time. Banks are heterogeneous in two dimensions: in their risk-sharing needs, and in their ability to trade large quantities of the asset, what we call their trading capacity. Specifically, we assume that trade size in an OTC bilateral meeting is an increasing function of *both* counterparties' capacities while, in the centralized market, it is an increasing function of a bank's own trading capacity. Different trading capacities represent differences in funding constraints, access to collateral pool, risk-management technology, or trading expertise.

After deriving general theoretical properties of equilibrium trading patterns and participation incentives, we analyze versions of the model that can be solved in closed form. In our leading specification, banks differ continuously in terms of their capacities, have identical risk sharing need, and face participation costs inducing exclusive participation decisions. As

[1] For example, regulators have mandated that some swaps trade multilaterally on platforms called "swap execution facilities." In 2009, G20 Leaders agreed that "all standardized OTC derivative contracts should be traded on exchanges or electronic trading platforms." And, as of June 2017, "12 jurisdictions have in force comprehensive assessment standards or criteria for determining when products should be platform traded, and an appropriate authority regularly assesses transactions against these criteria" (Financial Stability Board, 2017).

[2] For example, Biais and Green (2006) note that "more than 1000 bond issues are still listed on the Exchange" but "the overwhelming majority of trades are conducted over the counter." Riggs, Onur, Reiffen, and Zhu (2018) document that Swap Execution Facilities allow investors to use different execution mechanisms that differ in their degree of centralization. They find that the most centralized mechanism, a limit-order book, attracts very little volume. Holden, Lu, Lugovskyy, and Puzzello (2021) show how in the Chinese Foreign Exchange market, trading activity migrated away from the traditional centralized Limit Order Book to a newly created OTC trading venue.

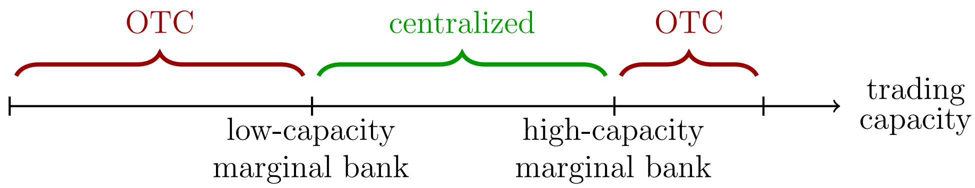

Figure 1: Non-monotonic participation with heterogeneous capacities. Low capacity and high capacity banks choose to participate in the OTC market, while intermediate capacity banks choose to participate in the centralized market. There are two marginal banks, one with a low capacity and one with high capacity.

illustrated by Figure 1, we find that participation decisions are non-monotonic in capacity: banks participate in the OTC market if their capacity is either small or large, and participate in the centralized market if their capacity is intermediate. Low-capacity banks find the OTC market attractive because they can demand intermediation services from high-capacity counterparties. Namely, while low-capacity banks trade at worse terms in the OTC market, their high-capacity counterparties help them establish larger positions than in the centralized market. Correspondingly, high-capacity banks find the OTC market attractive because they can profit from price dispersion by supplying intermediation services to low-capacity banks. Banks with intermediate capacity neither demand nor supply enough intermediation services, and find the centralized market more attractive because it allows them to escape the price discrimination of bilateral bargaining.

The equilibrium participation decisions and trading patterns resemble the observation that OTC markets have two layers of core-periphery structure, one between customers and dealers, and one between dealers (Hollifield, Neklyudov, and Spatt, 2017; Li and Schürhoff, 2019). We argue that low-capacity banks represent customers in practice, while high-capacity banks represent peripheral and core dealers. In equilibrium, peripheral dealers provide intermediation services to customers only, while core dealers provide intermediation services to both customers and peripheral dealers. Finally, the result that banks with intermediate capacity participate in the centralized market echoes the evidence of Holden, Lu, Lugovskyy, and Puzzello (2021) who study parallel OTC and centralized trading in the Chinese Foreign Exchange Market. They show that medium banks participate more in the centralized market than large banks, which corroborates our theoretical prediction that intermediate-capacity banks have the strongest incentives to participate in the centralized market.

Next, we study the welfare impact of reallocating marginal banks from the OTC to the centralized market. When participation is exclusive, this mechanically leads to a decrease in

OTC market participation, with two effects going in opposite directions. On the one hand, matches are destroyed because the marginal bank, who now participates only in the centralized market, no longer meets other banks in the OTC market. On the other hand, matches are created between the banks who no longer meet the marginal bank. We show that the first effect dominates for high-capacity marginal banks, while the second effect dominates for low-capacity marginal banks. Indeed, when the low-capacity marginal bank is reallocated to the centralized market, the trades destroyed have *smaller size* than the trades created, and vice versa for the marginal high-capacity bank. Hence, our welfare analysis suggests that marginal customers should participate more in centralized markets and, perhaps counter-intuitively, marginal peripheral dealers should participate more in OTC markets. A calculation of the social optimum confirms these findings.

In the last part of the paper we study three alternative specifications of the model. These examples illustrate how the analysis of welfare depends on equilibrium participation patterns and on banks' underlying heterogeneity.

In the first specification, participation is non-exclusive and the marginal bank is indifferent between participating in the OTC market only and participating simultaneously in the OTC and the centralized market. In that case, we find that making the marginal OTC banks participate in the centralized on top of the OTC market is welfare reducing. The reason is that, since the bank continues to participate in the OTC market, there is no match creation and destruction. The only effect at play is that, by participating in the centralized market and purchasing extra risk-sharing services, the bank reduces its transaction surplus aggregated across its OTC counterparties. Therefore, we end up with an unambiguous welfare loss in this specification.

Our second specification generates participation patterns similar to the exclusive case (in Figure 1), except that the participation of high-capacity banks is non-exclusive: they trade both in the OTC and the centralized market. We find that the welfare analysis turns out to be very similar to that in the exclusive model, because in this case if a marginal bank changes its optimal decision and moves to the centralized market, it leaves the OTC market. Therefore, in spite of non-exclusivity, reallocating marginal banks to the centralized market induces match creation and destruction in the OTC market.

Taken together, our two examples with non-exclusivity show that our main welfare effects are not driven by exclusivity per se. Instead, they depend on whether reallocating marginal banks to the centralized market induce match creation and destruction in the OTC market.

Finally, in the third specification, we consider the polar case in which banks have heterogeneous risk sharing needs but homogeneous capacities, and assume exclusive participation. We obtain that banks with strong risk sharing needs find it optimal to participate in the centralized market, while banks with moderate risk sharing needs find it optimal to participate in the OTC market. Amongst the OTC market participant, banks with moderate risk sharing needs supply intermediation services to banks with strong risk sharing needs. In this context, moving the marginal bank to the centralized market is in fact welfare reducing, because the trades destroyed have *larger value* than the trades created. An implication of these findings is that, in order to evaluate whether encouraging trade in a centralized trading venue is welfare improving, it is crucial to empirically distinguish an economy in which banks differ mostly in terms of trading capacity, from an economy in which banks differ mostly in terms of their risk sharing needs. Our examples suggest the following empirical distinctions. When banks differ mostly in terms of trading capacity, the per-dealer gross trading volume can be much larger than the per-customer gross volume, and the net trading volume of dealers can be large. In contrast, when banks differ mostly in terms of risk sharing needs, dealers and customers have comparable gross trading volume, but dealers have lower net trading volume. Hence, considering trading-volume patterns in the real-world OTC markets, our analytical examples suggest that banks differ mostly in terms of their trading capacity.

Literature review

This paper builds on Atkeson, Eisfeldt, and Weill (2015, henceforth AEW), who have developed a tractable framework, using insights from both the search- and network-theoretic literature, to study entry and trading patterns in an OTC market. We generalize AEW in two ways. First, while AEW only considered the margin of participation between autarky and the OTC market, we add a new margin: between the OTC and the centralized markets. This is clearly essential to analyze our main research question. Second, we allow banks to differ in a new dimension, their trading capacities. Heterogeneity in capacities gives rise to rich and realistic participation and trading patterns in OTC and centralized market, and generate new insights about welfare. Finally, the mathematical framework is also more general since we consider general distributions over both risk endowments and trading capacities, instead of discrete distributions over risk endowment only in AEW. While this introduces some technical difficulties, it also has advantages: it provides tools and results that are likely to be useful in other applications, it clarifies the economic forces at play, and it leads to closed-form characterizations of equilibrium for some important cases of interest.

A branch of the literature compares the costs and benefits associated with centralized and decentralized trading structures without endogenous participation decision. See, for example, Geromichalos and Herrenbrueck (2016), Liu, Vogel, and Zhang (2018), Li and Song (2019), Vogel (2019), Glode and Opp (2020), and Colliard, Foucault, and Hoffmann (2021). Another branch of the literature has studied the trade-off between exclusive participation in a centralized or a decentralized market. For example, Yavaş (1992), Gehrig (1993), Rust and Hall (2003), Miao (2006), Lee and Wang (2018), and Yoon (2018) considered models in which investors can search for OTC trading counterparties (customers or exogenously specified dealers) or trade with market-makers in a centralized venue, and showed that either venue may dominate from a welfare perspective depending on conditions.[3] While these papers only considered the way customers trade off between OTC and centralized markets, we also consider the trade-off faced by dealers. That is, in our model, both trading roles and trading venues are endogenous, while the rest of the literature took trading roles as given. This is important because in our main analytical example, we show that the dealer segment of the OTC market is too small while the customer segment is too large from a normative perspective. In addition, we also study non-exclusive participation, i.e., the possibility that investors participate simultaneously in two markets. We show that, in some cases, the normative analysis of non-exclusive participation is conceptually different from that of exclusive participation.[4]

Many recent papers have studied theoretically the emergence of traders with different degree centrality based on alternative assumptions regarding their heterogeneity. A non-exhaustive list of papers includes Babus (2009), Neklyudov (2012), Hugonnier, Lester, and Weill (2014), Afonso and Lagos (2015), Farboodi, Jarosch, and Shimer (2015), Farboodi, Jarosch, and Menzio (2016), Chang and Zhang (2016), Wang (2016), Babus and Kondor (2018), Bethune, Sultanum, and Trachter (2018), and Üslü (2019). Our paper builds on their insights with a different modeling framework to study equilibrium and socially optimal participation in OTC

[3]The majority of these papers assume indivisible assets, and so, trade sizes are fixed at one, not leaving any room for heterogeneity in trading capacity. Accordingly, the conditions they derive for welfare improvement are in terms of agents' valuations, the counterpart of risk sharing needs in our model. The other papers with divisible assets such as Yoon (2018) assume agents who are not restricted in their trade size, i.e., agents have "deep pockets." Hence, the conditions for welfare improvement in our leading analytical example with heterogeneous trading capacity are entirely complementary to what has been shown in the literature so far.

[4]Praz (2014) also studied non-exclusive trade within a dynamic equilibrium asset pricing framework, but without endogenous participation. In his model, investors trade two correlated assets in two markets, the first one in a centralized market, and the second one in a decentralized search market.

vs. centralized market. The advantage of our static modeling framework is that it allows for a rigorous, transparent, and simple characterization of the composition externalities induced by participation decisions. In addition, the participation and trading patterns endogenously generate two layers of core-periphery structures in our model: one between customers and dealers, and one between periphery and core dealers. This is reminiscent of OTC markets in practice. While agents differing from one another continuously in terms of their degree centrality in the OTC market is common in the literature,[5] an endogenous discontinuity in centrality arising due to the presence of a centralized venue is, to the best of our knowledge, a new result.

There is also an IO literature studying endogenous market participation. Mankiw and Whinston (1986) study entry to a product market with imperfect competition and find that an entrant's private incentive to enter may be larger than its social incentive depending on how it affects the incumbent's endogenous supply. McAfee and McMillan (1987), Levin and Smith (1994), and Menezes and Monteiro (2000) study auctions with endogenous participation decision and derive results markedly different from what the auction models with an exogenous set of bidders imply. For example, Levin and Smith (1994) show that limiting the number of potential bidders may be socially desirable, which is contrary to the predictions of models without endogenous participation. Bulow and Klemperer (2009) compare bidders' sequential entry decision to an auction and to a sequential sale mechanism with negotiation. They find that although the sequential sale mechanism always dominates in terms of social welfare, sellers usually prefer the auction. Similar to these studies, we also highlight a discrepancy between the privately and socially optimal outcomes when participation is endogenous. Naturally, our model significantly differs from these studies, because we model bilateral vs. multilateral trading of a perfectly divisible financial asset in a large market, while the mentioned studies focus on trading indivisible products in small markets.

Biais and Mariotti (2005), Axelson (2007), Rostek and Yoon (2018), and Babus and Hachem (2019) study how the market structure affects the optimal security design problem of asset issuers. A few papers have explored the manner in which market fragmentation may emerge as an equilibrium outcome due to information and price-setting frictions, and may dominate a centralized exchange. See Kawakami (2017), Malamud and Rostek (2017), Babus and Parlatore (2017), and Cespa and Vives (2018). We do not seek to explain fragmentation per-se, nor study

[5]See Hugonnier, Lester, and Weill (2014), Farboodi, Jarosch, and Shimer (2015), and Üslü (2019), for example.

asset issuance. Instead, we study the privately and socially optimal decisions of investors to participate in two given trading venues. Finally, we also contribute to the assignment literature because we study heterogeneous agents who make market participation decision.[6] Relative to this literature, our assignment model has two distinct features. First, participation is allowed to be non-exclusive. Second, since the OTC market is frictional, participation incentives are, in part, driven by incentives to demand or supply intermediation services.

The rest of the paper is organized as follows. In Section 2, we lay out our model of participation in an OTC and in a centralized market. In Section 3, we define an equilibrium and study its general properties. In Section 4, we consider analytical examples of the general model, under alternative assumptions about investor heterogeneity and participation costs. Then, we derive our main normative results regarding the social gain/loss from increasing the participation of customers in the centralized market.

2 Model

We generalize the model of Atkeson, Eisfeldt, and Weill (2015, henceforth AEW) in two ways. First, banks are heterogeneous in two dimensions: their risk-sharing need and their trading capacity. Second, banks can participate in two markets: an OTC market and a centralized market.

In the next two sections our presentation of the model is deliberately abstract. Our goal is to derive general properties, to highlight that the model is sufficiently flexible to capture a rich two-dimensional heterogeneity in banks' trading needs and trading ability, irrespective of microfoundations and functional forms, and to demonstrate that its results can be applied broadly to many OTC markets. We offer pencil-and-paper examples later, in Section 4.

2.1 Time, agents, and assets

There are four dates $t \in \{0, 1, 2, 3\}$, one good consumed at the terminal date, $t = 3$, and one divisible risky asset with normally distributed payoff. There is a measure one of traders who have Constant Absolute Risk Aversion (CARA), with the common coefficient η, over time-3 consumption. We assume that traders are organized into a measure one of large coalitions, called "banks." While many non-bank institutions such as hedge funds and insurance companies

[6] Some classical examples of the assignment game include Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981), and Bikhchandani and Ostroy (2002). See Núñez and Rafels (2015) for a recent survey on assignment markets.

trade in asset markets in reality, we employ the label bank, for brevity, to refer to all of them. We assume that banks differ in two dimensions: their risk-sharing needs and their trading capacities.

Differences in risk-sharing needs are generated by heterogeneity in banks' *endowment* of the risky asset, $\omega \in [0,1]$, where we normalized the upper bound of the endowment to 1. Since CARA banks have incentives to trade so as to equalize their holdings,[7] banks have stronger risk-sharing needs if their initial endowment is very large or small relative to the economy-wide average.

We assume as well that banks are heterogeneous in their *trading capacities*, denoted by $k \in [\underline{k}, \overline{k}]$. Banks with larger k can trade larger quantities in both the OTC market and the centralized markets, in a manner to be precisely explained shortly.

We let F denote the exogenous joint cumulative distribution of endowments and trading capacities, (ω, k), over the set $[0,1] \times [\underline{k}, \overline{k}]$, equipped with its Borel σ-algebra. We assume that, given capacity, the distribution of endowments is symmetric: $dF(\omega, k) = dF(1-\omega, k)$ for all $\omega \in [0,1]$.

2.2 Participation

At $t = 0$, banks make one of the four market participation decisions: they can choose to trade the risky asset in a decentralized OTC market with bilateral bargaining, $\pi = \mathrm{o}$, in a centralized market with price taking, $\pi = \mathrm{c}$, or in both markets at the same time, $\pi = \mathrm{oc}$. They can also stay in autarky, $\pi = \mathrm{a}$. We let $\Pi \equiv \{\mathrm{o}, \mathrm{c}, \mathrm{oc}, \mathrm{a}\}$ be the set of all possible participation decisions. After its participation decision, a bank's type is summarized by the triple $x \equiv (\omega, k, \pi)$. We let $\omega(x)$, $k(x)$ and $\pi(x)$ denote the endowment, capacity, and participation decision of type x. The cost of participation of type x is denoted by $C(x) \geq 0$. We assume for simplicity that $C(x)$ only depends on π and we normalize the cost of autarky to zero.

On aggregate, banks' collective participation decisions induce an endogenous measure N over the set X of all possible bank types, which we call *the participation path*. The participation path must satisfy a basic conservation condition:

$$dF(\omega, k) = \sum_{\pi \in \Pi} dN(\omega, k, \pi), \tag{1}$$

[7]Indeed, this is the allocation of assets in any Pareto optimum and, by the First Welfare Theorem, in any competitive equilibrium.

where $dN(\omega, k, \pi)$ denotes the measure of banks with endowment ω, capacity k, and participation decision π. This consistency condition states that the marginal distribution over endowment and trading capacity, (ω, k) must be consistent with the exogenous distribution F. Finally, anticipating a property of equilibrium, we guess throughout that the participation path is *symmetric*: banks with symmetric endowment and identical capacity make identical participation decisions, $dN(\omega, k, \pi) = dN(1-\omega, k, \pi)$.

2.3 Trading and payoffs

The timing of trade after participation decisions have been made is as follows. At $t = 1$, banks who chose $\pi \in \{\mathrm{o}, \mathrm{oc}\}$ trade in the OTC market. At $t = 2$, banks who chose $\pi \in \{\mathrm{c}, \mathrm{oc}\}$ trade in the centralized market. At $t = 3$, every bank consolidates all its traders' positions, and the risky asset pays off. We now describe trades and payoffs in detail.

OTC market trades. Let $X_\mathrm{o} \equiv \pi^{-1}(\{\mathrm{o}, \mathrm{oc}\})$ denote the set of banks' types participating in the OTC market, and assume positive OTC market participation, $N(X_\mathrm{o}) > 0$. Then, at $t = 1$, all banks with type $x \in X_\mathrm{o}$ send their traders to the decentralized OTC market, where they are paired uniformly to bargain over a bilateral trade. When a trader from a type-x bank is paired with a trader from a type-x' bank, the trader of type x buys a quantity $\gamma(x, x')$ of assets from the trader of type x', in exchange for the payment $P_\mathrm{o}(x, x')\,\gamma(x, x')$. A positive $\gamma(x, x')$ is an outright purchase, and a negative an outright sale. OTC market trades must satisfy an elementary bilateral feasibility constraint and a bilateral capacity constraint:

$$\gamma(x, x') + \gamma(x', x) = 0 \text{ for all } (x, x') \in X^2 \tag{2}$$

$$-\Gamma(x', x) \leq \gamma(x, x') \leq \Gamma(x, x'). \tag{3}$$

for some continuous, positive-valued and symmetric function $\Gamma(x, x')$. We rule out trade between types who do not participate in the OTC market by assuming that $\Gamma(x, x') = 0$ if $(x, x') \notin X_\mathrm{o}^2$. If both types participate in the OTC market, $(x, x') \in X_\mathrm{o}^2$, we assume that $\Gamma(x, x')$ only depends on, and is increasing in capacities (k, k'). The capacity constraint is crucial to our analysis because it prevents banks from fully sharing their risk by trading only in the OTC market. In practice, banks differ in their ability or willingness to let their traders take large positions for reasons such as differences in funding constraints, access to collateral pool, risk-management technology, or trading expertise. Taking stock, a collection of OTC market bilateral trades, $\gamma : X^2 \to \mathbb{R}$, is *feasible* if it is measurable and if it satisfies (2) and (3).

Centralized market trades. Let $X_{\mathrm{c}} \equiv \pi^{-1}(\{\mathrm{c}, \mathrm{oc}\})$ denote the set of banks' type participating in the centralized market and assume positive participation, $N(X_{\mathrm{c}}) > 0$. Then, at time $t = 2$, banks with types $x \in X_{\mathrm{c}}$ can trade multilaterally at some fixed price P_{c} in the centralized market. But they are still subject to a capacity constraint: this is natural because the economic forces underlying such constraint, such as risk-management concerns, are presumably at play in the centralized market as well.

With this in mind, we let a collection of centralized market trades be described by some measurable function $\varphi : X \to \mathbb{R}$. Centralized market trades are *feasible* if:

$$\int \varphi(x)\, dN(x) = 0 \tag{4}$$

$$-\Phi(x) \leq \varphi(x) \leq \Phi(x), \tag{5}$$

Condition (4) is the market-clearing condition in the centralized market, while condition (5) is the capacity constraint faced by banks in the centralized market, where $\Phi(x)$ is a positive and continuous function. We rule out trades by banks who do not participate in the centralized market by assuming that $\Phi(x) = 0$ if $x \notin X_{\mathrm{c}}$. If $x \in X_{\mathrm{c}}$, we assume that $\Phi(x)$ only depends on, and is increasing in the bank's own capacity k.

Consolidation and payoffs. After trading in the OTC and the centralized market, a bank consolidates all its trades. The asset's random payoff, denoted by v, realizes. Each trader then receives a consumption equal to the average per-trader payoff of her bank:

$$-C(x) + \omega(x)v + \int \gamma(x, x') \left[v - P_{\mathrm{o}}(x, x')\right] \, dN(x' \,|\, \mathrm{o}) + \varphi(x) \left(v - P_{\mathrm{c}}\right),$$

where $\omega(x)$ is the endowment of a bank of type x, and $N(x' \,|\, \mathrm{o}) \equiv N(x')/N(X_{\mathrm{o}})$. The first term is the participation cost incurred by type x. The second term is the payoff of the bank's asset endowment. The third term is the net payoff of OTC-market trades. Finally, the fourth term is the net payoff of centralized-market trades. To calculate the certainty equivalent corresponding to this payoff, we define the bank's *post-trade* exposure to the risky asset:

$$g(x) \equiv \omega(x) + \int \gamma(x, x')\, dN(x' \,|\, \mathrm{o}) + \varphi(x). \tag{6}$$

The first term is the initial endowment, the second term is the exposure gained via OTC-market trades, and the third term is the exposure gained via centralized-market trades. Then, the certainty-equivalent payoff of the bank writes:

$$-C(x) + U\left[g(x)\right] - \int \gamma(x,x')P_{\mathrm{o}}(x,x')\,dN(x' \mid \mathrm{o}) - \varphi(x)P_{\mathrm{c}}, \tag{7}$$

where $U(g) \equiv \mathbb{E}\left[v\right] g - \frac{\eta}{2}\mathbb{V}\left[v\right] g^2$ is the mean-variance payoff that obtains with CARA utility, absolute risk aversion η, and normally distributed asset payoff.[8]

3 Equilibrium

In this section, we define an equilibrium in two steps. First, we define an equilibrium conditional on participation decisions, summarized by the participation path N. Second, we define equilibrium participation decisions, N, given rational expectations about subsequent equilibrium trades.

3.1 Equilibrium trades given participation

We assume for simplicity that participation is positive in all markets, $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$ (as will be clear, it is straightforward to extend the analysis to the other cases).

Optimal trading in the OTC market. We assume that, in the OTC market, a trader maximizes his marginal impact on his bank's certainty-equivalent payoff, (7):

$$\gamma(x,x')\left\{U_g\left[g(x)\right] - P_{\mathrm{o}}(x,x')\right\},$$

where $U_g(\cdot)$ is the derivative of $U(\cdot)$, and where an individual trader takes others' decisions as given, as summarized by the post-trade exposure, $g(x)$. In other words, traders view themselves small relative to their bank's coalition, and do not coordinate their trades with other traders in the same bank coalition.[9]

[8]Quadratic payoffs are important to ensure that participation incentives and decisions are appropriately symmetric in endowment. That being said, a number of results regarding equilibrium conditional on participation go through even if v is not normally distributed and payoffs are not quadratic, i.e., the concavity of U suffices.

[9]This approach is used extensively in monetary economics literature as well as by AEW. See Lucas (1990), Andolfatto (1996), Shi (1997), and Shimer (2010), among others. It is also the continuum-population analogue of the Nash-in-Nash solution (i.e., the Nash equilibrium in Nash bargains) used in bilateral oligopoly settings with interdependent payoffs. See Horn and Wolinsky (1988), Stole and Zwiebel (1996), Crawford and Yurukoglu (2012), and Collard-Wexler, Gowrisankaran, and Lee (2019), among others. With the assumption of the

Assuming that bilateral trades are the outcome of symmetric Nash bargaining between the two traders, we obtain the following optimality conditions:

$$\gamma(x,x') = \begin{cases} \Gamma(x,x') & \text{if } g(x) < g(x') \\ \in [-\Gamma(x',x), \Gamma(x,x')] & \text{if } g(x) = g(x') \\ -\Gamma(x',x) & \text{if } g(x) > g(x') \end{cases} \tag{8}$$

for all $(x,x') \in X_{\mathrm{o}}^2$. That is, if the type-$x$ trader expects a lower post-trade exposure than the type-x' trader, then he should purchase some asset. Given that the type-x trader views himself as small relative to his coalition, he finds it optimal to purchase as much as feasible given the bilateral trading capacity constraint (3). The asset price between x and x' is set to split the bilateral gains from trade in half:

$$P_{\mathrm{o}}(x,x') = \frac{1}{2}\left\{U_g\left[g(x)\right] + U_g\left[g(x')\right]\right\}. \tag{9}$$

One sees from (8) that OTC market trades tend to bring banks' post-trade exposures closer together, in that banks with small exposures tend to buy from banks with high exposures. However, in general, banks do not equalize their post-trade exposures, for two reasons. First, the trading capacity constraint (3) limits the size of OTC market trades. Second, the bilateral trading protocol implies that traders in the same bank will trade in opposite direction depending on who they meet. For example, type-x traders purchase from type-x' traders if $g(x) < g(x')$, but they sell if $g(x) > g(x')$. Trades of the same size going in opposite direction net out to zero, and so do not contribute to the equalization of post-trade exposures.

Optimal trading in the centralized market. Taking first-order conditions with respect to $\varphi(x)$, one sees that the optimality condition in the centralized market is:

$$\varphi(x) = \begin{cases} \Phi(x) & \text{if } g(x) < U_g^{-1}(P_{\mathrm{c}}) \\ \in [-\Phi(x), \Phi(x)] & \text{if } g(x) = U_g^{-1}(P_{\mathrm{c}}) \\ -\Phi(x) & \text{if } g(x) > U_g^{-1}(P_{\mathrm{c}}) \end{cases} \tag{10}$$

continuum of banks and traders – thus, by allowing each bank to optimize participation and each trader to optimize trading decisions while taking as given decisions of others – we provide abstract results in Propositions 1 and 2 robust to how the various features of the model are microfounded.

That is, the bank trades as much as allowed by the trading capacity constraint, buying if its post-trade exposure is less than $U_g^{-1}(P_{\mathrm{c}})$, and selling otherwise.

Definition of equilibrium given participation. An equilibrium given positive participation in all markets, $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$, is a collection $(\gamma, \varphi, g, P_{\mathrm{o}}, P_{\mathrm{c}})$ of feasible OTC market bilateral trades, γ, feasible centralized market trades, φ, post-trade exposures, g, OTC market prices, P_{o}, and a centralized market price, P_{c}, such that (6), (8), (9), and (10) hold.

Notice that our definition of equilibrium requires that the optimality conditions (8) and (10) hold *everywhere*, even for sets of types that are measure zero according to N. This simply means that banks' trading decisions must be optimal both on and off the participation path, which is crucial to evaluate the value of all possible participation decisions and solve for equilibrium.[10]

Existence. To establish existence, we show that an equilibrium allocation, (γ, φ, g), solves a planning problem. Namely, we consider the social planning problem:

$$W^{\star}(N) = \sup \int \{U\left[g(x)\right] - C(x)\}\; dN(x),$$

with respect to square-integrable OTC market trades, γ, centralized market trades, φ, feasible N-almost everywhere, and post-trade exposures g generated by (γ, φ) according to (6). We obtain:

Proposition 1. *There exists an equilibrium given positive participation in all markets. All equilibria solve the planning problem given participation. The equilibrium is essentially unique in the sense that all equilibria share the same post-trade risk exposures, g, prices, $(P_{\mathrm{o}}, P_{\mathrm{c}})$, and certainty-equivalent payoff, (7). Finally, equilibrium post-trade exposures are continuous in x, symmetric and increasing in endowment, increasing in capacity if $\omega \leq 1/2$ and decreasing if $\omega \geq 1/2$, larger when a bank participates in the centralized market in addition to the OTC market if $\omega \leq 1/2$, and smaller if $\omega \geq 1/2$.*

The existence proof starts from the observation that all equilibria solve the planner's problem, which follows by direct comparison of the planner's first-order conditions with the equilibrium optimality condition (8)-(10). Next, using standard results on convex optimization

[10] Suppose, for example, that some banks with endowments and trading capacities (ω, k) in some set A only participate in the centralized market. That is, $N(A \times \{\mathrm{c}\}) > 0$ but $N(A \times \{\mathrm{o}, \mathrm{oc}\}) = 0$. To verify whether participating only in the centralized market is indeed optimal, banks $(\omega, k) \in A$ evidently need to compare the value of *all* participation decisions, $\pi \in \{\mathrm{o}, \mathrm{c}, \mathrm{oc}\}$. This means that we need to solve for trades, (γ, φ), and payoffs for all types $x \in A \times \{\mathrm{o}, \mathrm{c}, \mathrm{oc}\}$, even if some of these types are in measure zero according to N.

in infinite dimensional vector spaces (see, for example, Proposition 1.2, Chapter II in Eckland and Témam, 1987), we establish that the planner's problem has at least one solution. Finally, we show that an appropriately selected solution of the planner's problem is the basis of an equilibrium. The key difficulty in completing this step is that the planner's problem has many solutions, since it only cares about those types that have positive measure according to the participation path N. Specifically, the planner only needs to determine trading behavior on the participation path, while our definition of equilibrium requires to determine trading behavior both on and off that path.[11] However, the planner's problem uniquely determines aggregate market conditions: the post-trade exposures of all counterparties that can be met with positive probability in the OTC market, $g(x)$, and the price in the centralized market, P_{c}. This allows us to calculate optimal trading behavior given any off-path participation decision.

The Proposition also establishes intuitive properties of post-trade exposures given any symmetric participation path, N. In particular, banks' post-trade exposures are symmetric around the perfect risk-sharing benchmark, $\frac{1}{2}$. Moreover, banks who start with endowments further away from $\frac{1}{2}$ have more difficulties to share risk, in the sense that they end up with a post-trade exposure that is also further away from $\frac{1}{2}$. Finally, banks share risk more effectively if they can trade more, either because they have larger capacities or because they participate in the centralized market in addition to the OTC market.

3.2 Equilibrium participation

The certainty equivalent of a type-x bank, before participation cost, can be written:

$$U\left[g(x)\right] - \int \gamma(x, x')\, P_{\mathrm{o}}(x, x')\, dN(x' \,|\, \mathrm{o}) - \varphi(x)\, P_{\mathrm{c}}. \tag{11}$$

The first term is the certainty equivalent utility over post-trade exposure. The second term is the total cost for OTC market trades, and the third term is the total cost of centralized market trades. Using (6), (8) and (9), this formula can be re-written conveniently as follows.

[11] Continuing with the example of Footnote 10, consider banks with endowments and trading capacities (ω, k) in some set A who only participate in the centralized market, $N(A \times \{\mathrm{c}\}) > 0$ and $N(A \times \{\mathrm{o}, \mathrm{oc}\}) = 0$. As argued above, the equilibrium requires to determine their payoffs and trades on and off the participation path, that is, for all participation decisions $\pi \in \{\mathrm{c}, \mathrm{o}, \mathrm{oc}\}$. But since these banks only participate in the centralized market, the types $x \in A \times \{\mathrm{o}, \mathrm{oc}\}$ are in zero measure, have zero weight in the planner's objective, and so have indeterminate socially optimal trades.

Lemma 1. *Assume that participation is positive in all markets. Then, the certainty equivalent of a bank of type $x \in X$, (11), can be written*

$$U\left[\omega(x)\right] + \mathrm{MPV}(x), \;\; \textit{with } \mathrm{MPV}(x) = S(x) - \frac{B(x)}{2},$$

where $\mathrm{MPV}(x)$ *is the marginal private value,* $S(x)$ *the full appropriation surplus, and* $B(x)$ *the bargaining surplus, defined as:*

$$\begin{aligned} S(x) &\equiv U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - \int U_g\left[g(x')\right]\gamma\left(x, x'\right) dN(x' \,|\, \mathrm{o}) \\ B(x) &\equiv \int \left| U_g\left[g(x)\right] - U_g\left[g(x')\right] \right| \Gamma(x, x')\, dN(x' \,|\, \mathrm{o}). \end{aligned} \tag{12}$$

Moreover, the marginal private value increases with k, decreases with $\omega \in \left[0, \frac{1}{2}\right]$, and symmetrically increases with $\omega \in \left[\frac{1}{2}, 1\right]$.

The marginal private value, or MPV, is the net certainty-equivalent payoff relative to autarky. It can be split into two components.

The first component of the MPV, $S(x)$, is what we call the full appropriation surplus. It is the value of changing exposure, assuming that all assets are bought and sold at the counterparty's marginal value, which is the Walrasian price P_{c} in the centralized market and the counterparty bank's marginal value $U_g\left[g(x')\right]$ in the OTC market.

But the MPV is smaller than the full appropriation surplus because, for OTC market trades, a bank faces price impact through bilateral bargaining. This can be seen from the fact that Equation (9) depends positively on both counterparties' marginal value. As a result, when a type-x trader expects a lower post-trade exposure than her counterparty, she buys for more than her counterparty marginal value. Vice versa, when she expects a higher post-trade exposure she sells for less than her counterparty's marginal value. The second component of the MPV is the sum of all these OTC bargaining-induced losses for a bank of type-x: it is equal to half of the bargaining surplus, $B(x)/2$, due to the symmetry in bargaining powers.

Definition of an equilibrium with positive participation. An equilibrium with positive participation in both markets is a positive measure, N, over the set of banks' types, X, satisfying the following three conditions. First, participation is positive in both markets: $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$. Second, the participation path must satisfy (1), that is, it must be consistent with the primitive exogenous distribution of risk endowment and trading capacities, F. Third, the participation path must be generated by optimal participation decisions, that is:

$$\int \left(\mathrm{MPV}(x) - C\,(x) - \max_{\pi' \in \Pi} \left\{ \mathrm{MPV}\,(\omega(x), k(x), \pi') - C(\omega(x), k(x), \pi') \right\} \right) dN(x) = 0.$$

It is conceptually more subtle to define an equilibrium in which participation is zero in one or in both markets, $N(X_{\mathrm{o}}) = 0$ or $N(X_{\mathrm{c}}) = 0$. Indeed, in that case one needs to specify a bank's rational belief regarding its payoff if it chooses to enter a market in which no one else participates.[12] For the remainder of this paper, we will focus on equilibria in which participation is positive in all markets.

3.3 Efficient participation: a first-order approach

In this section, we evaluate the welfare impact of a *marginal change* in banks' participation. This is useful for at least two reasons. First, it allows to determine whether changes in N, such as encouraging more participation in the centralized market, would improve equilibrium welfare. Second, it delivers the first-order necessary conditions of a social optimum.

Let us start with some arbitrary symmetric participation path such that $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$ and consider changes in N of the form:

$$N + \varepsilon\,(n^+ - n^-),$$

where ε is a small positive number, while (n^+, n^-) is a pair of positive and symmetric measures that increase and decrease participation across markets. The pair (n^+, n^-) of positive and symmetric measures is *admissible* if it satisfies two natural conditions. First, the participation path $N + \varepsilon\,(n^+ - n^-)$ must satisfy (1), so as to conserve the distribution of endowments and trading capacities.[13] Second, the new participation path $N + \varepsilon\,(n^+ - n^-)$ must remain positive for all ε small enough. Formally, we require that n^- is absolutely continuous with respect to N, with a bounded Radon-Nikodym derivative.

From Proposition 1, we know that equilibrium social welfare given the participation path $N + \varepsilon\,(n^+ - n^-)$ solves an optimization problem: it is equal to $W^\star\left[N + \varepsilon\,(n^+ - n^-)\right]$, the

[12]One possible choice of beliefs is to assume that, if no one else participates in a market, then the payoff of participation is zero. But this creates coordination failures: no participation is always an optimal choice if the market is expected to be empty. Another choice is to assume that some infinitesimal exogenous measure of banks participate in all markets at no cost. Finally, one could also attempt to specify beliefs in the spirit of subgame perfection, as in a competitive search equilibrium. That is, if a bank chooses to enter in an empty market, it expects to attract the banks who have most incentives to enter.

[13]Equivalently, (n^+, n^-) must satisfy the conservation equation: $\sum_{\pi \in \Pi} dn^+(\omega, k, \pi) = \sum_{\pi \in \Pi} dn^-(\omega, k, \pi)$.

maximized value of the social planner's objective given the participation path $N + \varepsilon\,(n^+ - n^-)$. This observation allows us to use Envelope Theorems to calculate the derivative of social welfare with respect to ε. Precisely, adapting arguments from Milgrom and Segal (2002), we obtain:

Proposition 2. *Assume participation is positive in all markets and N is symmetric, and consider any admissible (n^+, n^-). Then, the function $\varepsilon \mapsto W^\star\left[N + \varepsilon\,(n^+ - n^-)\right]$ is right-hand differentiable at $\varepsilon = 0$, with derivative:*

$$\frac{d}{d\varepsilon}\left[W^\star(N + \varepsilon\,(n^+ - n^-)\right](0^+) = \int \{\mathrm{MSV}(x) - C\,(x)\}\,\left[dn^+(x) - dn^-(x)\right]$$
$$\textit{with} \quad \mathrm{MSV}(x) \equiv \mathrm{MPV}(x) + \mathbb{I}_{\{x \in X_\mathrm{o}\}} \frac{1}{2}\left[B(x) - \bar{B}\right],$$

where $\mathrm{MPV}(x)$ is the marginal private value, defined in Lemma 1, $B(x)$ is the equilibrium bargaining surplus given participation path N, defined in (12), and $\bar{B} = \int B(x')\,dN(x' \,|\, \mathrm{o})$ is the average equilibrium bargaining surplus across banks who participate in the OTC market.

The Proposition shows that, when a bank participates exclusively in the centralized market, then the marginal social and private values are equalized, $\mathrm{MSV}(x) = \mathrm{MPV}(x)$. It is intuitive that price taking aligns private and social incentives, even in the presence of capacity constraints.

However, the Proposition also shows that when a bank participates in the OTC market, $x \in X_\mathrm{o}$, there is a wedge between the marginal social value and the marginal private value, $\mathrm{MSV}(x) - \mathrm{MPV}(x) = \frac{1}{2}\left[B(x) - \bar{B}\right]$. As in the classical welfare analysis of matching models (see, e.g. Hosios, 1990), the wedge arises because OTC market prices do not incorporate the social value and cost of match creation and destruction induced by OTC market participation.

Participation induces match creation simply because a new participant trades with incumbents. The social value of match creation is equal to the bargaining surplus, $B(x)$. However, when they bargain, banks only appropriate half of the social value of match creation. The other half of the bargaining surplus, $B(x)/2$, drives a wedge between the MSV and the MPV.

But match creation has an opportunity cost: when a new participant matches with incumbents, incumbents match less together. This is what we call match destruction. To calculate the quantity and social value of these destroyed matches, notice first that the creation of a match between a new participant and an incumbent requires just one incumbent trader. The destruction of a match between incumbents frees up exactly two incumbent traders. Hence, the quantity of match destroyed per match created is equal to one half. Moreover, the matching protocol implies that matches are destroyed at random in the populations of incumbents. Hence, the average social value of a match destroyed is equal to the average bargaining surplus. Taken together, these observations imply that the social cost of match destruction is equal to half of the average bargaining surplus, $\bar{B}/2$.

We now study the welfare implications of reallocating a marginal bank from the OTC to the centralized market. We show that the analysis depends crucially on whether the reallocation induces match creation and destruction in the OTC market.

Marginal bank reallocation with match creation and destruction. Consider an equilibrium in which a marginal bank is indifferent between participation in the OTC market, with a type x such that $\pi(x) \in \{\text{o}, \text{oc}\}$, and exclusive participation in the centralized market, with a type x' such that $\pi(x') = \text{c}$ (we provide an analytical example of such equilibria in Section 4). If this bank changes its participation decision from $\pi \in \{\text{o}, \text{oc}\}$ to $\pi = \text{c}$, its type becomes x' instead of x and Proposition 2 implies that welfare changes by

$$\Delta W = \text{MSV}(x') - C(x') - (\text{MSV}(x) - C(x)).$$

Using the indifference condition $\text{MPV}(x') - C(x') = \text{MPV}(x) - C(x)$ between the OTC market and the centralized market,

$$\Delta W = \text{MSV}(x') - \text{MPV}(x') - [\text{MSV}(x) - \text{MPV}(x)] = \frac{1}{2}\left[-B(x) + \bar{B}\right], \tag{13}$$

where the second equality follows because, as noted before, exclusive participation in the centralized market aligns private with social values. To understand this formula, recall that when the marginal bank is reallocated to the centralized market, it no longer matches with infra-marginal OTC banks, with social cost equal to the bargaining surplus, $B(x)$. But infra-marginal OTC banks substitute their match with the marginal bank by matches amongst themselves, with social value equal to the average bargaining surplus, $\bar{B}$. The formula shows that, if $B(x) < \bar{B}$, then the reallocation of marginal banks from the OTC market to the centralized market is welfare improving. Inspecting the bargaining-surplus formula of Lemma 1, it is clear that $B(x)$ will be smaller than $\bar{B}$, and so, $\Delta W > 0$, if:

1. the trades of the marginal bank have a sufficiently small size, measured by $\Gamma(x, x')$, relative to the average,

2. the trades of the marginal bank create a small enough surplus per quantity traded, measured by $|U_g[g(x)] - U_g[g(x')]|$, relative to the average.

These two conditions depend on endogenous outcomes, specifically on the participation path N and on the post-trade exposure, $g(x)$. In the next section, we develop analytical examples that illustrate how these two conditions depend on exogenous parameters.

Marginal bank reallocation <u>without</u> match creation and destruction. Now let us consider an equilibrium in which a marginal bank is indifferent between trading exclusively in the OTC market, with a type x such that $\pi(x) = \text{o}$, and trading in both the OTC and the centralized market, with a type x' such that $\pi(x') = \text{oc}$. Proceeding as above we obtain a different formula for the social value of reallocation:

$$\Delta W = \frac{1}{2} \left[-B(x) + B(x') \right].$$

Relative to (13), this new formula replaces the average surplus, $\bar{B}$, by the bargaining surplus of the bank when it trades in the OTC and the centralized market at the same time. This is because, when participation is non exclusive, there is no match creation and destruction: the bank continues to trade in the OTC market. But its bargaining surplus changes, since it has access to the centralized market.

The formula of Lemma 1 now suggests a different condition for a welfare improvement: the reallocation of a marginal bank to from $\pi = \text{o}$ to $\pi = \text{oc}$ must increase its surplus per quantity traded in the OTC market. However, Proposition 1 implies that this condition can never be satisfied because a bank who trades in the centralized market on top of the OTC market moves closer to the full risk sharing benchmark, and so, reduces its surplus per trade with other OTC banks, $B(x') < B(x)$.

4 Analytical examples

In this section we study tractable parametric examples. This allows us to fill gaps left in the previous section: in particular, we establish equilibrium existence at the participation stage, we characterize patterns of participation and trade, and we compare systematically the equilibrium and the social optimum.

4.1 Heterogeneous capacities and exclusive participation

In our main example, we assume that capacities are heterogeneous across banks: they are distributed according to a continuous and strictly positive density $f(k)$ over the compact interval

$[\underline{k}, \overline{k}]$. To focus on the implications of heterogeneous capacities, we assume that risk-sharing needs are the same for all banks: namely, the endowment distribution has just two points, $\omega = 0$, or $\omega = 1$, with equal probability, and is independent from capacities. Therefore, $\omega = 0$ banks are the natural buyers while $\omega = 1$ banks natural sellers. While they trade in opposite directions, $\omega = 0$ and $\omega = 1$ banks have identical participation incentives as long as they have identical capacities.[14]

It is not obvious how to best specify the bilateral trading capacity constraint, $\Gamma(k, k')$, since we do not provide precise micro-foundations. We argue that it should satisfy a natural property: the quantity traded in a bilateral trade should depend positively on the capacities of both counterparties. For example, in practice, there is much more customer-to-dealer volume than customer-to-customer volume. Viewed through the lens of our model, this means that low-capacity customers are able to substantially increase the size of their transactions when they trade with high-capacity dealers.[15] To capture such effect in a tractable way, we assume that, in the OTC market, bilateral trades are subject to the separable capacity constraint $\Gamma(k, k') = (k + k')/2$.[16] Likewise, in the centralized market, the capacity constraint is $\Phi(k) = (k + K)/2$, where K represents the capacity of the centralized exchange. We assume that $K \in (0, 1)$ so that the centralized market is imperfect.[17]

Finally, we assume for now that banks pay identical cost $C \geq 0$ to participate either in the OTC market ($\pi = \text{o}$) or in the centralized market ($\pi = \text{c}$). As we focus on equilibria with symmetric participation, the centralized market price is $U_g(\frac{1}{2})$.

Equilibrium conditional on participation. We first show that:

[14]This property means that the same equilibrium would obtain if banks made their participation decision ex ante, before learning about their endowment. In this sense, participation does not depend on transitory changes in endowments but only on capacity.

[15]For a theoretical motivation, consider the recent work of Üslü (2019), who studies a dynamic model where trade quantities are determined bilaterally on the margin without any exogenous restrictions. He shows that, as a result of dealers' *endogenous* willingness to trade in large quantities, trading with a dealer enables a customer to trade in large quantities that would not be possible in a trade with another customer.

[16]Since we are allowing for a general density function, $f(k)$, the restriction imposed by this functional form is not linearity but separability. Formally, consider any separable capacity constraint of the form $h(\ell) + h(\ell')$, for some strictly increasing and differentiable function h and some capacity ℓ that is distributed according to some continuous and strictly positive density. Then, after the change of variable $k = 2h(\ell)$, this separable constraint becomes equivalent to the linear constraint considered here.

[17]One may be concerned that our result rely strongly on the separability of the capacity constraint. To alleviate this concern, in Online Appendix D, we also consider the "max" capacity constraint, that is $\Gamma(k, k') = \max\{k, k'\}$ and $\Phi(k) = \max\{k, K\}$, and we show our main results still hold.

Lemma 2. *Given any N with exclusive participation, an $\omega = 0$ bank with capacity k attains the following post-trade exposure and marginal private values:*

$$g(0,k,\mathrm{c}) = \frac{1}{2}\min\{k+K,1\} \quad \text{and} \quad \mathrm{MPV}(0,k,\mathrm{c}) = \frac{|U_{gg}|}{2} g(0,k,\mathrm{c})\left[1-g(0,k,\mathrm{c})\right]$$
$$g(0,k,\mathrm{o}) = \bar{g}_{\mathrm{o}} \quad \text{and} \quad \mathrm{MPV}(0,k,\mathrm{o}) = \frac{|U_{gg}|}{8}\left(2\bar{g}_{\mathrm{o}} + k\left(1-2\bar{g}_{\mathrm{o}}\right)\right).$$

where $\bar{g}_{\mathrm{o}} \equiv \frac{1}{2}\min\{\mathbb{E}[k' \,|\, \mathrm{o}], 1\}$ and $\mathbb{E}[k' \,|\, \mathrm{o}] \equiv \int k(x')\, dN(x' \,|\, X_{\mathrm{o}})$ is the average capacity amongst OTC market participants. Finally, the post-trade exposures of $\omega = 1$ banks are symmetric to that of $\omega = 0$ banks, and their marginal private values are the same.

Given that the centralized market price is $U_g(\frac{1}{2})$, centralized market trades bring post-trade exposures as close as possible to $\frac{1}{2}$, subject to capacity constraints.

A perhaps surprising result is that, regardless of N, the OTC market equilibrium has an *atom property*: banks with identical endowment equalize their post-trade exposures, even though they have different capacities. For example, a bank with $k \geq 1$ in the OTC market has sufficient capacity to attain the full-risk sharing post-trade exposure, $\frac{1}{2}$. Yet it chooses a lower one, $\bar{g}_{\mathrm{o}} < \frac{1}{2}$.

To gain intuition for this atom property, consider banks' equilibrium trades when $\bar{g}_{\mathrm{o}} < \frac{1}{2}$. Since $\bar{g}_{\mathrm{o}} < 1 - \bar{g}_{\mathrm{o}}$, an $\omega = 0$ bank always buys quantity $(k+k')/2$ from an $\omega = 1$ bank. This implies that $\omega = 0$ banks with high capacity purchase large quantities from $\omega = 1$ banks, while $\omega = 0$ banks with low capacity purchase small quantities. This gives $\omega = 0$ banks incentive to trade together in an effort to equalize their exposures. The Lemma shows that this leads to complete equalization: banks with identical ω have identical post-trade exposures, regardless of their capacity k.[18].

One can in fact verify that, when two $\omega = 0$ banks meet, an equilibrium trade is that the k bank purchases the quantity

$$\frac{k'-k}{2} \tag{14}$$

from the k' bank. This means that high-capacity $\omega = 0$ banks always sell to low-capacity $\omega = 0$ banks. In that sense, high capacity banks play the role of intermediaries: they buy from $\omega = 1$

[18]This result is not special to the separable trading capacity constraint. One can show for example that it holds whenever $\Gamma(k,k')$ is submodular in capacities. See Appendix C

banks and re-sell to $\omega = 0$ banks. Low capacity banks, on the other hand, play the role of customers: they buy both from $\omega = 1$ banks and from other $\omega = 0$ banks.[19]

Optimal participation. Lemma 2 reveals as well that the MPVs of participating in the two markets have different properties, as shown in Figure 2. While the MPV of centralized market participation is a concave and bounded function of k, the MPV of OTC market participation is linear and increasing in k, with a strictly positive slope if $\bar{g}_{\mathrm{o}} < \frac{1}{2}$. Taken together and keeping in mind that participation costs are the same in both markets, one sees from Figure 2 that the equation

$$\mathrm{MPV}(0, k, \mathrm{o}) = \mathrm{MPV}(0, k, \mathrm{c})$$

has at most two intersections.[20] One easily verifies that one of these is $k^{\star\star} = 1$. Let us denote the other intersection, if it exists, by $k^\star < 1$. We obtain:

Lemma 3. *If banks participate in both markets and if $\bar{g}_{\mathrm{o}} < \frac{1}{2}$, then optimal participation decisions are:*

$$\pi(0, k) = \pi(1, k) = \begin{cases} \mathrm{o} & \text{if } k \in [\underline{k}, k^\star] \\ \mathrm{c} & \text{if } k \in (k^\star, 1) \\ \mathrm{o} & \text{if } k \in \left[1, \bar{k}\right], \end{cases}$$

for some $k^\star \in [\underline{k}, 1 - K]$. Moreover $\bar{g}_{\mathrm{o}} > (k + K)/2$ for all $k \in [\underline{k}, k^\star]$.

Consider first the trade-off faced by $\omega = 0$ banks with low capacity, $k < k^\star$. The Lemma shows that, for these banks $\bar{g}_{\mathrm{o}} > (k + K)/2$, which means that the OTC market allows them to trade larger quantities than the centralized market. Indeed, in the centralized market, these banks would have to rely solely on their own trading capacity, while in the OTC market they can benefit from the intermediation services provided by $\omega = 0$ banks with high capacities

[19] Because all $\omega = 0$ banks have the same post-trade exposures, $\bar{g}_{\mathrm{o}}$, the optimal bilateral trades are in fact indeterminate. However, they cannot be arbitrary. In particular, the net trades are determinate: one can easily confirm that high-capacity $\omega = 0$ banks sell to other $\omega = 0$ banks, while low-capacity banks buy from other $\omega = 0$ banks.

[20] If participation costs are not equal, then the equations remain almost the same: one simply need to subtract $C(\mathrm{o})$ and $C(\mathrm{c})$ on the left- and right-hand sides. In Figure 2 this merely shifts the two MPV curves down, so it does not impact the qualitative prediction about participation patterns.

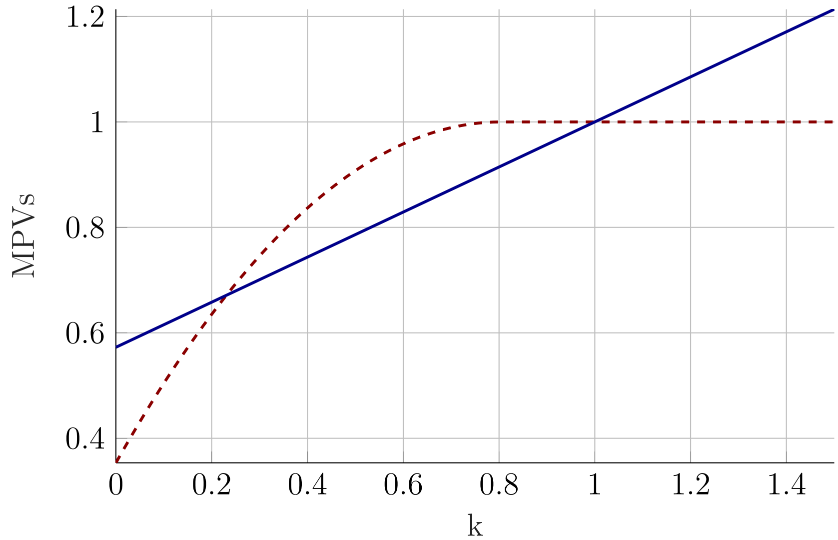

Figure 2: The MPV of centralized (red dashed curve) and OTC market participation (blue plain line, as functions of capacity, k.

$k > 1$. But these services are costly because bargaining creates price impact: $k < k^{\star}$ banks buy at the high price $U_g(\bar{g}_{\text{o}})$ instead of the lower centralized market price $U_g(\frac{1}{2})$.[21]

Next, consider the trade-off faced by $\omega = 0$ banks with high capacities, $k > 1$. These banks could obtain full risk sharing and attain a post-trade exposure of $\frac{1}{2}$ in the centralized market, at a price $U_g(\frac{1}{2})$. Yet, they prefer to enter the OTC market because they can profit from price dispersion by providing intermediation services. Namely, they are net sellers to other $\omega = 0$ banks and are able to bargain a high price $U_g(\bar{g}_{\text{o}}) > U_g(\frac{1}{2})$.

In summary, banks participate in the OTC market if their capacity is either small or large, and participate in the centralized market if their capacity is intermediate. Low-capacity banks find the OTC market attractive because they can trade with high-capacity banks, from whom they demand intermediation services. Vice versa, high-capacity banks find the OTC market attractive because they can profit from price dispersion by supplying intermediation services to low-capacity banks. Banks with intermediate capacity neither demand nor supply sufficient intermediation services, and find the centralized market more attractive because the trade is multilateral at a fixed price instead of bilateral at dispersed prices.

The optimal participation patterns above are reminiscent of the observation that OTC markets have two layers of "core-periphery" structure, one between customers and dealers, and one between dealers (Hollifield, Neklyudov, and Spatt, 2017; Li and Schürhoff, 2019;

[21] As it turns out, there is no price impact in bilateral meetings between $\omega = 0$ and $\omega = 1$ banks, since the price is $\frac{1}{2}\left[U_g(\bar{g}_{\text{o}}) + U_g(1 - \bar{g}_{\text{o}})\right] = U_g(\frac{1}{2})$.

Hendershott, Li, Livdan, and Schürhoff, 2020). To be more precise, let us map banks with $k \leq k^\star$ to customers in practice, and banks with $k \geq 1$ to peripheral and core dealers in order of increasing capacity. That is, the marginal dealers ($k \simeq 1$) are the most peripheral ones. Then, according to the equilibrium bilateral trades of Equation (14), peripheral dealers provide intermediation services to customers only, and core dealers provide intermediation services to both customers and peripheral dealers. Therefore, as emphasized by many studies, a core-periphery structure arises both in the customer-to-dealer market, and in the inter-dealer market.

Another implication of our analysis is that core dealers have the strongest incentives to participate in the OTC market, while peripheral dealers are almost indifferent between operating as a small dealer in the OTC market and using centralized trading just to satisfy their own trading need. Holden, Lu, Lugovskyy, and Puzzello (2021) study how, in 2006, the Chinese Foreign Exchange Market introduced an OTC trading venue in addition to their existing centralized limit order book. They document in particular that large banks migrated more to the OTC market than medium banks. This corroborates our theoretical prediction that high-capacity banks have the strongest incentives to participate in the OTC market.

Existence and uniqueness. Establishing the existence of an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$ boils down to solving the equation:

$$\mathrm{MPV}(0, k^\star, \mathrm{o} \,|\, \bar{g}_{\mathrm{o}}(k^\star)) = \mathrm{MPV}(0, k^\star, \mathrm{c}\,), \tag{15}$$

where we made the marginal private value of participating in the OTC market an explicit function of $\bar{g}_{\mathrm{o}}(k^\star) = \frac{1}{2} \min \left\{\mathbb{E}\left[k' \,|\, k' \leq k^\star \text{ or } k' \geq 1\right], 1\right\}$. Working out properties of this equation, we establish:

Proposition 3. *Suppose that the market participation cost C is small enough. If $\overline{k} \leq 1$, only the centralized market can be active. If $\overline{k} > 1$, then there exists a unique equilibrium with two active markets. Moreover, $\bar{g}_{\mathrm{o}} < \frac{1}{2}$ if and only if $\mathbb{E}\left[k' \,|\, k' < 1 - K \text{ or } k' \geq 1\right] < 1$.*

If $\overline{k} \leq 1$ then large-capacity banks cannot supply sufficient intermediation services to attract low-capacity banks in the OTC market, and only the centralized market can be active. If $\mathbb{E}\left[k' \,|\, k' < 1 - K \text{ or } k' \geq 1\right] \geq 1$, then there is sufficient capacity in the OTC market to attain full risk sharing and $\bar{g}_{\mathrm{o}} = \frac{1}{2}$. Otherwise, there is imperfect risk sharing.

Comparative statics. How does the size of the OTC market depend on the riskiness of the asset? In our setting, an increase in risk translates into an increase in $|U_{gg}|$, the curvature of the certainty equivalent.

Lemma 4. *Suppose that* $\mathbb{E}\left[k' \mid k' < 1 - K \text{ or } k' \geq 1\right] < 1$ *and* $C(\mathrm{o}) \simeq C(\mathrm{c})$*. Then:*

- *Participation in the OTC market decreases in* $|U_{gg}|$ *if* $C(\mathrm{o}) < C(\mathrm{c})$*;*
- *Participation in the OTC market increases in* $|U_{gg}|$ *if* $C(\mathrm{o}) > C(\mathrm{c})$*.*

The intuition is that, when the asset is riskier, the marginal bank becomes more willing to receive risk-sharing services in the market with highest participation cost. This is because the participation cost differential per unit of risk becomes smaller as the asset becomes riskier. Hence, if $C(\mathrm{o}) < C(\mathrm{c})$, an increase in risk causes banks to participate more in the centralized market, and vice versa.

If one believes that safer securities tend to trade in OTC markets, while riskier securities tend to trade in centralized markets, then Lemma 4 suggests that $C(\mathrm{o}) < C(\mathrm{c})$. Consistent with this view, de Rourea, Moench, Pelizzon, and Schneider (2020) study the market for Bunds (German sovereign bonds) where exchange and OTC trading coexist, and find that trading on the exchange is more likely on days with high intraday volatility or for Bunds with long maturities.

Welfare. As argued in Section 3.3, moving the marginal bank to the centralized market is optimal if and only if its bargaining surplus $B(k)$ is smaller than the average $\bar{B}$. In turn, one easily sees that

$$B(0, k, \mathrm{o}) - \bar{B} = \frac{1}{2} \frac{U_g(\bar{g}_\mathrm{o}) - U_g(1 - \bar{g}_\mathrm{o})}{2} \frac{k - \mathbb{E}\left[k' \mid \mathrm{o}\right]}{2}.$$

Suppose there is partial risk sharing, $\bar{g}_\mathrm{o} < \frac{1}{2}$, and consider moving the marginal low-capacity bank, $k^\star$, from the OTC to the centralized market. From Lemma 2 and Lemma 3 we obtain that $\bar{g}_\mathrm{o} = \mathbb{E}\left[k' \mid \mathrm{o}\right]/2 > (k^\star + K)/2$, which implies that $\mathbb{E}\left[k' \mid \mathrm{o}\right] > k^\star$. Therefore, moving the $k^\star$ bank to the centralized market is welfare improving. In contrast, moving the marginal high-capacity bank, $k^{\star\star} = 1$, is welfare reducing. This follows immediately because $\bar{g}_\mathrm{o} < \frac{1}{2}$ implies that $\mathbb{E}\left[k' \mid \mathrm{o}\right] < 1 = k^{\star\star}$.

To summarize, marginal dealers impose a positive externality on others thanks to their higher-than-average trading capacity, while marginal customers impose a negative externality on others due to their lower-than-average trading capacity. Indeed, marginal dealers create larger surplus than they destroy: they engage in large trades in the OTC market, while destroying, on average, smaller trades. The opposite is true for marginal customers. Accordingly, the social planner wants more participation in the OTC market from marginal dealers and less participation from marginal customers, relative to the equilibrium.

One can also provide an analysis of socially optimal participation patterns, when the social planner's problem is to choose a symmetric participation path N in order to maximize utilitarian welfare. Using the marginal social value formula of Proposition 2 to analyze the first-order conditions of this problem, we obtain:

Proposition 4. *Suppose the market participation cost C is small enough and that $\bar{k} > 1$. Then, there exists a solution of the planner's problem in which banks participate in both markets, and banks with $k \geq 1$ participate in the OTC market. The planner's solution features partial risk sharing if and only if $\mathbb{E}\left[k' \mid k' < 1 - K \text{ or } k' \geq 1\right] < 1$. Moreover, in this partial risk-sharing case, the social optimum is characterized by two thresholds $\underline{k} \leq k^{\star}_{\text{opt}} \leq k^{\star\star}_{\text{opt}} < 1$, such that*

$$\pi_{\text{opt}}(0,k) = \pi_{\text{opt}}(1,k) = \begin{cases} \text{o} & \text{if } k \in \left[\underline{k}, k^{\star}_{\text{opt}}\right] \\ \text{c} & \text{if } k \in \left(k^{\star}_{\text{opt}}, k^{\star\star}_{\text{opt}}\right) \\ \text{o} & \text{if } k \in \left[k^{\star\star}_{\text{opt}}, \bar{k}\right]. \end{cases}$$

The Proposition first shows that the social optimum and the equilibrium have qualitatively similar participation patterns: participation is non monotonic and characterized by two thresholds. But, whenever there is partial risk sharing, the thresholds are not the same, implying that the equilibrium is not a social optimum. In particular, since $k^{\star\star}_{\text{opt}} < 1$, the planner finds it optimal to bring more dealer (high capacity) banks in the OTC market.

The planner's first-order conditions provide closed-form solutions for entry thresholds as a function of $\bar{g}_{\text{o}}$, which facilitates the numerical computation of a social optimum. Figure 3 plots the participation thresholds in the social optimum as a function of the centralized market capacity, K. One sees that, in a social optimum, there is less participation of low-capacity banks and, as we already know from the Proposition, more participation of large-capacity banks. The effect on OTC market size turns out to be ambiguous: in this numerical example, the socially optimal OTC market is larger than the equilibrium for low K, and smaller for large K. Finally, Figure 4 shows that, conditional on participating in the OTC market, risk sharing improves in

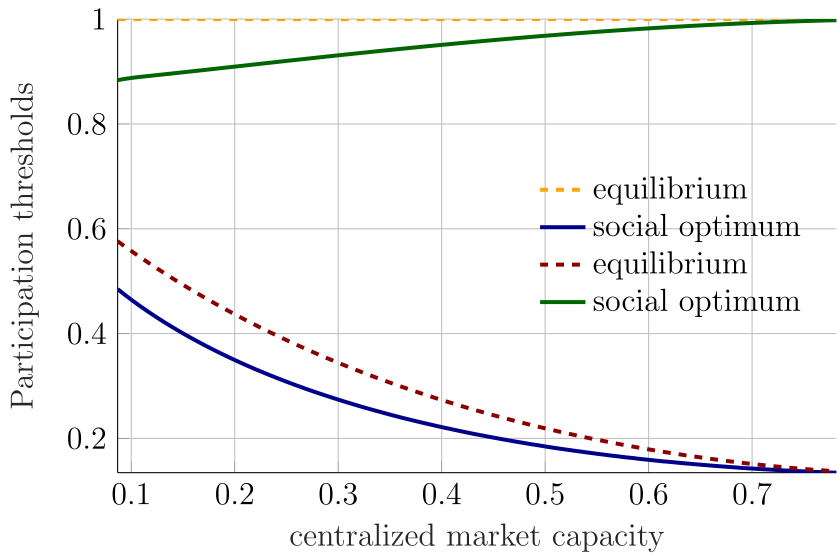

Figure 3: The participation thresholds as a function of the centralized market capacity, in the equilibrium vs. social optimum. The distribution of capacities is assumed to be uniform over $[0, \bar{k}]$, with $\bar{k} = 1.496$.

the social optimum, in the sense that the post-trade exposure $\bar{g}_o$ becomes closer to $1/2$, the full risk sharing benchmark.

4.2 Exclusive vs. non-exclusive participation

In this section we study whether the welfare results of Section 4.1, in which participation is taken to be exclusive, are robust to non-exclusive participation. Our two examples below confirm an observation made earlier in Section 3.3. Namely, the welfare result do not depend on exclusive participation per se, but on whether reallocating a marginal bank induces match creation and destruction in the OTC market. If the reallocation does not induce match creation and destruction, then it is always welfare reducing. If the reallocation induces match creation and destruction, then its effect is similar to the one discussed in the exclusive case.

Allowing for non-exclusive participation makes it harder to characterize equilibrium analytically. First, in contrast with Section 4.1, post-trade exposures in the OTC market are in general not equalized. Second, with more participation choices, the equilibrium becomes potentially more complex and so is harder to characterize. In order to simplify the analysis, we focus on a region of the parameter space such that participation is non-exclusive for a small fraction of

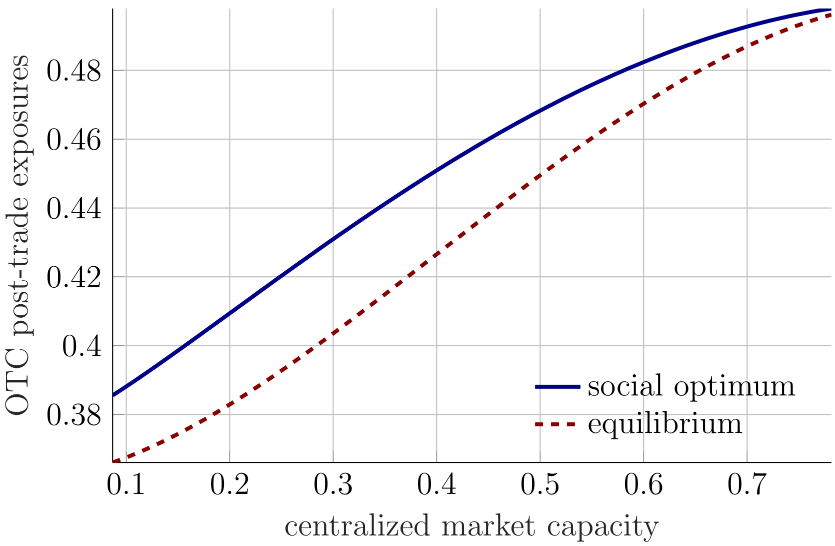

Figure 4: The OTC post-trade exposure as a function of the centralized market capacity, in the equilibrium vs. social optimum. The distribution of capacities is assumed to be uniform over $[0, \bar{k}]$, with $\bar{k} = 1.496$.

banks. As will be clear below, this leads to a simple distribution of post-trade exposures in the OTC market and to intuitive equilibrium participation choices.

4.2.1 A first example of non-exclusive participation

As in the previous section, we assume heterogeneous capacities, distributed according to the density $f(k)$ over some interval $[\underline{k}, \overline{k}]$. We also keep the same trading capacity constraint. The difference with the previous section is the structure of participation costs. We assume that participating in the OTC market is free, $C(\mathrm{o}) = 0$, and participating in the centralized market, *in addition or instead*, entails a cost $C(\mathrm{oc}) = C(\mathrm{c}) = C$ which is relatively large. A natural interpretation is that of a pre-existing OTC market structure challenged by a novel electronic trading platform. Notice that this cost structure implies immediately that participation in the centralized market will always be non-exclusive: since $C(\mathrm{oc}) = C(\mathrm{c})$, banks that choose to participate in the centralized also optimally participate in the OTC market.

Proposition 5. *Suppose that capacities are distributed over the compact interval $[\underline{k}, \overline{k}]$, with $\underline{k} > 0$, $\overline{k} + K < 1$, and density $f(k)$ such that $\bar{k} f(\bar{k}) > \frac{1}{2}$ and average capacity $\int k f(k)\, dk < 1$.*

Then there is a participation cost C and a participation threshold $k^\star$ such that, in equilibrium,

$$\pi(0,k)=\pi(1,k)=\begin{cases}\text{o} & \text{if } k\in[\underline{k},k^\star]\\ \text{oc} & \text{if } k\in\left(k^\star,\overline{k}\right].\end{cases}$$

Moreover, there are two atoms $\bar{g}_{\text{o}}<\bar{g}_{\text{oc}}$ such that

$$g(0,k,\text{o})=\bar{g}_{\text{o}} \text{ if } k\in[\underline{k},k^\star] \text{ and } g(0,k,\text{oc})=\bar{g}_{\text{oc}} \text{ if } k\in(k^\star,\overline{k}].$$

Finally, the post-trade exposures of $\omega=1$ banks are symmetric to that of $\omega=0$ banks.

In the equilibrium of the Proposition, the atom property continues to hold, but conditional on endowment and participation decisions: that is, the atoms are different for banks who participate in the OTC market only, and for banks who participate in both markets. In particular, $\omega=0$ banks who only participate in the OTC market have post-trade exposure $\bar{g}_{\text{o}}$, while $\omega=0$ banks who participate in both markets have higher post-trade exposure $\bar{g}_{\text{oc}}$, reflecting the fact that participating in the centralized market expands risk-sharing opportunities. In turn, high-capacity banks have the strongest incentives to participate in the centralized market because they can purchase larger quantities and re-sell them to others in the OTC market.

In such an equilibrium, making the marginal bank, with capacity $k^\star$, participate in the centralized market in addition to the OTC market is welfare reducing. To see this, recall from the Proposition that when banks participate in the centralized market in addition to the OTC market, their post trade exposures become closer to the median, $1/2$. But this means that their bargaining surplus per quantity traded in the OTC market becomes smaller, on average. Indeed, the bargaining surplus of a bank with post-trade exposure g is proportional to the absolute distance between g and the post-trade exposures of other banks. As is well known, such an average distance is minimized if g is the median post-trade exposure.

4.2.2 A second example of non-exclusive participation

In this section, we show that the welfare analysis of non-exclusive and exclusive participation can, in some cases, be similar. This example illustrates that our results do not depend on exclusivity per se, but on whether the reallocation of banks to the centralized market induces match creation and destruction in the OTC market.

In the example, the participation patterns are the same as in the exclusive case, but with one difference: high-capacity banks participate both in the OTC and the centralized market instead of participating exclusively in the OTC market. In order to generate such participation patterns, we assume that the cost of participating in both market, $C(\mathrm{oc})$, is not too large, while participating exclusively in one market is free, that is $C(\mathrm{o}) = C(\mathrm{c}) = 0$. For tractability, we consider the following "max" capacity constraint,[22]

$$\Gamma(k, k') = \frac{1}{2} \max \{k, k'\} .$$

In Appendix A.10, we study equilibria when capacities are uniformly distributed over the compact interval $[0, 1]$, that the capacity of the centralized exchanged is nil, $K = 0$, and that the non-exclusive participation cost is $C(\mathrm{oc}) = |U_{gg}|/16$. After deriving equilibrium post-trade exposures and marginal private values in closed form, we verify numerically that there are are participation thresholds $k^\star < k^{\star\star}$ such that, in equilibrium,

$$\pi(0, k) = \pi(1, k) = \begin{cases} \mathrm{o} & \text{if } k \in [\underline{k}, k^\star] \\ \mathrm{c} & \text{if } k \in (k^\star, k^{\star\star}) \\ \mathrm{oc} & \text{if } k \in \left[k^{\star\star}, \bar{k}\right] . \end{cases}$$

In the OTC market:

$$g(0, k, \mathrm{o}) = \bar{g}_{\mathrm{o}} < \frac{1}{2} \text{ if } k \in [0, k^\star],$$
$$g(0, k, \mathrm{oc}) = 1 \text{ if } k \in [k^{\star\star}, 1],$$

and the post-trade exposures of $\omega = 1$ banks are symmetric to that of $\omega = 0$ banks.[23] Moreover, the average bargaining surplus is in between the bargaining surpluses of the marginal banks,

$$B(0, k^\star, \mathrm{o}) < \bar{B} < B(0, k^{\star\star}, \mathrm{oc}).$$

The participation patterns are thus similar to the one in the example of Section 4.1, with the difference that high-capacity banks participate non-exclusively in both the OTC and the centralized market. We find that the post-trade exposures of low-capacity banks in the OTC market are equalized at $\bar{g}_{\mathrm{o}} < \frac{1}{2}$, while the post-trade exposures of the high-capacity banks who participate non-exclusively in the OTC and the centralized market are equalized at $\bar{g}_{\mathrm{oc}} = \frac{1}{2}$.

[22] In Section D, we show that this specification preserves the two layers of "core-periphery" structure and the welfare implications of our leading example.

[23] We also verified that the equilibrium existence is robust to small variations of $C(\mathrm{oc})$ around $|U_{gg}|/16$.

As it turns out, the welfare analysis of this equilibrium is also similar to the exclusive example of Section 4.1. This is because moving a marginal bank to the centralized market, from $\pi = \mathrm{o}$ to $\pi = \mathrm{c}$ or from $\pi = \mathrm{oc}$ to $\pi = \mathrm{c}$, induces match creation and destruction in the OTC market. As before, the welfare effect is governed by the comparison between the bargaining surplus of the marginal bank, and the average bargaining surplus in the OTC market. Hence, we find that moving a low-capacity marginal bank ($k^\star$) to the centralized market is welfare improving, while moving a high-capacity marginal bank ($k^{\star\star}$) to the centralized market is welfare reducing.

4.3 Heterogeneous capacities vs. heterogeneous endowments

In this section, we show that the welfare analysis depends on the type of heterogeneity: if banks are heterogeneous in endowment, then moving customer banks to the centralized market is welfare reducing instead of welfare improving. We establish this result under the following assumptions. First, banks now are heterogeneous in their endowment: the distribution of ω across banks is uniform over the interval $[0, 1]$. Second, banks are homogeneous in their trading capacities: the trading capacity constraint is $\Gamma(x, x') = \Phi(x) = k$ for all x and x' and some $k < \frac{1}{2}$. Third, participation costs $C(\pi)$ induce exclusive participation: optimal participation choices are either $\pi = \mathrm{o}$ or $\pi = \mathrm{c}$. We also require in this case that $C(\mathrm{o}) < C(\mathrm{c})$, otherwise the centralized market would always dominate the OTC market.[24]

We guess and verify that, under parameter restrictions to be determined, there exists a symmetric equilibrium in which extreme-ω banks, who have the strongest risk-sharing needs, participate exclusively in the centralized market, which is the most efficient trading venue. Middle-ω banks, on the other hand, participate exclusively in the OTC market.[25] Formally, and keeping in mind that participation is symmetric, there is some $\omega^\star \in [0, \frac{1}{2}]$ such that banks

[24]Our ranking of participation costs is that a centralized market typically imposes more stringent regulatory requirements than OTC markets, involves reengineering of participants' infrastructure to prepare for electronic trading, and requires costly membership to a central clearing party (CCP). Securities and Exchange Commission (2011) discusses technological, regulatory, and disclosure costs of trading in centralized Swap Execution Facilities (SEF). In a survey about the incremental costs resulting from the mandate to migrate trade to SEFs, ISDA Research Staff (2011) reports that Buy-Side users "expect to spend an average of \$2.1 million in technology, \$1.3 million amending client/counterparty documentation and \$200 thousand (annually) in additional regulatory reporting." Finally, Duffie, Li, and Lubke (2010) state that "beyond demonstrating its financial strength and providing margin, each CCP member must also contribute capital to a pooled CCP guarantee fund. The guarantee fund is an additional layer of defense, after initial margin, to cover losses stemming from the failure of a member to perform on a cleared derivative."

[25]These participation patterns are similar to the one obtained by Gehrig (1993) and Miao (2006), who consider models in which investors differ in their private valuation, which is conceptually analogous to our assumption that banks differ in their initial endowment. In these papers, however, all investors participate as customers in the OTC market.

with $\omega \in [0, \omega^\star) \cup (1-\omega^\star, 1]$ participate in the centralized market, and banks with $\omega \in [\omega^\star, 1-\omega^\star]$ participate in the OTC market. Notice that, while these participation patterns are non-monotonic in endowment, they are in fact monotonic in risk-sharing need: two banks with symmetric endowment ω and $1-\omega$ have in fact identical risk-sharing needs.

Lastly, we guess and verify later that $\omega^\star$ satisfies $\omega^\star + k < \frac{1}{2}$. As will be clear below, this ensures that the marginal bank shares risk imperfectly in the OTC market, and so faces meaningful trade-off between the OTC and the centralized market.

Equilibrium conditional on participation. We first characterize trading patterns in the OTC market.

Lemma 5. *Given our conjectured participation patterns, post-trade exposures in the OTC and the centralized market are:*

$$\begin{aligned}
&g(\omega, k, \mathrm{c}) = \min\left\{\omega + k, \frac{1}{2}\right\} \text{ for } \omega \leq \frac{1}{2} \\
&g(\omega, k, \mathrm{c}) = \max\left\{\omega - k, \frac{1}{2}\right\} \text{ for } \omega \geq \frac{1}{2} \\
&g(\omega, k, \mathrm{o}) = \omega + k\left[1 - 2N(\omega \,|\, \mathrm{o})\right],
\end{aligned}$$

where $N(\omega \,|\, \mathrm{o})$ is the cumulative distribution of endowments of OTC market participants.

Figure 5 illustrates the post-trade exposures conditional on $\pi = \mathrm{c}$ and $\pi = \mathrm{o}$. To understand the formula for post-trade exposures in the centralized market, recall that the centralized market price is $U_g(\frac{1}{2})$. Therefore, an $\omega \in [0, \frac{1}{2}]$ bank buys as much as allowed by its trading capacity, subject to not exceeding a post-trade exposure of $\frac{1}{2}$. We obtain the formula for the post-trade exposures in the OTC market by guessing that $g(\omega, k, \mathrm{o})$ is strictly increasing in ω. Then, the optimality condition (8) implies that an ω trader always sells k units to $\omega' < \omega$ traders, and purchases k units from $\omega' > \omega$ traders:

$$g(\omega, k, \mathrm{o}) = \omega - kN(\omega \,|\, \mathrm{o}) + k\left[1 - N(\omega \,|\, \mathrm{o})\right], \tag{16}$$

Given our assumed participation decisions and given uniform distribution,

$$N(\omega \,|\, \mathrm{o}) = \begin{cases} 0 & \text{if } \omega \in [0, \omega^\star) \\ \dfrac{\omega - \omega^\star}{1 - 2\omega^\star} & \text{if } \omega \in [\omega^\star, \frac{1}{2}], \end{cases}$$

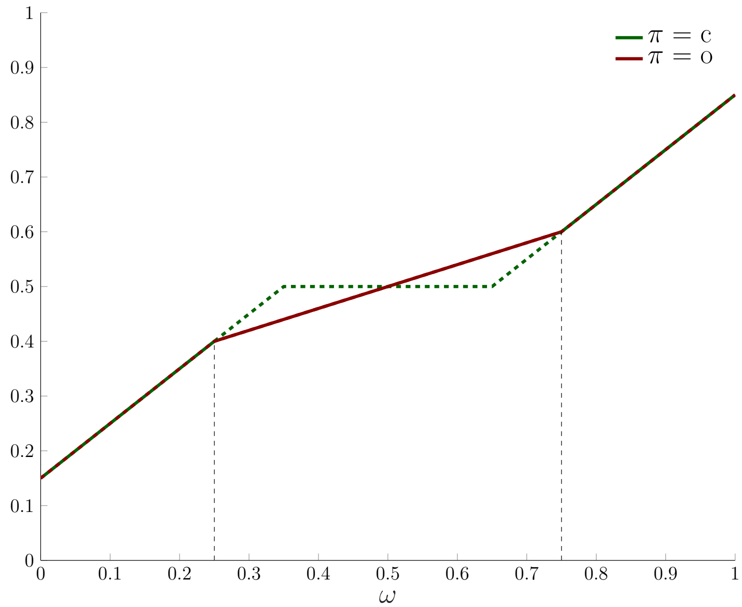

Figure 5: Post-trade exposures as a function of endowment in the example with heterogeneous endowment and exclusive participation. Plain lines show the post-trade exposure on the participation path, that is, conditional on a bank's actual participation decision. Dotted lines show the post-trade exposure off the participation path, conditional on making the alternative exclusive market participation decision.

and, by symmetry, $N(\omega \,|\, \mathrm{o}) = 1 - N(1 - \omega \,|\, \mathrm{o})$ for $\omega \geq \frac{1}{2}$. Plugging the expression for the conditional distribution in (16), and keeping in mind that $\omega^\star + k < \frac{1}{2}$, one easily sees that $g(\omega, k, \mathrm{o})$ is strictly increasing, so our guess is verified.[26]

As in AEW, banks with extreme ω trade like "customers," in the sense that most of their trades go in the same direction. Specifically, low-ω banks mostly purchase assets, and high-ω banks mostly sell assets. Middle-ω banks, on the other hand, trade like "intermediaries." They trade in all directions, buying from high-ω banks and selling to low-ω banks.

Optimal participation and equilibrium existence. Next, we determine the optimal participation decision by studying the relative incentives to participate in the centralized vs. the OTC market.

Lemma 6. *Given our conjectured participation patterns, the difference between the* MPV *of participating in the centralized market and that of participating in the OTC market* $\mathrm{MPV}(\omega, k, \mathrm{c}) - \mathrm{MPV}(\omega, k, \mathrm{o})$ *is strictly U-shaped and symmetric around* $\omega = \frac{1}{2}$.

[26]As before, the formula (16) applies on and off the participation path, for all banks: the one who actually decide to participate in the OTC market (in the support of $N(\omega \,|\, \mathrm{o})$) and the one who do not (outside the support). Figure 5 shows $g_{\mathrm{o}}(\omega)$, with plain lines for the banks in the support and with dotted lines for the banks outside the support.

Lemma 6 shows that extreme-ω banks' incentives to participate in the centralized market dominates their incentives to participate in the OTC market, which confirms our guess about participation patterns.

Let us focus on the trade-off faced by the marginal bank. In the OTC market, this bank only meets counterparties with strictly larger post-trade exposures, so it purchases k in all of its bilateral meeting. Hence, its post-trade exposure is

$$g(\omega^\star, k, \mathrm{o}) = \omega^\star + k.$$

Given that the capacity constraint applies in all markets, the bank reaches the same post-trade exposure in the centralized market, $g(\omega^\star, k, \mathrm{c}) = \omega^\star + k$. Yet, the MPVs are not the same because terms of trade are different. In the centralized market, the price is $U_g(\frac{1}{2})$.[27] In the OTC market, the average price paid by the marginal bank is:

$$\frac{1}{2}\left[U_g\left(g(\omega^\star, k, \mathrm{c})\right) + U_g\left(\frac{1}{2}\right)\right],$$

since the average marginal value across all OTC counterparties is $U_g(\frac{1}{2})$. So one sees that, because of bargaining, the marginal bank buys at higher prices in the OTC market than in the centralized market. All in all, and keeping in mind that $C(\mathrm{o}) < C(\mathrm{c})$, the marginal bank trades off better terms of trade in the centralized market, with lower participation costs in the OTC market.

To obtain the equilibrium equation for $\omega^\star$, we calculate the difference between the MPV in the two markets. As argued above, it only reflects differences in terms of trade: it is equal to the quantity times the price difference between the centralized and the OTC market:

$$\mathrm{MPV}(\omega^\star, k, \mathrm{c}) - \mathrm{MPV}(\omega^\star, k, \mathrm{o}) = k\frac{|U_{gg}|}{2}\left(\frac{1}{2} - \omega^\star - k\right) = C(\mathrm{c}) - C(\mathrm{o}).$$

It is then straightforward to adapt the argument of Lemma 6 and show that the conjectured participation patterns are optimal.

[27] Notice that there are multiple market-clearing prices in the centralized market, since the trading capacity constraint is binding for all banks. However, $U_g(\frac{1}{2})$ is the only price consistent with symmetric participation incentives, and so with a symmetric participation equilibrium.

Welfare analysis. An application of the general envelope analysis of Proposition 2 to this special case implies that reallocating banks near the $\omega^{\star}$ margin from the OTC to the centralized market is welfare improving if and only if $B(\omega^{\star}, k, \mathrm{o}) - \bar{B} < 0$, which does not hold in equilibrium.

The result arises because the marginal bank has the strongest unfulfilled risk-sharing needs. In equilibrium, the distance between its post-trade exposure and the post-trade exposures of other banks participating in the OTC market is largest. Correspondingly, it creates a larger bargaining surplus per quantity traded than any other bank participating in the OTC market. Hence, by reallocating a marginal bank from the OTC market to the centralized market, the social planner destroys matches in which the surplus per quantity traded is large, and creates matches in which the surplus per quantity traded is small. Since we are assuming here that banks have the same capacity, the trade size is the same in all matches. Hence, the reallocation of the marginal bank reduces welfare.

We conclude that, with heterogeneity in endowments, the centralized market is inefficiently large. Encouraging further centralized market participation is welfare reducing.

4.4 Empirical implications of different heterogeneities

Our analytical examples suggest that, when participation is exclusive, increasing customer participation in centralized market may be welfare improving when banks differ mostly in their ability to take large positions (trading capacity), but is welfare reducing when banks differ mostly in their risk-sharing needs (endowment). Therefore, it is crucial to empirically distinguish an economy in which banks differ mostly in terms of their trading capacities, from an economy in which banks differ mostly in terms of their risk-sharing needs. To do so, we study banks' net and gross OTC trading volume, defined as:

$$NV(x) \equiv \int \gamma(x, x')\, dN(x' \,|\, \mathrm{o}) = g(x) - \omega(x),$$
$$GV(x) \equiv \int |\gamma(x, x')|\; N(x' \,|\, \mathrm{o}).$$

Using results from Sections 4.1 and 4.3, we obtain the following proposition.

Proposition 6. *In our analytical example with heterogeneous trading capacity and homogeneous risk-sharing need, and given the bilateral trades (14):*

$$\frac{\partial NV}{\partial k}(\omega, k, \mathrm{o}) = 0 \text{ *and* } \frac{\partial GV}{\partial k}(\omega, k, \mathrm{o}) > 0.$$

In our analytical example with homogeneous trading capacity and heterogeneous risk-sharing need,

$$\frac{\partial NV}{\partial \omega}(\omega, k, \mathrm{o}) < 0 \text{ for } \omega < \frac{1}{2}, \ \frac{\partial NV}{\partial \omega}(\omega, k, \mathrm{o}) > 0 \text{ for } \omega > \frac{1}{2}, \text{ and } \frac{\partial GV}{\partial \omega}(\omega, k, \mathrm{o}) = 0.$$

The Proposition implies that, in our example with heterogeneous trading capacity, the net volume is independent of capacity, k. Indeed, banks with identical endowment have identical post-trade exposures, and so have identical net volume. The gross volume, on the other hand, is increasing in k. In contrast, in our example with heterogeneous endowment, the net volume is largest for banks with extreme endowments, and smallest for banks with intermediate endowments. Indeed intermediate-endowment banks provide intermediation services precisely because they do not need to use their capacity to change their net exposures. The gross volume, on the other hand, is the same for all banks. This is because all banks have the same trading capacity, k, and because there are strict gains from trade in all bilateral matches. Therefore, the same quantity is traded in all bilateral matches for all banks, leading to constant gross volume.

Stylized facts. Empirical evidence suggests that dealers concentrate a very large fraction of gross volume (Bech and Atalay, 2010; Di Maggio, Kermani, and Song, 2017; Hollifield, Neklyudov, and Spatt, 2017; Li and Schürhoff, 2019). This observation holds after controlling for natural measure of bank size (see for example Atkeson, Eisfeldt, and Weill, 2013, for the CDS market), which is relevant for our model in which all banks have the same number of traders and hence the same size. This observation is better in line with the heterogeneous capacity model, in which the gross volume of dealers is larger than that of customers since $\frac{\partial GV}{\partial k} > 0$ according to Proposition 6. In the heterogeneous endowment model, by contrast, all agents have the same gross volume.

Siriwardane (2018) reports empirical evidence about net volume in the context of CDS markets: he finds that dealers also concentrate a very large fraction of net buying and net selling volumes. This is in contradiction with the heterogeneous endowment model in which dealers tend to have lower net volume than customers since $\frac{\partial NV}{\partial \omega} < 0$ for $\omega \leq 1/2$ according to Proposition 6. Again, it can be argued that the heterogeneous capacity model is better in line with that piece of evidence, because dealers' net volume is as high as customers' in the heterogeneous capacity model.

5 Conclusion

We developed a model of costly participation of heterogeneous banks in an OTC and/or a centralized market. The equilibrium generates rich participation and trading patterns and can be used to evaluate the social value of policies that reallocate banks across markets. We highlight the two main determinants of welfare change stemming from a reallocation of marginal banks: (i) whether the reallocation leads to match creation and destruction in the OTC market and (ii) the type of heterogeneity. For cases where there is no match creation/destruction from reallocating a marginal bank, we find that such interventions are always welfare reducing. However, we also find that if banks differ in their ability to take large positions and if there is match creation and destruction, reallocating marginal OTC market customers to the centralized market is welfare improving, but reallocating marginal dealers is welfare reducing.

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A Selected proofs

A.1 Proof of Proposition 1

A.1.1 The planner's problem

The planner's objective is

$$W(\varphi, \gamma \,|\, N) = \int U\left[g(x \,|\, \varphi, \gamma)\right]\, dN(x).$$

where we ignore participation cost to simplify notations (because they are sunk), and where we make it explicit that post-trade exposures depend on centralized and OTC market trades (φ, γ) via (6). Given that the measure N is finite, $N(X) < \infty$, repeat applications of the Cauchy Schwartz inequality show that the planner's objective is continuous in (γ, φ). Because $(\gamma, \varphi) \mapsto W(\gamma, \varphi \,|\, N)$ is continuous, it is lower semi-continuous. Clearly, the function is also concave and the constraint set is closed and bounded. Existence of a solution then follows from an application of Proposition 1.2, Chapter II in Eckland and Témam (1987).

The uniqueness of post-trade exposures follows because the objective is strictly concave in post-trade exposures.

The derivation of the first-order condition is based on arguments from Chapter 8 in Luenberger (1969). Following the notations of Luenberger, we let Ω denote the set of square-integrable trades (γ, φ) that are feasible N-almost everywhere, and we view (4) as a parametric linear equality constraint. The separating-hyperplane argument of Theorem 1 in Chapter 8.3, together with its generalization to linear equality constraint in Problem 7, imply that:

Lemma 7. *There exists some $g_c \in \mathbb{R}$ such that the solution to the planner's problem maximizes the Lagrangian $L(\varphi, \gamma) \equiv W(\varphi, \gamma) - U_g(g_c) \int \varphi(x)\, dN(x)$.*

We provide a detailed proof in online Appendix B.3. We now derive first-order condition by applying a variational argument to the Lagrangian. We start with first-order conditions with respect to centralized market trades. Given any optimal (φ, γ) and we let:

$$\begin{aligned}
\hat{\varphi}(x) &= \varphi(x) + \varepsilon\left[\Phi(x) - \varphi(x)\right]\mathbb{I}_{\{g(x)<g_c\}} - \varepsilon\left[\Phi(x) + \varphi(x)\right]\mathbb{I}_{\{g(x)>g_c\}} \\
&\equiv \varphi(x) + \varepsilon\Delta_{\varphi}(x),
\end{aligned}$$

and we note that $(\hat{\varphi}, \gamma)$ is feasible for the planning problem, as long as $\varepsilon \in [0,1]$. Hence, for small ε, we obtain that up to second-order terms:

$$
\begin{aligned}
&L(\hat{\varphi},\gamma) - L(\varphi,\gamma) = \varepsilon \int U_g\left[g(x)\right]\Delta_\varphi(x)\,dN(x) - \varepsilon U_g\left(g_{\mathrm{c}}\right)\int \Delta_\varphi(x)\,dN(x) \\
=&\varepsilon \int \left[\Phi(x) - \varphi(x)\right]\left[U_g[g(x)] - U_g(g_{\mathrm{c}})\right]\mathbb{I}_{\{g(x)<g_{\mathrm{c}}\}} \\
-&\varepsilon \int \left[\Phi(x) + \varphi(x)\right]\left[U_g[g(x)] - U_g(g_{\mathrm{c}})\right]\mathbb{I}_{\{g(x)>g_{\mathrm{c}}\}}.
\end{aligned}
$$

Optimality implies that the sum of these two terms is negative. But each integrand is positive by construction. Hence, each integrand is equal to zero N almost everywhere. In other words, we obtain that (10) must hold N almost everywhere. Similarly, in the OTC market, let

$$
\begin{aligned}
\hat{\gamma}(x,x') &= \gamma(x,x') + \varepsilon\left[\Gamma(x,x') - \gamma(x,x')\right]\mathbb{I}_{\{g(x)<g(x')\}} - \varepsilon\left[\Gamma(x',x) + \gamma(x,x')\right]\mathbb{I}_{\{g(x)>g(x')\}} \\
&\equiv \gamma(x,x') + \varepsilon\Delta_\gamma(x,x').
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\frac{L(\hat{\gamma},\varphi) - L(\gamma,\varphi)}{N(X_{\mathrm{o}})} = \varepsilon \int U_g\left[g(x)\right]\int \Delta_\gamma(x,x')\,dN(x'\,|\,\mathrm{o})\,dN(x\,|\,X_{\mathrm{o}}) \\
=&\frac{\varepsilon}{2}\int\int U_g\left[g(x)\right]\Delta_\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&+\frac{\varepsilon}{2}\int\int U_g\left[g(x')\right]\Delta_\gamma(x',x)\,dN(x'\,|\,\mathrm{o})\,dN(x\,|\,\mathrm{o}) \\
=&\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\Delta_\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
=&\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\left[\Gamma(x,x') - \gamma(x,x')\right]\mathbb{I}_{\{g(x)<g(x')\}}\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&-\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\left[\Gamma(x',x) + \gamma(x,x')\right]\mathbb{I}_{\{g(x)>g(x')\}}\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}).
\end{aligned}
$$

If $\gamma(x,x')$ is optimal, this must be negative. Since both integrands are positive, they must be zero N almost everywhere. In other words, (8) holds N almost everywhere.

A.1.2 Proof of equilibrium existence

Step 1: the partial equilibrium determination of post-trade exposures. To establish existence, we start from a solution of the planner's problem and modify it so that optimality conditions hold everywhere instead of almost everywhere. Economically, this amounts to determine the post-trade exposures of any (ω, k) for any participation decision π. Formally, we consider some arbitrary $x \in X$, and we fix some arbitrary centralized market price, $U_g(h_{\mathrm{c}})$, and any arbitrary function for the post-trade exposures, $h(x')$, of other banks in the OTC market. Taken together, these fully determine expectations about terms of trades in both markets. We then seek a solution to the problem:

$$g = \omega(x) + \int \gamma(x, x')\, dN(x' \,|\, \mathrm{o}) + \varphi(x), \tag{17}$$

where centralized and OTC market trades $\varphi(x)$ and $\gamma(x, x')$ are chosen optimally given h_{c} and $h(x')$. That is:

$$\varphi(x) = \begin{cases} \Phi(x) & \text{if } g < h_{\mathrm{c}} \\ \in [-\Phi(x), \Phi(x)] & \text{if } g = h_{\mathrm{c}} \\ -\Phi(x) & \text{if } g > h_{\mathrm{c}} \end{cases} \tag{18}$$

Likewise:

$$\gamma(x, x') = \begin{cases} \Gamma(x, x') & \text{if } g < h(x') \\ \in [-\Gamma(x', x), \Gamma(x, x')] & \text{if } g = h(x') \\ -\Gamma(x', x) & \text{if } g > h(x'). \end{cases} \tag{19}$$

The main result is that:

Lemma 8. *The fixed point problem (17)-(19) has a unique solution. Moreover, this solution remains the same:*

- *If $h(x')$ is changed but remains the same $N(\cdot \,|\, \mathrm{o})$-almost everywhere;*
- *If the OTC optimality conditions (19) only hold $N(\cdot \,|\, \mathrm{o})$-almost everywhere.*

Step 2: modifiying a solution of the planner's problem into an equilibrium. As noted in the text, the planner's problem only determines trades and post-trade exposures N-almost everywhere, that is, on the participation path. The next step is to construct an equilibrium by modifying a solution to the planner's problem so that trading decisions are optimal off the participation path:

Lemma 9. *Let (γ, φ, g) denote a solution to the planner's problem. Then there is some $(\hat{\gamma}, \hat{\varphi}, \hat{g})$, equal to (γ, φ, g) N-almost everywhere, that is the basis of an equilibrium.*

The proof builds on Lemma 8 which allows to determine trades and post-trade exposure for all participation decisions, given aggregate market conditions. The details are in online Appendix B.5

A.1.3 Properties of the equilibrium post-trade exposures

A useful result about equilibrium post-trade exposures, $g(x)$, is that they are equi-continuous in $x \in X$.

Lemma 10. *Suppose that the trading capacity constraints $\Gamma(x, x')$ and $\Phi(x)$ are continuous. Then, there exists a positive and continuous function $G(x_2, x_1)$, satisfying $G(x, x) = 0$ for all x, and such that, for all x_1 and x_2: $|g(x_1) - g(x_2)| \leq G(x_2, x_1)$.*

What remains to show is the following lemma.

Lemma 11. *In symmetric economies, the equilibrium post-trade exposures, g, have following properties:*

- *Symmetric: For $\omega \leq \frac{1}{2}$, $g(\omega, k, \pi) = 1 - g(1-\omega, k, \pi)$.*
- *Increasing in endowment: $\omega < \omega' \Rightarrow g(\omega, k, \pi) \leq g(\omega', k, \pi)$.*
- *Less than $\frac{1}{2}$ for $\omega \leq \frac{1}{2}$: $g(\omega, k, \pi) \leq \frac{1}{2}$.*
- *Greater than $\frac{1}{2}$ for $\omega > \frac{1}{2}$: $g(\omega, k, \pi) \geq \frac{1}{2}$.*
- *Increasing in capacity for $\omega \leq \frac{1}{2}$: $k < k' \Rightarrow g(\omega, k, \pi) \leq g(\omega, k', \pi)$.*
- *Decreasing in capacity for $\omega > \frac{1}{2}$: $k < k' \Rightarrow g(\omega, k, \pi) \geq g(\omega, k', \pi)$.*
- *Increasing in centralized market participation for $\omega \leq \frac{1}{2}$: $g(\omega, k, \mathrm{o}) \leq g(\omega, k, \mathrm{oc})$.*
- *Decreasing in centralized market participation for $\omega > \frac{1}{2}$: $g(\omega, k, \mathrm{o}) \geq g(\omega, k, \mathrm{oc})$.*

The proofs of Lemma 10 and 11 are in the online Appendix B.6 and B.7.

A.2 Proof of Lemma 1

Define the Marginal Private Value, $\mathrm{MPV}(x)$, as the certainty equivalent payoff of a bank of type x net of the autarky value and before the participation cost. Then, (7) implies

$$
\begin{aligned}
\mathrm{MPV}(x) =& U[g(x)] - U[\omega(x)] - \int \gamma(x,x') P_{\mathrm{o}}(x,x')\, dN(x' \,|\, \mathrm{o}) - \varphi(x) P_{\mathrm{c}} \\
=& U[g(x)] - U[\omega(x)] - \int \gamma(x,x') \frac{U_g[g(x)] + U_g[g(x')]}{2}\, dN(x' \,|\, \mathrm{o}) - \varphi(x) P_{\mathrm{c}} \\
=& U[g(x)] - U[\omega(x)] - \varphi(x) P_{\mathrm{c}} - \int \gamma(x,x') U_g\left[g(x')\right]\, dN(x' \,|\, \mathrm{o}) \\
& - \int \gamma(x,x') \frac{U_g[g(x)] - U_g[g(x')]}{2}\, dN(x' \,|\, \mathrm{o}) \\
=& \overbrace{U[g(x)] - U[\omega(x)] - \varphi(x) P_{\mathrm{c}} - \int \gamma(x,x') U_g\left[g(x')\right]\, dN(x' \,|\, \mathrm{o})}^{=S(x)} \\
& - \frac{1}{2} \underbrace{\int \Gamma(x,x') \left| U_g[g(x)] - U_g\left[g(x')\right] \right|\, dN(x' \,|\, \mathrm{o})}_{=B(x)},
\end{aligned}
$$

where the second equality obtains by using (9), the third equality by rearranging, and the last one by using (8), which establishes the decomposition stated in the Lemma. Next, we prove the properties of the marginal private value stated in the following Lemma, whose proof is in the online Appendix B.

Lemma 12. *The marginal private value increases with k, decreases with $\omega \in \left[0, \frac{1}{2}\right]$, and symmetrically increases with $\omega \in \left[\frac{1}{2}, 1\right]$.*

A.3 Proof of Proposition 2

Given any collection of trades (γ, φ) and any ε, we define the post-trade exposure by:

$$g(x, \gamma, \varphi, \varepsilon) \equiv \omega(x) + \int \gamma(x, x') \, \frac{dN(x') + \varepsilon dn(x')}{N(X_{\mathrm{o}}) + \varepsilon n(X_{\mathrm{o}})} + \varphi(x).$$

Social welfare before participation costs is:

$$W(\gamma, \varphi, \varepsilon) \equiv \int U \left[g(x, \gamma, \varphi, \varepsilon)\right] \, (dN(x) + \varepsilon \, dn(x)) \, ,$$

where the function W implicitly depends on $N + \varepsilon \, n$. The corresponding Lagrangian is

$$L(\gamma, \varphi, \varepsilon) \equiv W(\gamma, \varphi, \varepsilon) - U_g \left(\frac{1}{2}\right) \int \varphi(x) \, \left[dN(x) + \varepsilon dn(x)\right] .$$

Notice that Lagrange multiplier for the centralized market resource constraint is $U_g\left(\frac{1}{2}\right)$ because any admissible reallocation must maintain symmetry. Letting Λ denote the set of trades (γ, φ) that are feasible $N + \varepsilon n$ almost everywhere, and using notation that closely follow Milgrom and Segal (2002), the planner's problem is:

$$W^\star(\varepsilon) = \sup_{(\gamma,\varphi)\in\Lambda} W(\gamma, \varphi, \varepsilon) = \sup_{(\gamma,\varphi)\in\Lambda} L(\gamma, \varphi, \varepsilon)$$

where the second equality uses the fact that the solution to the planner's problem maximizes the Lagrangian. We know from our earlier results that the planner's problem has at least a solution. That is, the maximum correpondence

$$\Lambda^\star(\varepsilon) = \{(\gamma, \varphi) \in \Lambda \, : \, L(\gamma, \varphi, \varepsilon) = W^\star(\varepsilon)\}.$$

is not empty. The rest of the proof extends arguments from Milgrom and Segal (2002), and is organized as follows. In Section B.1.1 we first show that the planner's value, $W^\star(\varepsilon)$, is right-hand differentiable at $\varepsilon = 0$. In Section B.1.2, we establish that the right-hand derivative maximizes marginal social value with respect to all socially optimal trades, $(\gamma, \varphi) \in \Lambda^\star(0)$:

$$\frac{dW^\star}{d\varepsilon}(0^+) = \max_{(\gamma^\star,\varphi^\star)\in\Lambda^\star(0)} \frac{\partial L}{\partial \varepsilon}(\gamma^\star, \varphi^\star, 0). \tag{20}$$

The maximization problem on the right-hand side of (33) is not trivial because the set $\Lambda^\star(0)$ of socially optimal trades can be large. Indeed, these trades are not determined for entrants in the OTC market, that is, for types which have positive measure according to n but not to N. In Section B.1.3, we find the solution to this maximization problem: we show that it is solved by equilibrium trades. That is, to calculate the marginal social value, one should assume that the entrants in the OTC market follow their equilibrium trade.

We end this section with an explicit calculation of the partial derivative of L evaluated at equilibrium trade:

$$\frac{\partial L}{\partial \varepsilon}(\gamma, \varphi, 0) = \int U\left[g(x)\right]\, dn(x) + \int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi(x)\, dn(x).$$

Using that

$$\frac{\partial g}{\partial \varepsilon}(x) = \int \gamma(x, x')\, \frac{dn(x')N(X_{\mathrm{o}}) - dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}}) + \varepsilon\, n(X_{\mathrm{o}})\right]^2},$$

the second term can be simplified as follows:

$$\begin{aligned}
&\int\!\!\int U_g\left[g(x)\right] \gamma(x, x') \frac{dn(x')N(X_{\mathrm{o}}) - dN(x')n(X_{\mathrm{o}})}{N(X_{\mathrm{o}})^2}\, dN(x) \\
=&\int\!\!\int U_g\left[g(x')\right] \gamma(x', x) \frac{dN(x')}{N(X_{\mathrm{o}})}\, dn(x) - n(X_{\mathrm{o}}) \int\!\!\int \; U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_{\mathrm{o}})^2} \\
=&-\int\!\!\int U_g\left[g(x')\right] \gamma(x, x') \frac{dN(x')}{N(X_{\mathrm{o}})}\, dn(x) - n(X_{\mathrm{o}}) \int\!\!\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_{\mathrm{o}})^2}
\end{aligned} \tag{21}$$

where the first equality follows by exchanging the name of variables, namely replacing x by x' in the first term, and the second equality follows by bilateral feasibility, i.e. $\gamma(x', x) = -\gamma(x, x')$. The first term of (21) can be rewritten as follows:

$$\begin{aligned}
&-\int\!\!\int P_{\mathrm{o}}(x, x')\gamma(x, x') dN(x' \,|\, \mathrm{o}) dn(x) + \int\!\!\int \left(P_{\mathrm{o}}(x, x') - U_g\left[g(x')\right]\right) \gamma(x, x')\, dN(x' \,|\, \mathrm{o}) dn(x) \\
=&-\int\!\!\int P_{\mathrm{o}}(x, x')\gamma(x, x') dN(x'|\mathrm{o}) dn(x) + \frac{1}{2}\int\!\!\int \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right) \gamma(x, x') dN(x'|\mathrm{o}) dn(x) \\
=&-\int\!\!\int P_{\mathrm{o}}(x, x')\gamma(x, x') dN(x'|\mathrm{o}) dn(x) + \frac{1}{2}\int\!\!\int \left|U_g\left[g(x)\right] - U_g\left[g(x')\right]\right| \Gamma(x, x') dN(x'|\mathrm{o}) dn(x) \\
=&-\int\!\!\int P_{\mathrm{o}}(x, x')\gamma(x, x')\, dN(x' \,|\, \mathrm{o})\, dn(x) + \frac{1}{2}\int B(x)\, dn(x),
\end{aligned}$$

where: the first line obtains by using the definition of $N(x \,|\, \mathrm{o})$ and subtracting and adding bilateral OTC payments, $P_{\mathrm{o}}(x, x')\gamma(x, x')$; the second line by substituting the expression (9) for $P_{\mathrm{o}}(x, x')$; the third line by using the optimality condition for bilateral trades (8); and the fourth line by using the definition of the bargaining surplus.

The second term of (21) can be simplified as follows:

$$
\begin{aligned}
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x)\right]\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&\quad -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x')\right]\gamma(x',x)\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x)\right]\gamma(x,x')dN(x\,|\,\mathrm{o})dN(x'\,|\,\mathrm{o}) \\
&\quad + \quad \frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x')\right]\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,o) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int \left(U_g\left[g(x)\right]-U_g\left[g(x')\right]\right)\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int \left|U_g\left[g(x)\right]-U_g\left[g(x')\right]\right|\Gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{1}{2}\int_{X_{\mathrm{o}}}\bar{B}\,dn(x).
\end{aligned}
$$

where: the first equality follows by using the definition of $N(x\,|\,\mathrm{o})$, breaking the integral into two identical halves and exchanging the name of variables in the second term, replacing x by x' in the second half; the second equality follows by bilateral feasibility $\gamma(x',x) = -\gamma(x,x')$; the third equality by collecting terms; the fourth equality using the optimality condition (8) for bilateral trades; and the fifth equality by using the definition of the average bargaining surplus.

The formula of the Proposition follows from collecting all terms, using the definition of the certainty equivalent value in (11) and its expression in terms of MPV, and by re-defining function $\varepsilon \mapsto W^\star\left[N + \varepsilon\left(n^+ - n^-\right)\right]$ to include participation costs and with a particular choice of reallocation $n = n^+ - n^-$.

A.4 Proof of Lemma 2

Post-trade exposures in the OTC market. The proof that post-trade exposures are equalized proceeds as usual by guess and verify. Namely, if the atom property holds and post-trade exposures of $\omega = 0$ banks are equal to some $\bar{g}_{\mathrm{o}} < 1/2$, then by symmetry the post-trade exposures of $\omega = 1$ banks are strictly above $1/2$. Hence, in any meeting between an $\omega = 0$ bank of capacity k and an $\omega = 1$ bank of capacity k', the $\omega = 0$ bank purchase $(k+k')/2$ assets from the $\omega = 1$ bank. Hence, the post-trade exposure of $\omega = 0$ banks with capacity k writes:

$$
\bar{g}_{\mathrm{o}} = \frac{1}{2}\int \gamma(k,k')\,dN(k'\,|\,\mathrm{o}) + \frac{1}{2}\int \frac{k+k'}{2}\,dN(k'\,|\,\mathrm{o}), \tag{22}
$$

where the first term is the net trade with other $\omega = 0$ banks and the second term is the net trade with $\omega = 1$ banks. Integrating across all k in the OTC market and keeping in mind that the aggregate net trade between all $\omega = 0$ banks must be zero, we obtain that the atom is given by:

$$\int \bar{g}_{\rm o} \, dN(k \,|\, {\rm o}\,) = \bar{g}_{\rm o} = \frac{\mathbb{E}\,[k' \,|\, {\rm o}\,]}{2}. \tag{23}$$

Conversely, suppose that $\mathbb{E}\,[k' \,|\, {\rm o}\,] < 1$. One can directly verify that the bilateral trades $\gamma(k, k') = (k' - k)/2$ satisfy the capacity constraint and make the post-trade exposures of all $\omega = 0$ equal to $\bar{g}_{\rm o}$ given by (23). Since $\omega = 0$ banks equalize their exposures, any feasible trade size is optimal.

Next suppose that $\mathbb{E}\,[k' \,|\, {\rm o}\,] \geq 1$. Then let $\alpha \equiv \mathbb{E}\,[k' \,|\, {\rm o}\,]^{-1} \leq 1$. Then one can directly verify that if an $\omega = 0$ bank of capacity k trades $\alpha(k + k')/2$ with $\omega = 1$ banks of capacity k', and $\alpha(k' - k)$ with $\omega = 0$ banks of capacity k', then its post-trade exposure is equal to $1/2$. Since $\alpha \leq 1$, these trades satsify the capacity constraint. Since all banks equalize their exposures to $1/2$, any feasible trade size is optimal.

Marginal private values in the OTC market. To calculate the MPV, consider first the full appropriation surplus:

$$\begin{aligned} S(0, k, {\rm o}) =& U(\bar{g}_{\rm o}) - U(0) \\ & - \frac{1}{2} \int U_g(\bar{g}_{\rm o}) \gamma(k, k') \, dN(k' \,|\, {\rm o}) - \frac{1}{2} \int \frac{k + k'}{2} U_g(1 - \bar{g}_{\rm o}) \, dN(k' \,|\, {\rm o}). \end{aligned}$$

Then, subtracting half of the bargaining surplus:

$$\begin{aligned} {\rm MPV}(0, k, {\rm o}) =& U(\bar{g}_{\rm o}) - U(0) \\ & - \frac{1}{2} \int U_g(\bar{g}_{\rm o}) \gamma(k, k') \, dN(k' \,|\, {\rm o}) - \frac{1}{2} \int \frac{k + k'}{2} \frac{U_g(\bar{g}_{\rm o}) + U_g(1 - \bar{g}_{\rm o})}{2} \, dN(k' \,|\, {\rm o}). \end{aligned}$$

Subtracting and adding $U_g(\bar{g}_{\rm o})\bar{g}_{\rm o}$ and using formula (22) for $\bar{g}_{\rm o}$, we obtain:

$$\begin{aligned} {\rm MPV}(0, k, {\rm o}) &= U(\bar{g}_{\rm o}) - U(0) - U_g(\bar{g}_{\rm o})\bar{g}_{\rm o} + \frac{1}{2} \int \frac{U_g(\bar{g}_{\rm o}) - U_g(1 - \bar{g}_{\rm o})}{2} \frac{k + k'}{2} \, dN(k' \,|\, {\rm o}) \\ &= \frac{|U_{gg}|}{2} \bar{g}_{\rm o}^2 + \frac{1}{2} |U_{gg}| \frac{1 - \bar{g}_{\rm o} - \bar{g}_{\rm o}}{2} \frac{k + \mathbb{E}\,[k \,|\, {\rm o}\,]}{2}, \end{aligned}$$

which simplifies to the formula shown in the Lemma after replacing $\bar{g}_{\rm o} = \mathbb{E}\,[k' \,|\, {\rm o}\,]/2$, if less than $1/2$, and otherwise by setting $\bar{g}_{\rm o} = 1/2$.

Post-trade exposure in the centralized market. Given a price of $U_g(1/2)$, it is clear that the optimal trade of an $\omega = 0$ bank in the centralized market is the smaller of $(k + K)/2$ and $1/2$.

Marginal private values in the centralized market. The marginal private value is calculated as:

$$\begin{aligned}\mathrm{MPV}(0,k,\mathrm{c}) &= U(g(0,k,\mathrm{c})) - U(0) - U_g(1/2)g(0,k,\mathrm{c}) \\ &= U(g(0,k,\mathrm{c})) - U(0) - U_g(g(0,k,\mathrm{c}))g(0,k,\mathrm{c}) + \left(U_g(g(0,k,\mathrm{c})) - U_g(1/2)\right)g(0,k,\mathrm{c}) \\ &= \frac{|U_{gg}|}{2}g(0,k,\mathrm{c})^2 + |U_{gg}|\left(\frac{1}{2} - g(0,k,\mathrm{c})\right)g(0,k,\mathrm{c}),\end{aligned}$$

which simplifies to the formula of the Lemma.

A.5 Proof of Lemma 3

We first need to show that $k^\star < 1-K$. This follows because, when $k = 1-K$, then a bank can attain perfect risk-sharing in the centralized market, $g_{\mathrm{c}}(1-K) = 1/2$, which one can verifies makes the centralized market a strictly better choice than the OTC market.

Next, we need to show that $\bar{g}_{\mathrm{o}} > (k^\star + K)/2$. Given our maintained assumption that there is imperfect risk-sharing, $\bar{g}_{\mathrm{o}} < 1/2$, we have that $\mathrm{MPV}(0,k^\star,\mathrm{o}) < |U_{gg}|/8$, the full risk-sharing MPV. Since, in addition, $\mathrm{MPV}(0,k^\star,\mathrm{o}) = \mathrm{MPV}(0,k^\star,\mathrm{c})$, it follows that risk-sharing is imperfect in the centralized market as well. Hence, $g(\omega,k^\star,\mathrm{c}) = (k+K)/2$. Using the indifference condition again, we obtain:

$$\begin{aligned}&\mathrm{MPV}(0,k^\star,\mathrm{o}) = \mathrm{MPV}(0,k^\star,\mathrm{c}) \\ \Leftrightarrow\ & 2\bar{g}_{\mathrm{o}} + k^\star(1-2\bar{g}_{\mathrm{o}}) = (k^\star+K)\left(2-(k^\star+K)\right) \\ \Leftrightarrow\ & 2\bar{g}_{\mathrm{o}}\left(1-k^\star)\right) = (k^\star+K)\left(2-(k^\star+K)\right) - k \\ \Leftrightarrow\ & \left(2\bar{g}_0 - (k^\star+K)\right)(1-k^\star) = (k^\star+K)\left(2-(k^\star+K)\right) - k^\star - (1-k^\star)(k^\star+K) \\ \Leftrightarrow\ & \left(2\bar{g}_0 - (k^\star+K)\right)(1-k^\star) = K(1-(k^\star+K)) > 0,\end{aligned}$$

which implies that $\bar{g}_{\mathrm{o}} > (k^\star+K)/2$.

A.6 Proof of Proposition 3

When $\overline{k} \leq 1$. Then there are no banks with $k > 1$ in the OTC market and so follows from Lemma 3 that:

$$\bar{g}_{\mathrm{o}} = \frac{1}{2}\min\left\{\mathbb{E}\left[k' \,|\, k' \leq k^\star\right], 1\right\} \leq \frac{k^\star}{2}.$$

According to Lemma 3, if there were positive participation in both market, then we would have that $\bar{g}_{\mathrm{o}} > k^\star/2$. Hence, if $\overline{k} \leq 1$, there does not exist an equilibrium with positive participation in both markets.

When $\overline{k} > 1$: equilibrium such that $\bar{g}_{\mathbf{o}} < 1/2$. In this paragraph, we show that, there exists an equilibrium with $\bar{g}_{\text{o}} < 1/2$ if and only if $\mathbb{E}\left[k' \,|\, k' \leq 1 - K \text{ and } k' \geq 1\right] < 1$, and that this equilibrium is unique.

It is clear that, if an equilibrium with $\bar{g}_{\text{o}} < 1/2$ exists, then

$$\bar{g}_{\text{o}} = \hat{g}(k^\star) < \frac{1}{2} \text{ where } \hat{g}(x) \equiv \frac{1}{2}\mathbb{E}\left[k' \,|\, k' \leq x \text{ or } k' \geq 1\right].$$

Taking derivatives, we find that $\hat{g}'(x)$ is proportional to:

$$\begin{aligned} h(x) &= x\left(\int_{\underline{k}}^{x} f(k)\,dk + \int_{1}^{\overline{k}} f(x)\,dk\right) - \left(\int_{\underline{k}}^{x} kf(k)\,dk + \int_{1}^{\overline{k}} kf(x)\,dk\right) \\ &= \mathbb{P}\left(k \leq x \text{ or } k \geq 1\right) \times \left(x - \mathbb{E}\left[k \,|\, k \leq k \text{ and } x \geq 1\right]\right). \end{aligned} \tag{24}$$

Moreover, $h'(x) = F(x) + 1 - F(1) > 0$. Therefore, $h(\underline{k}) < 0$ and $h(x)$ changes sign at most once.

With this calculation in mind we show that, if an equilibrium with $\bar{g}_{\text{o}} < 1/2$ exists, then

$$\mathbb{E}\left[k' \,|\, k' \leq 1 - K \text{ or } k' \geq 1\right] < 1.$$

Indeed, if the opposite inequality holds, then one easily verifies that $\hat{g}(1-K) \geq 1/2$ and $h(1-K) \leq 0$. Moreover, since $h'(x) > 0$, it follows that $h(x) \leq 0$ for all $x \leq 1 - K$. Hence, $\hat{g}(x)$ is decreasing over $[\underline{k}, 1-K]$ and $\hat{g}(x) \geq 1/2$ over $[\underline{k}, 1-K]$. But we have reached a contradiction because we have shown that, in an equilibrium with $\bar{g}_{\text{o}} < 1/2$, $\bar{g}_{\text{o}} = \hat{g}(k^\star) < 1/2$.

Next, we show the converse: if $\mathbb{E}\left[k' \,|\, k' \leq 1 - K \text{ or } k' \geq 1\right] < 1$, then there exists an equilibrium with $\bar{g}_{\text{o}} < 1/2$. Indeed, evaluating the equilibrium equation at $k^\star = \underline{k}$ and keeping in mind that $\overline{k} > 1$, we obtain that $\hat{g}(\underline{k}) > 1/2$, which implies that $\text{MPV}(\underline{k} \,|\, \hat{g}(\underline{k})\,\text{o}\,) > \text{MPV}(\underline{k} \,|\, \text{c}\,)$. At $k^\star = 1-K$, on the other hand, since $\hat{g}(1-K) < 1/2$, one can directly verify that $\text{MPV}(1-K \,|\, \hat{g}(1-K), \text{o}\,) < \text{MPV}(1-K \,|\, \text{c}\,)$. Taken together, this means that the equilibrium equation has a unique solution $k^\star < 1 - K$. Since, at a solution, the marginal bank obtains imperfect risk sharing in the centralized market, we must have that $\hat{g}(k^\star) < 1/2$.

Finally, we show that the equilibrium just identified must be unique. Indeed, differentiating the equilibrium equation (15) with respect to $k^\star$, we then obtain:

$$\frac{\partial \text{MPV}}{\partial k^\star}(0, k^\star, \text{o} \,|\, \hat{g}(k^\star)) - \frac{\partial \text{MPV}}{\partial k^\star}(0, k^\star, \text{c}) + \frac{\partial \text{MPV}}{\partial \bar{g}_{\text{o}}}(0, k^\star, \text{o} \,|\, \hat{g}(k^\star)\,)\frac{d\hat{g}}{dk^\star}(k^\star).$$

The difference between first two terms is strictly negative. Recall indeed that, holding $\bar{g}_{\text{o}} = \hat{g}(k^\star)$ constant, $k^\star$ is the smallest of two intersections between two functions: $\text{MPV}(0, k, \text{o})$, which is linear and strictly increasing in k, and $\text{MPV}(0, k, \text{c})$, which is strictly concave. Hence, it must be the case that the first function crosses the second from above. The third term is also negative. Indeed, since $k^\star \leq 1$, the partial derivative of the MPV with respect to $\bar{g}_{\text{o}}$ is positive. From Lemma 3, we have that $k^\star/2 \leq \hat{g}(k^\star)$ or, equivalently, that $k^\star \leq \mathbb{E}\left[k \,|\, k \leq k^\star \text{ or } k \geq 1\right]$. It follows from equation (24) that $d\hat{g}/dk^\star(k^\star) \leq 0$. This establishes the claim.

Equilibrium such that $\bar{g}_{\mathbf{o}} = 1/2$. We already know that, for such an equilibrium to exist, we must have $\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1\right] \geq 1$. Moreover, if such an equilibrium exists, then $\text{MPV}(k \,|\, \text{o}) = |U_{gg}|/8$, which is strictly greater than $\text{MPV}(k \,|\, \text{c})$ for $k < 1-K$, and equal to $\text{MPV}(k \,|\, \text{c})$ for $k \geq 1-K$. Therefore the following banks participate in the OTC markets: all banks with capacities $k < 1-K$ participate in the OTC market, and some measurable subset B of banks in $[1-K, \overline{k}]$. Next, we verify that we can pick some measurable $B \subseteq [1-K, \overline{k}]$ such that $\bar{g}_{\text{o}} = 1/2$. From Lemma 2, this is equivalent to:

$$\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ and } k' \in B\right] \geq 1,$$

for some measurable $B \subseteq [1-K, \overline{k}]$. Equivalently:

$$\int_{\underline{k}}^{1-K} (k-1)\, f(k)\, dk + \int_{k \in B} (k-1)\, f(k)\, dk \geq 0.$$

Clearly, the second term is maximized if $B = [1, \overline{k}]$, in which case the inequality simplifies to our maintained assumption that $\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1\right] \geq 1$. Hence, we can find some measurable $B \in [1-K, \overline{k}]$ such that $\bar{g}_{\text{o}} = 1/2$.

A.7 Proof of Lemma 4

Assume that $C(\text{o}) \simeq C(\text{c})$. Then, the analysis of post-trade exposures and optimal participation patterns is the same as before. Assuming that $\bar{g}_{\text{o}} < 1/2$, the equilibrium equation can be written

$$H(k) - \Delta = 0$$

where

$$\begin{aligned} H(k) &\equiv \frac{1}{8}\left(2\hat{g}(k) + k\left(1 - 2\hat{g}(k)\right)\right) - \frac{1}{2} g(0,k,\text{c})\left(1 - g(0,k,\text{c})\right) \\ \Delta &\equiv \frac{C(\text{o}) - C(\text{c})}{|U_{gg}|}. \end{aligned}$$

If $\mathbb{E}\left[k' \,|\, k' < 1-K \text{ and } k' \geq 1\right] < 1$ and $\Delta = 0$, we know from Proposition 3 that the equilibrium equation has two solutions which we denoted by $k^\star(0) < 1-K$ and $k^{\star\star}(0) = 1$. Going through th same steps as in the proof of 3, it is immediate to show that, when $\Delta \simeq 0$, then there are also two solutions: a unique solution $k^\star(\Delta)$ in the interval $(\underline{k}, 1-K)$, and a unique solution $k^{\star\star}(\Delta)$ in the interval $(1-K, \overline{k})$.

The proof of existence established that $H'(k^\star(0)) < 0$. Differentiating at $k^{\star\star}(0) = 1$, one can verify that $H'(k^{\star\star}(0)) > 0$ since $\bar{g}_\text{o} < 1/2$. An application of the Implicit Function Theorem shows that, for $\Delta \simeq 0$, $k^\star(\Delta)$ is increasing in Δ and $k^{\star\star}(\Delta)$ is decreasing in Δ. The result follows.

A.8 Proof of Proposition 4

Preliminary result: the social planner's objective. With symmetric participation, the planner's objective can be written:

$$\begin{aligned} W(N) =& \frac{1}{2}\int \left[U(0) + U(1)\right] \, dN(k,\text{a}) + \frac{1}{2}\int \left[U(\bar{g}_\text{o}) + U(1-\bar{g}_\text{o}) - 2C\right] \, dN(k,\text{o}) \\ &+ \frac{1}{2}\int \left[U(g_\text{c}(k)) + U(1-g_\text{c}(k)) - 2C\right] \, dN(k,\text{c}), \end{aligned} \tag{25}$$

where $\bar{g}_\text{o}$ and $g_\text{c}(k)$ are given by Lemma 2.

Preliminary results: the marginal social values. Direct calculations show that the marginal social values are:

$$\begin{aligned} \text{MSV}(0,k,\text{o}) &= \frac{|U_{gg}|}{8}\left(2k\left(1-2\bar{g}_\text{o}\right) + 4\bar{g}_\text{o}^2\right) \\ \text{MSV}(0,k,\text{c}) &= \frac{|U_{gg}|}{2} g(0,k,\text{c})\left(1 - gg(0,k,\text{c})\right). \end{aligned}$$

If C is small enough, all banks participate in some market. To show this claim suppose that some banks choose autarky and consider the participation path obtained by moving all banks in autarky to the centralized market:

$$d\hat{N}(k,\text{a}) = 0, d\hat{N}(k,\text{o}) = dN(k,\text{o}), \text{ and } d\hat{N}(k,\text{c}) = dN(k,\text{c}) + dN(k,\text{a}),$$

The associated change in welfare is:

$$\begin{aligned} W(\hat{N}) - W(N) &= \frac{1}{2}\int \left[U(g(0,k,\text{c})) - U(0) + U(1-g(0,k,\text{c})) - U(1) - 2C\right] \, dN(k,\text{a}) \\ &= \frac{1}{2}\int \left[|U_{gg}| g(0,k,\text{c})\left(1-g(0,k,\text{c})\right) - 2C\right] \, dN(k,\text{a}), \end{aligned}$$

where the second line follows from direct calculations. Since $g(0,k,\text{c}) \geq K/2$, the result follows.

Without loss, banks with $k \geq 1$ participate in the OTC market. If there is no participation in the OTC market, then consider the following alternative participation path. All banks with $k \geq 1$ participate in the OTC market, and all bank with $k < 1$ participate in the centralized market. The $k \geq 1$ banks who now participate in the OTC market have the same payoff as in the centralized market: their post-trade exposure is $\bar{g}_{\text{o}} = 1/2$. The $k < 1$ banks who participate in the centralized market keep the same post-trade exposure as well. Hence, social welfare is the same.

If there is some participation in the OTC market, then a marginal increase in the participation of a $k > 1$ bank in the OTC market create a marginal change in social welfare equal to:

$$\begin{aligned}\text{MSV}(0, k, \text{o}) - \text{MSV}(0, k, c) &= \frac{|U_{gg}|}{8}\left(2k\left(1 - 2\bar{g}_{\text{o}}\right) + 4\bar{g}_{\text{o}}^2 - 1\right) \\ &= \frac{|U_{gg}|}{8}\left(1 - 2\bar{g}_{\text{o}}\right)\left[2k - \left(1 + 2\bar{g}_{\text{o}}\right)\right] \geq 0,\end{aligned}$$

since $k \geq 1$ and $\bar{g}_{\text{o}} \leq 1/2$. Hence, increasing participation of $k \geq 1$ banks always increase social welfare weakly. It follows that welfare increases if we modify the participation path so that all $k \geq 1$ participate in the OTC market, but all other participation decisions remain the same.

Existence of a solution to the planner's problem. Given our previous result that it is never optimal to leave a bank in autarky, the planner's problem is to choose a participation path N to maximize social welfare (25), subject to the constraint that all banks participate in either the OTC or the centralized market, that is: $N(B, \text{o}) + N(B, \text{c}) = F(B)$ for all Borel sets B of $[\underline{k}, \overline{k}]$, and where F is the exogenous distribution of capacities in the bank population. By Theorem 14.1 in Aliprantis and Border (2006), the above constraint can be equivalently written as:

$$\int h(k)\;\left[dN(k, \text{o}) + dN(k, \text{c})\right] = \int h(k)\, dF(k), \tag{26}$$

for all continuous functions h over $[\underline{k}, \overline{k}]$. Given that we know that it is weakly optimal to make all $k \geq 1$ banks participate in the OTC market, we add the constraint:

$$\int \rho(k)\, dN(k, \text{o}) = \sum_{\pi \in \Pi} \int \rho(k) \mathbb{I}_{\{\pi = \text{o}\}}\, dN(k, \pi) = \int \rho(k)\, dF(k), \tag{27}$$

where

$$\rho(k) = \begin{cases} 0 & \text{if } k \leq 1 \\ (k-1)/\varepsilon & \text{if } k \in [1, 1+\varepsilon] \\ 1 & \text{if } k \in [1+\varepsilon, \overline{k}]. \end{cases}$$

It is clear that this constraint is satisfied for all N such that all $k \geq 1$ banks participate in the OTC market. Now, notice that the constraints (26) and (27) are defined by integrals against continuous functions. Therefore, they define a constraint set that is compact with respect to the topology of weak convergence. Moreover, for any N in the constraint set (27) holds, it follows that for any N in this constraint set:

$$\int dN(k,\mathrm{o}) \geq \int \rho(k)\, dN(k,\mathrm{o}) = \int \rho(k)\, dF(k) > 0.$$

Therefore, the function:

$$N \mapsto \mathbb{E}\left[k \,|\, \mathrm{o}\right] = \frac{\int k\, dN(k,\mathrm{o})}{\int \, dN(k,\mathrm{o})}$$

is continuous at any N in the constraint set. By implication, post-trade exposures $\bar{g}_{\mathrm{o}}$ are continuous in N, and so is the planner's objective $W(N)$. The existence of a solution then follows from the fact that the planner's problem is to maximize a continuous function over a compact constraint set.

Full risk sharing obtains if and only if $\mathbb{E}\left[k \,|\, k \leq 1-K \text{ or } k \geq 1\right] \geq 1$. For the "if" part, suppose that the planner's problem features full risk sharing, and consider the planner's solution such that all $k \geq 1$ banks participate in the OTC market. Then, for all banks who participate in the centralized market, $g_{\mathrm{c}}(k) = 1/2$ implying that $k \geq 1-K$. Therefore, the set of banks who participate in the OTC market is $k \in [\underline{k}, 1-K]$ and $[1,\overline{k}]$. Since there is full risk sharing, $\bar{g}_{\mathrm{o}} = 1/2$ and

$$\mathbb{E}\left[k \,|\, k \leq 1-K \text{ or } k \geq 1\right] \geq 1.$$

For the "only if" part, notice that, given that no bank stays in autarky, the value of the planner's problem is bounded above by that of full risk sharing, net of participation costs. The result follows because, if $\mathbb{E}\left[k \,|\, k \leq 1-K \text{ or } k \geq 1\right] \geq 1$, then risk-sharing is attained in equilibrium, and so is feasible for the planner.

Participation threshold in the partial risk-sharing case. Direct calculation show that the marginal social values are:

$$\mathrm{MSV}(0,k,\mathrm{o}) = \frac{|U_{gg}|}{8}\left(2k\left(1-2\bar{g}_{\mathrm{o}}\right) + 4\bar{g}_{\mathrm{o}}^2\right)$$
$$\mathrm{MSV}(0,k,\mathrm{c}) = \frac{|U_{gg}|}{2} g(0,k,\mathrm{c})\left(1-g(0,k,\mathrm{c})\right).$$

Proposition 2 imply that $\pi = \mathrm{o}$ if $\mathrm{MSV}(k \,|\, \mathrm{o}) > \mathrm{MSV}(k \,|\, \mathrm{c})$ and $\pi = \mathrm{c}$ if the opposite strict inequality holds.

Given partial risk sharing, $\bar{g}_{\mathrm{o}} < 1/2$ and $\mathrm{MSV}(k \,|\, \mathrm{o})$ is linear and strictly increasing in k. Moreover, when $k > 1-K$, $\mathrm{MSV}(0,k,\mathrm{c})$ is constant and when $k \leq 1-K$, $\mathrm{MSV}(0,k,\mathrm{c})$ is strictly concave in k. Therefore, $\mathrm{MSV}(0,k,\mathrm{o}) - \mathrm{MSV}(0,k,\mathrm{c})$ has at most two roots, implying that socially optimal participation is characterized by two thresholds as claimed. Moreover, one easily verifies that $\mathrm{MSV}(0,1,\mathrm{o}) - \mathrm{MSV}(0,1,\mathrm{c}) > 0$, implying that the upper threshold is strictly less than one.

A.9 Proof of Proposition 5

We maintain the assumption stated in Proposition and proceed as follows. First, given some participation threshold $k^\star < \bar{k}$ but close to $\bar{k}$, we solve for the post-trade exposures of $\pi = \text{o}$ and $\pi = \text{oc}$. Second, given the post-trade exposures, we solve for the MPV$'s$. Third, we verify that that the slope of $\text{MPV}(k \,|\, \text{oc})$ is larger than that of $\text{MPV}(k \,|\, \text{o})$, so that these MPV cross only once. Fourth, we set C so that the crossing occurs at $k^\star$. Let us first define

$$\begin{aligned}
\bar{g}_{\text{o}} &\equiv \frac{\mathbb{E}\left[k' \,|\, \text{o}\right]}{2} + \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{\mathbb{E}\left[k' \,|\, \text{oc}\right]}{2} \\
\bar{g}_{\text{oc}} &\equiv \frac{\mathbb{E}\left[k' \,|\, \text{oc}\right] + K}{2} + \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{\mathbb{E}\left[k' \,|\, \text{oc}\right]}{2}.
\end{aligned} \tag{28}$$

Then, the post-trade exposures for $\pi = \text{o}$ are given by the following Lemma:

Lemma 13. *For all $k^\star$ sufficiently close to $\bar{k}$, the post trade exposures are*

$$\begin{aligned}
g_{\text{o}}(k) &= \bar{g}_{\text{o}} \\
g_{\text{oc}}(k) &= \max\left\{\bar{g}_{\text{o}}, \min\left\{\bar{g}_{\text{oc}}, \frac{k+K}{2} + \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{k + \mathbb{E}\left[k' \,|\, \text{oc}\right]}{2}\right\}\right\}
\end{aligned} \tag{29}$$

In particular, $g_{\text{oc}}(k) = \bar{g}_{\text{oc}}$ for all $k \in [k^\star, \bar{k}]$.

Next, we study post-trade exposures in the limit $k^\star \to \bar{k}$:

Lemma 14. *As $k^\star \to \bar{k}$, the post-trade exposures admit the expansion*

$$\begin{aligned}
g_{\text{o}}(k) &= \bar{g}_{\text{o}}^\dagger + o(1) \\
g_{\text{oc}}(k) &= g_{\text{oc}}^\dagger(k) + o(1),
\end{aligned}$$

where the $\bar{g}_{\text{o}}^\dagger \equiv \frac{\mathbb{E}[k]}{2}$ and $g_{\text{oc}}^\dagger(k) \equiv \max\left\{g_{\text{o}}, \min\left\{\frac{k+K}{2}, \frac{1}{2}\right\}\right\}$.

Next, consider the marginal private values:

Lemma 15. *The marginal private value of $\pi = \text{o}$ is:*

$$\begin{aligned}
\text{MPV}(k \,|\, \text{o}) = & \frac{|U_{gg}|}{2} \bar{g}_{\text{o}}^2 \\
& + \frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\text{o}}\right) \left(k + \mathbb{E}\left[k' \,|\, \text{o}\right]\right) \\
& + \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\text{o}}\right) \left(k + \mathbb{E}\left[k' \,|\, \text{oc}\right]\right).
\end{aligned}$$

The marginal private value of $\pi = \mathrm{oc}$ is:

$$\begin{aligned}\mathrm{MPV}(k \,|\, \mathrm{oc}) = & \frac{|U_{gg}|}{2} g_{\mathrm{oc}}(k)^2 + \frac{|U_{gg}|}{4} \left(1 - 2g_{\mathrm{oc}}(k)\right)(k + K) \\ & + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\mathrm{o}}\right) \left(k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]\right) \\ & + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2g_{\mathrm{oc}}(k)\right) \left(k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]\right).\end{aligned}$$

As for the post-trade exposures, we can derive expansions for the marginal private values as $k^\star \to \bar{k}$. We first define:

$$\begin{aligned}\mathrm{MPV}^\dagger(k \,|\, \mathrm{o}) & \equiv \frac{|U_{gg}|}{2} (g_{\mathrm{o}}^\dagger)^2 + \frac{|U_{gg}|}{8} \left(1 - 2g_{\mathrm{o}}^\dagger\right)(k + \mathbb{E}[k]) \\ \mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) & \equiv \frac{|U_{gg}|}{2} (g_{\mathrm{oc}}^\dagger(k))^2 + \frac{k + K}{2} \frac{|U_{gg}|}{2} \left(1 - 2g_{\mathrm{oc}}^\dagger(k)\right) + \frac{|U_{gg}|}{8} \left(1 - 2g_{\mathrm{o}}^\dagger\right)(k + \mathbb{E}[k]).\end{aligned}$$

Then, we obtain:

Lemma 16. *As $k^\star \to \bar{k}$, $\mathrm{MPV}(k \,|\, \mathrm{o}) = \mathrm{MPV}^\dagger(k \,|\, \mathrm{o}) + o(1)$, $\mathrm{MPV}(k \,|\, \mathrm{oc}) = \mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) + o(1)$, and*

$$\frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{o}) \geq \frac{d\mathrm{MPV}^\dagger}{dk}(k \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}^\dagger}{dk}(k \,|\, \mathrm{o}) + o(1).$$

Since $\mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) - \mathrm{MPV}^\dagger(k \,|\, \mathrm{o})$ is concave, and since our maintained assumption that $K + \bar{k} < 1$ implies that:

$$\frac{d\mathrm{MPV}^\dagger}{dk}(\bar{k} \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}^\dagger}{dk}(\bar{k} \,|\, \mathrm{o}) > 0,$$

we obtain that, for all $k^\star$ close enough to $\bar{k}$, $\mathrm{MPV}(k \,|\, \mathrm{oc}) - \mathrm{MPV}(k \,|\, \mathrm{o})$ is strictly increasing in $k \in [\underline{k}, \bar{k}]$. This means that if we pick C such that

$$C = \mathrm{MPV}(k^\star \,|\, \mathrm{oc}) - \mathrm{MPV}(k^\star \,|\, \mathrm{o}),$$

then, for all $k^\star$ close enough to $\bar{k}$, $\pi = \mathrm{o}$ is optimal for $k \in [0, k^\star)$, and $\pi = \mathrm{oc}$ is optimal for $k \in [k^\star, \bar{k}]$.

A.10 Derivation of the equilibrium of Section 4.2.2

Assume that there are thresholds $0 < k^\star < k^{\star\star} < 1$ such that the participation patterns is $[0, k^\star] = X_{\mathrm{o}} \backslash X_{\mathrm{oc}}$, $(k^\star, k^{\star\star}) = X_{\mathrm{c}} \backslash X_{\mathrm{oc}}$, and $[k^{\star\star}, 1] = X_{\mathrm{oc}}$. Given the uniform distribution for capacities, in the OTC market the masses of exclusive and non-exclusive participants are respectively

$$m_{\mathrm{o}} = k^\star, \text{ and } m_{\mathrm{oc}} = 1 - k^{\star\star}.$$

The average post-trade exposure of banks with $\omega = 0$ and capacity $k \in [0, k^\star]$, participating only in the OTC market, computes as

$$\begin{aligned}
\bar{g}_\mathrm{o} &= \frac{1}{2}\frac{m_\mathrm{o}}{m_\mathrm{oc}+m_\mathrm{o}}\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle|(k',k'')\in(X_\mathrm{o}\backslash X_\mathrm{oc})^2\right] \\
&\quad+\frac{m_\mathrm{oc}}{m_\mathrm{oc}+m_\mathrm{o}}\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle|k'\in X_\mathrm{o}\backslash X_\mathrm{oc}, k''\in X_\mathrm{oc}\right] \\
&=\frac{1}{2}\frac{m_\mathrm{o}}{m_\mathrm{oc}+m_\mathrm{o}}\frac{1}{2}\frac{2k^\star}{3}+\frac{m_\mathrm{oc}}{m_\mathrm{oc}+m_\mathrm{o}}\frac{1}{2}\frac{1+k^{\star\star}}{2} \\
&=\frac{1}{2}\left(\frac{m_\mathrm{o}}{m_\mathrm{oc}+m_\mathrm{o}}\frac{k^\star}{3}+\frac{m_\mathrm{oc}}{m_\mathrm{oc}+m_\mathrm{o}}\frac{1+k^{\star\star}}{2}\right) \\
&=\frac{1}{2}\left(\frac{k^\star}{1-k^{\star\star}+k^\star}\frac{k^\star}{3}+\frac{1-k^{\star\star}}{1-k^{\star\star}+k^\star}\frac{1+k^{\star\star}}{2}\right)
\end{aligned}$$

In the following lemma, we derive the post trade exposure of any bank given its participation decision.

Lemma 17. *Given a participation pattern, $\{k^\star, k^{\star\star}\}$, the post trade exposures of a bank with capacity $k \in [0, 1]$, and endowment $\omega = 0$, that participates exclusively in the centralized exchange or in the OTC market, or non exclusively in both markets, respectively, are*

$$\begin{aligned}
g(0,k,\mathrm{c}) &= g_\mathrm{c}(k) = \frac{k}{2} \\
g(0,k,\mathrm{o}) &= g_\mathrm{o}(k) = \min\left[\max\left(\bar{g}_\mathrm{o}, \frac{m_\mathrm{oc}}{m_\mathrm{oc}+m_\mathrm{o}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle|k'\in X_\mathrm{oc}\right]\right), \frac{1}{2}\right] \\
g(0,k,\mathrm{oc}) &= g_\mathrm{oc}(k) = \min\left[\max\left(\bar{g}_\mathrm{o}, \frac{k}{2}+\frac{m_\mathrm{oc}}{m_\mathrm{oc}+m_\mathrm{o}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle|k'\in X_\mathrm{oc}\right]\right), \frac{1}{2}\right]
\end{aligned}$$

Then, in next lemma, we obtain the marginal private value of different participation decisions of any bank.

Lemma 18. *Given a participation pattern, $\{k^\star, k^{\star\star}\}$, the gross marginal private value of entry, of a bank with capacity $k \in [0, 1]$ and endowment $\omega = 0$, exclusively in the centralized exchange, exclusively* *in the OTC market, or non exclusively in both markets are, respectively,*

$$\begin{aligned}
&\mathrm{MPV}(0,k,\mathrm{c}) = \frac{|U_{gg}|}{2}\frac{k}{2}\left(1-\frac{k}{2}\right)\\
&\mathrm{MPV}(0,k,\mathrm{o})\\
&\quad= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-g_{\mathrm{o}}(k)\right)\left(\frac{1}{2}+\max\left(0,\bar{g}_{\mathrm{o}}-\frac{m_{\mathrm{oc}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{oc}}\right]\right)\right)\right]\\
&\qquad+\frac{|U_{gg}|}{2}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\frac{m_{\mathrm{o}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right]\\
&\mathrm{MPV}(0,k,\mathrm{oc})\\
&\quad= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-g_{\mathrm{oc}}(k)\right)\left(\frac{1}{2}-\frac{k}{2}+\max\left(0,\bar{g}_{\mathrm{o}}-\frac{k}{2}-\frac{m_{\mathrm{oc}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{oc}}\right]\right)\right)\right]\\
&\qquad+\frac{|U_{gg}|}{2}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\frac{m_{\mathrm{o}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right]
\end{aligned}$$

In order to fully derive the marginal value of entries as function of k, $k^\star$, and $k^{\star\star}$, can must compute the following objects,

$$\begin{aligned}
\mathbb{E}\left[\max(k,k'))\middle| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] &= \begin{cases}\frac{(k^\star)^2+k^2}{2k^\star} \text{ if } k<k^\star\\ k \text{ if } k\geq k^\star\end{cases}\\
&= \frac{(k^\star)^2+\min(k,k^\star)^2}{2k^\star}+\max(k-k^\star,0),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\max(k,k'))\middle| k'\in X_{\mathrm{oc}}\right] &= \begin{cases}\frac{1+k^{\star\star}}{2} \text{ if } k<k^{\star\star}\\ \frac{k(k-k^{\star\star})+\frac{1-k^2}{2}}{1-k^{\star\star}} = \frac{1+k^{\star\star}}{2}+\frac{(k-k^{\star\star})^2}{2(1-k^{\star\star})} \text{ if } k^{\star\star}\leq k\leq 1\\ k \text{ if } k>1\end{cases}\\
&= \frac{1+k^{\star\star}}{2}+\frac{(\min[1,\max(k,k^{\star\star})]-k^{\star\star})^2}{2(1-k^{\star\star})}+\max(k-1,0).
\end{aligned}$$

Using previous results, an equilibrium can be pinned down by finding $0<k^\star<k^{\star\star}<1$ such that

$$\begin{aligned}
&\max\left\{\mathrm{MPV}(0,k,\mathrm{oc})-|U_{gg}|/16,\mathrm{MPV}(0,k,\mathrm{o}),\mathrm{MPV}(0,k,\mathrm{c})\right\}\\
&=\begin{cases}\mathrm{MPV}(0,k,\mathrm{o}) \text{ if } k\in[0,k^\star]\\ \mathrm{MPV}(0,k,\mathrm{c}) \text{ if } k\in[k^\star,k^{\star\star}]\\ \mathrm{MPV}(0,k,\mathrm{oc})-|U_{gg}|/16 \text{ if } k\in[k^{\star\star},1]\end{cases}
\end{aligned}$$

As we have set $C(\mathrm{oc})=|U_{gg}|/16$, and as all the MPVs expressions are proportional to $|U_{gg}|$, one can see that the term $|U_{gg}|$ simplifies. We can therefore normalize $|U_{gg}|$ to one without loss of generality.

Mathematica finds a solution to the former equation system, $k^\star \simeq 0.202213$, $k^{\star\star} \simeq 0.932682$. Then, with *Mathematica*, in Figure 6 we use those values to plot the three (net) MPVs of entry, as functions of k, and verify that entry incentives are indeed consistent.

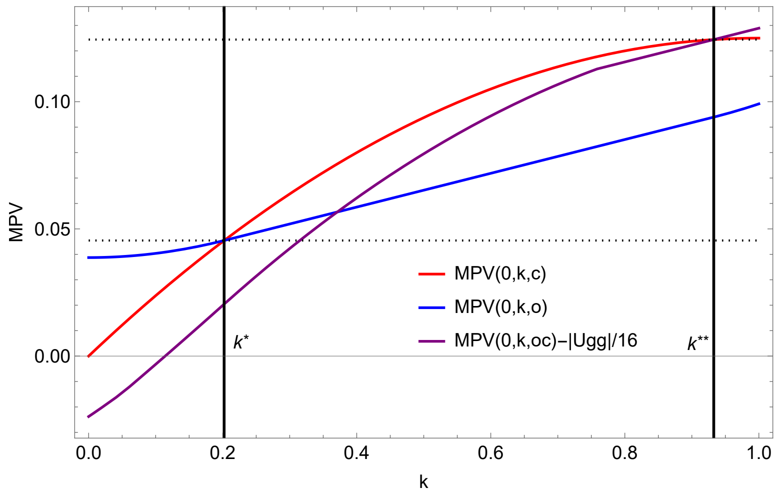

Figure 6: The net MPVs of participation in the centralized market (red curve), the OTC market (blue curve), and both markets (purple curve), as functions of capacity, k, when $k^\star = 0.202213$, $k^{\star\star} = 0.932682$.

In equilibrium, we should make sure that if a $k = 1$ bank participates exclusively in the OTC market, its post trade exposure does not reach $1/2$. By taking a "directional" positions with the banks in X_{oc}, the post-trade exposure of this bank is equal to $(m_{\text{oc}}/2)/(m_{\text{oc}} + m_{\text{o}})$, which is indeed lower. In equilibrium we should also verify that $\bar{g}_{\text{o}} < 1/2$, that $g_{\text{o}}(k) = \bar{g}_{\text{o}}$ for $k \in [0, k^\star]$, and that $g_{\text{oc}}(k) = 1/2$ for $k \in [k^{\star\star}, 1]$. With *Mathematica*, we find that $\bar{g}_{\text{o}} \simeq 0.145961$ and that the equilibrium post trade exposures are consistent as shown in Figure 7.

In the proof of Lemma 18, we derived the bargaining surplus of bank as a function of its trading capacity and its participation decision. Using those result, we can compute the equilibrium bargaining surpluses. First, the bargaining surplus of a bank which participates exclusively in the OTC markets with capacity $k \in [0, k^\star]$, is equal to

$$B(0, k, \text{o}) = |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\text{o}} \right) \mathbb{E} \left[\frac{\max(k, k')}{2} \middle| k' \in X_{\text{o}} \right].$$

Second, the bargaining surplus of a bank which participates exclusively in both markets, with capacity $k \in [k^{\star\star}, 1]$, is equal to

$$B(0, k, \text{oc}) = |U_{gg}| \frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \left(\frac{1}{2} - \bar{g}_{\text{o}} \right) \mathbb{E} \left[\frac{\max(k, k')}{2} \middle| k' \in X_{\text{o}} \backslash X_{\text{oc}} \right].$$

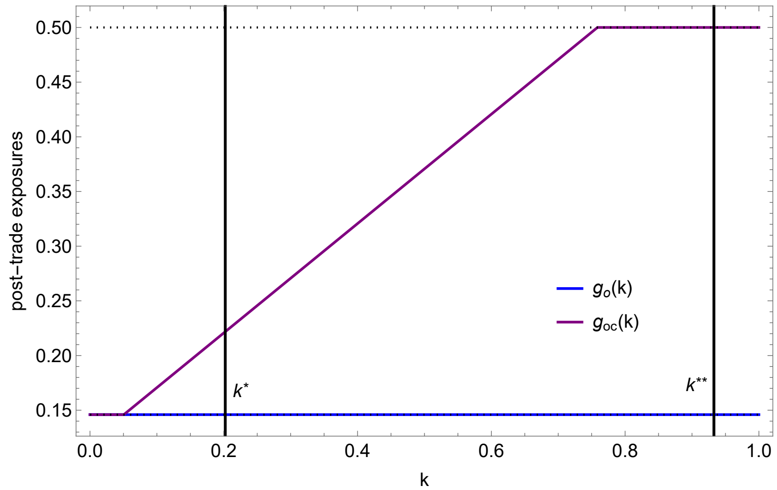

Figure 7: The post-trade exposures of a bank that participate exclusively in the OTC market (blue curve), or in both markets (purple curve), as functions of its capacity, k, when $k^\star = 0.202213,\ k^{\star\star} = 0.932682$.

Therefore, the average bargaining surplus computes as

$$\begin{aligned}\overline{B} &= \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \mathbb{E}\left[\frac{\max(k', k'')}{2} \middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}, k'' \in X_{\mathrm{o}}\right] \\ &+ \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \mathbb{E}\left[\frac{\max(k', k'')}{2} \middle| k' \in X_{\mathrm{oc}}, k'' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right] \\ &= |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} 2\bar{g}_{\mathrm{o}}\end{aligned}$$

With *Mathematica*, by plugging in $k^\star = 0.202213,\ k^{\star\star} = 0.932682$, and normalizing $|U_{gg}| = 1$, we find $\overline{B} = 0.0775387$. Next, we compute the bargaining surpluses of the marginal bank, that are

$$B(0, k^\star, \mathrm{o}) = |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \left(\frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{k^\star}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{1 + k^{\star\star}}{4}\right)$$

and

$$B(0, k^{\star\star}, \mathrm{oc}) = |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{k^{\star\star}}{2}$$

With *Mathematica*, we find $B(0, k^\star, \mathrm{o}) = 0.0695795$ and $B(0, k^{\star\star}, \mathrm{oc}) = 0.123867$ which implies that

$$B(0, k^\star, \mathrm{o}) < \overline{B} < B(0, k^{\star\star}, \mathrm{oc}).$$

Therefore moving the bank $k^\star$ (exclusively) to the centralized market is welfare improving, while moving the bank $k^{\star\star}$ (exclusively) to the centralized market is welfare deteriorating.

A.11 Proof of Proposition 6

Heterogeneity in trading capacities. Since all $\omega = 0$-banks who participate in the OTC market have the same exposure, the bilateral trades between them are not uniquely determined. We make the natural assumption that, when two $\omega = 0$-traders meet, they "swap" the exposures their banks acquired from $\omega = 1$-banks, and vice versa when two $\omega = 1$-traders meet. Precisely, let

$$\bar{\gamma}(k) \equiv \mathbb{E}\left[\frac{k+k'}{2} \,|\, \mathrm{o}\right] = \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2},$$

denote the net trade of a $(\omega = 0, k)$-bank with all $\omega = 1$-banks. We assume that, when an $(\omega = 0, k)$-trader meets an $(\omega = 0, k')$-trader, their bilateral trade is $\bar{\gamma}(k') - \bar{\gamma}(k) = \frac{k'-k}{2}$. It is easy to check that these bilateral exposures satisfy the trading capacity constraint. Moreover, when aggregated across all possible $\omega = 0$-counterparties, these swaps mechanically equalize exposure of all $\omega = 0$-banks who participate in the OTC market. Hence, these swaps implement the equilibrium post-trade exposure. Given our selection for bilateral trade, the net and gross volume are:

$$\begin{aligned} NV(0,k,\mathrm{o}) &= g = \frac{\mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} \\ GV(0,k,\mathrm{o}) &= \frac{1}{2}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} + \frac{1}{2}\frac{\mathbb{E}\left[|k'-k| \,|\, \mathrm{o}\right]}{2}. \end{aligned}$$

Heterogeneity in endowment. With heterogeneous endowments, the net and gross volume for a bank with endowment ω are:

$$\begin{aligned} NV(\omega,k,\mathrm{o}) &= k\left(1 - 2N\left[\omega \,|\, \mathrm{o}\right]\right) \\ GV(\omega,k,\mathrm{o}) &= k. \end{aligned}$$

Applying the Leibniz' rule to the formulas above whenever necessary, one obtains the (in)equalities stated in the Proposition.

Supplement to "A Theory of Participation in OTC and Centralized Markets"

This online appendix contains proofs omitted from the printed manuscript.

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B Other proofs

B.1 Omitted arguments from the proof of Proposition 2

In this section we proceed with weaker assumptions than in the proposition:

$$N(X_{\rm o}) + \varepsilon\, n(X_{\rm o}) > 0 \tag{30}$$

$$N(X_{\rm c}) + \varepsilon\, n(X_{\rm c}) > 0, \tag{31}$$

for all sufficiently small $\varepsilon > 0$. These two conditions hold in particular under the maintained assumption of the proposition, namely, if there is strictly positive participation in the market and if (n^+, n^-) is an admissible direction of reallocation. However they are more general: they allow us also calculate the marginal social value of creating some positive participation in an empty market: e.g. $N(X_{\rm c}) = 0$ and $n(X_{\rm c}) > 0$.

B.1.1 Right-hand differentiability

In this Section we show that:

Lemma 19. *Given any selection $(\gamma^\star, \varphi^\star)(\varepsilon)$ of the maximum correspondence:*

$$\frac{dW^\star}{d\varepsilon}(0^+) = \lim_{\varepsilon \to 0^+} \frac{\partial L}{\partial \varepsilon}((\gamma^\star, \varphi^\star)(\varepsilon), \varepsilon) \geq \max_{(\gamma^\star, \varphi^\star) \in \Lambda^\star(0)} \frac{\partial L}{\partial \varepsilon}((\gamma^\star, \varphi^\star), 0).$$

To that end, we check that the assumptions of Theorem 1, 3 and Corrolary 4 in Milgrom and Segal (2002) are satisfied in our setting. The main technical difficulty is to establish the result of Corrolary 4: the right-derivative can be calculated by taking the maximum of the partial derivative over all maximizers. This is a result that Milgrom and Segal (2002) provide in their Corollary 4 for continuous functions on compact choice sets. Since our choice set is only weakly compact, we must check that

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the required continuity properties hold in the weak topology. A useful preliminary result is to note that the functions g, $\partial g/\partial\varepsilon$, and $\partial^2 g/\partial\varepsilon^2$ are all uniformly bounded in $(x,\gamma,\varphi,\varepsilon)\in X\times\Lambda\times[0,\bar{\varepsilon}]$, for some $\bar{\varepsilon}$ small enough. For g, this follows directly because φ, γ, and ω are uniformly bounded. The first derivatives of g can be calculated explicitly as:

$$\frac{\partial g}{\partial\varepsilon}=\int\gamma(x,x')\,\frac{dn(x')N(X_{\mathrm{o}})-dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}})+\varepsilon\, n(X_{\mathrm{o}})\right]^2}. \tag{32}$$

The result then follows because γ is uniformly bounded, because N and n are finite signed measures, and from (30) and (31), which ensure that the numerators, $N(X_{\mathrm{o}})+\varepsilon n(X_{\mathrm{o}})$ and $N(X_{\mathrm{c}})+\varepsilon n(X_{\mathrm{c}})$ are bounded away from zero for small enough ε. Similar arguments imply uniform boundedness of the second derivative

$$\frac{\partial^2 g}{\partial\varepsilon^2}=-\,2n(X_{\mathrm{o}})\int\gamma(x,x')\,\frac{dn(x')N(X_{\mathrm{o}})-dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}})+\varepsilon\, n(X_{\mathrm{o}})\right]^3}.$$

Uniform boundedness allows us to apply Leibniz' rule to differentiate under the integral sign, and obtain the first and second partial derivative of the Lagrangian L with respect to ε:

$$\begin{aligned}
\frac{\partial L}{\partial\varepsilon}=&\int U\left[g(x,\gamma,\varphi,\varepsilon)\right]\,dn(x)+\int U_g\left[g(x,\gamma,\varphi,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\,\left[dN(x)+\varepsilon dn(x)\right]\\
&-U_g\left(\frac{1}{2}\right)\int\varphi(x)\,dn(x)\\
\frac{\partial^2 L}{\partial\varepsilon^2}=&2\int U_g\left[g(x,\gamma,\varphi,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\,dn(x)\\
&+\int U_{gg}\left[g(x,\gamma,\varphi,\varepsilon)\right]\left[\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\right]^2\,\left[dN(x)+\varepsilon dn(x)\right]\\
&+\int U_g(x,\gamma,\varphi,\varepsilon)\frac{\partial^2 g}{\partial\varepsilon^2}(x,\gamma,\varphi,\varepsilon)\left[dN(x)+\varepsilon dn(x)\right].
\end{aligned}$$

The uniform boundedness properties above imply that both $\partial L/\partial\varepsilon$ and $\partial^2 L/\partial\varepsilon^2$ are uniformly bounded in $(\gamma,\varphi,\varepsilon)$. This further implies that both L and $\partial L/\partial\varepsilon$ are Lipchitz continuous functions of ε, with Lipchitz coefficients that do not depend on γ or φ. Therefore, the equi-continuity and equi-differentiability properties required in Theorem 1 and 3 in Milgrom and Segal (2002) hold and Lemma 19 follows.

B.1.2 The right-hand derivative maximizes marginal social value

Next, we show that the inequality in Lemma 19 is, in fact, an equality:

Lemma 20. *The right-hand derivative maximizes marginal social value:*

$$\frac{dW^\star}{d\varepsilon}(0^+)=\max_{(\gamma^\star,\varphi^\star)\in\Lambda^\star(0)}\frac{\partial L}{\partial\varepsilon}(\gamma^\star,\varphi^\star,0). \tag{33}$$

We adapt the argument of Corollary 4 in Milgrom and Segal. To that end consider a sequence $\varepsilon_m \to 0^+$ and some associated sequence of trades $(\gamma_m^\star, \varphi_m^\star) \in \Lambda^\star(\varepsilon_m)$. Let $g_m^\star(x) \equiv g(x, \gamma_m^\star, \varphi_m^\star, \varepsilon_m)$ and $\partial g_m^\star / \partial \varepsilon(x) \equiv \partial g / \partial \varepsilon(x, \gamma_m^\star, \varphi_m^\star, \varepsilon_m)$. Similarly, let $g^\star(x) \equiv g(x, \gamma^\star, \varphi^\star, 0)$ and $\partial g^\star / \partial \varepsilon = \partial g / \partial \varepsilon(x, \gamma^\star, \varphi^\star, 0)$.

Weak convergence. Given that that bilateral exposures are uniformly bounded, the Riez Weak Compactness Theorem (Royden and Fitzpatrick, 2010, Section 19.4) allows us to successively extract weakly convergent subsequences, so that we can assume without loss of generality that (γ_m, φ_m) converges weakly to some $(\gamma^\star, \varphi^\star)$ for $\gamma^\star$ and $\varphi^\star$ in $L^2(N \times N)$, $L^2(N \times n)$, $L^2(n \times N)$ and $L^2(n \times n)$, and that the sequences of real numbers $\int U\left[g_m^\star(x)\right] \, dN(x)$, $\int U\left[g_m^\star(x)\right] \, dn(x)$ and $\int dU/dg[g_m^\star(x)] \partial g_m^\star / \partial \varepsilon(x) \, dN(x)$ all converge. It then follows from direct calculations using the explicit formula for partial derivatives shown in the proof of Lemma 19 that $g_m^\star(x)$ and $\partial g_m^\star / \partial \varepsilon(x)$ converge to $g^\star(x)$ and $\partial g^\star / \partial \varepsilon(x)$ weakly in $L^2(N)$ and $L^2(n)$.

Strong convergence and asymptotic optimality of post-trade exposures. Given that $g \mapsto \int U\left[g(x)\right] \, dN(x)$ is strongly continuous and convex, it is weakly upper semi-continuous (see Corollary 2.2 in Eckland and Témam, 1987), which implies that:

$$\begin{aligned} &\int U\left[g^\star(x)\right] \, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi^\star(x) \, dN(x) \\ &\qquad \geq \lim_{m \to \infty} \int U\left[g_m^\star(x)\right] \, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi_m^\star(x) \, dN(x). \end{aligned} \tag{34}$$

Given any $(\gamma, \varphi) \in \Lambda$, the optimality of $(\gamma_m^\star, \varphi_m^\star)$ given the distribution $N + \varepsilon_m n$ implies that:

$$\begin{aligned} &\int U\left[g_m^\star(x)\right] \, \left[dN(x) + \varepsilon_m dn(x)\right] - U_g\left(\frac{1}{2}\right) \int \varphi_m^\star(x) \, \left[dN(x) + \varepsilon_m dn(x)\right]) \\ &\qquad \geq \int U\left[g(x, \gamma, \varphi, \varepsilon_m)\right] \left[dN(x) + \varepsilon_m dn(x)\right] - U_g\left(\frac{1}{2}\right) \int \varphi(x) \, \left[dN(x) + \varepsilon_m dn(x)\right]. \end{aligned} \tag{35}$$

It can be easily checked that, holding (γ, φ) fixed, $g(x, \gamma, \varphi, \varepsilon_m) \to g(x, \gamma, \varphi, 0)$ strongly in $L^2(N)$. Given that $g \mapsto \int U\left[g(x)\right] \, dN(x)$ is strongly continuous, we can go to the limit in the inequality (35) and, combining with (34), we obtain:

$$\begin{aligned} &\int U\left[g^\star(x)\right] \, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi^\star(x) \, dN(x) \\ &\geq \lim_{m \to \infty} \int U\left[g_m^\star(x)\right] \, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi_m^\star(x) \, dN(x) \\ &\qquad \geq \int U\left[g(x, \gamma, 0)\right] \, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi(x) \, dN(x). \end{aligned}$$

It follows that $(\gamma^\star, \varphi^\star)$ is an optimum for $\varepsilon = 0$, i.e. $(\gamma^\star, \varphi^\star) \in \Lambda^\star(0)$. Taking the supremum over $(\gamma, \varphi) \in \Lambda$ implies that

$$\lim_{m\to\infty} \int U\left[g_m^\star(x)\right] dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dN(x)$$
$$= \int U\left[g^\star(x)\right] dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi^\star(x)\, dN(x).$$

Notice that, for all (γ, ϕ) maximizing the Lagrangian, we must have $\int \phi(x)\, dN(x) = 0$.[4] Taken together this implies that

$$\lim_{m\to\infty} \int U\left[g_m^\star(x)\right] dN(x) = \int U\left[g^\star(x)\right] dN(x).$$

Since $U[g]$ is quadratic and $g_m^\star \to g^\star$ weakly in $L^2(N)$, it follows that $\int \left[g_m^\star(x)\right]^2 dN(x) \to \int \left[g^\star(x)\right]^2 dN(x)$. Therefore $g_m^\star \to g^\star$ weakly in $L^2(N)$, and the $L^2(N)$ norm of g_m converges to that of $g^\star$. It thus follows that $g_m^\star \to g^\star$ strongly in $L^2(N)$.

The derivative maximizes marginal social value. With these results in mind, consider

$$\frac{\partial L}{\partial \varepsilon}(\gamma_m^\star, \varphi_m^\star, \varepsilon_m) = \int U\left[g_m^\star(x)\right] dn(x) + \int U_g\left[g_m^\star(x)\right] \frac{\partial g_m^\star}{\partial \varepsilon}\left[dN(x) + \varepsilon_m dn(x)\right]$$
$$- U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dn(x). \tag{36}$$

Using the weak upper semi continuity of $g \mapsto \int U[g(x)]\, dn(x)$ as above, we obtain that

$$\int U\left[g^\star(x)\right] dn(x) \geq \lim_{m\to\infty} U\left[g_m^\star(x)\right] dn(x). \tag{37}$$

Now recall that $dU/dg[g(x)]$ is linear, that $g_m^\star \to g^\star$ strongly in $L^2(N)$, that $\partial g_m^\star/\partial\varepsilon$ is uniformly bounded and converges weakly in $L^2(N)$ toward $\partial g^\star/\partial\varepsilon$. It thus follows that $dU/dg\left[g_m^\star\right]\partial g_m^\star/\partial\varepsilon$ converges weakly in $L^2(N)$ towards $dU/dg\left[g^\star\right]\partial g^\star/\partial\varepsilon$. Also recall that $\varphi_m^\star$ is uniformly bounded and converges weakly in $L^2(N \times N)$. Together with (37), this allows us to go to the limit as in (36) and

[4] Indeed, since the Lagrangian is strictly concave in g, it must be that the optimal post-trade exposures are equal, N almost everywhere, for all (γ, ϕ) maximizing the Lagragian. Letting (γ^e, φ^e) denote some equilibrium trades, this means that

$$g(x) = \omega(x) + \int \gamma(x, x')\, dN(x') + \varphi(x) = \omega(x) + \int \gamma^e(x, x')\, dN(x') + \varphi^e(x),$$

since we know from before that equilibrium trade solve the planner's problem and so maximize the Lagrangian as well. The result follow after integrating against $dN(x)$, keeping in mind that trades in the OTC market must be bilaterally feasible and so net out to zero.

obtain:

$$\frac{\partial L}{\partial \varepsilon}(\gamma^\star, \varphi^\star, 0) = \int U\left[g^\star(x)\right]\, dn(x) + \int U_g\left[g^\star(x)\right] \frac{\partial g^\star}{\partial \varepsilon} dN(x) \\ - U_g\left(\frac{1}{2}\right) \int \varphi^\star(x)\, dn(x) \geq \lim_{m\to\infty} \frac{\partial L}{\partial \varepsilon}(\gamma_m^\star, \varphi_m^\star, \varepsilon_m).$$

Combining with Lemma 19, the result follows.

B.1.3 Equilibrium exposures maximize the partial derivative

We now calculate the partial derivative:

$$\frac{\partial L}{\partial \varepsilon}(\gamma, \varphi, 0) = \int U\left[g(x)\right]\, dn(x) + \int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi(x)\, dn(x), \tag{38}$$

where, to simplify notations, we have let $g(x) \equiv g(x, \gamma, \varphi, 0)$ and $\partial g/\partial\varepsilon(x) \equiv \partial g/\partial \varepsilon\,(x, \gamma, \varphi, 0)$.

Next, we simpifly the second term of (38). If $N(X_\text{o}) > 0$, then the formula (32) implies that

$$\begin{aligned}
&\int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) = \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dn(x')N(X_\text{o}) - dN(x')n(X_\text{o})}{N(X_\text{o})^2}\, dN(x) \\
&= \int\int U_g\left[g(x')\right] \gamma(x', x) \frac{dN(x')}{N(X_\text{o})}\, dn(x) - n(X_\text{o}) \int\int\; U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} \\
&= -\int\int U_g\left[g(x')\right] \gamma(x, x') \frac{dN(x')}{N(X_\text{o})}\, dn(x) - n(X_\text{o}) \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2}
\end{aligned}$$

where the second-to-last equality follows by exchanging the name of variables, namely replacing x by x' in the first term, and the last equality follows by bilateral feasibility, i.e. $\gamma(x', x) = -\gamma(x, x')$. In addition, we have that:

$$\begin{aligned}
&- n(X_\text{o}) \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} \\
&= -\frac{n(X_\text{o})}{2} \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} - \frac{n(X_\text{o})}{2} \int\int U_g\left[g(x')\right] \gamma(x', x) \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} \\
&= -\frac{n(X_\text{o})}{2} \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} + \frac{n(X_\text{o})}{2} \int\int U_g\left[g(x')\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2} \\
&= -\frac{n(X_\text{o})}{2} \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\} \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_\text{o})^2},
\end{aligned}$$

where: the first equality follows by breaking the integral into two identical halves and exchanging the name of variables in the second term, replacing x by x' in the second half; the second equality follows by bilateral feasibility $\gamma(x', x) = -\gamma(x, x')$; and the third equality by collecting terms. Taken together we obtain that, if $N(X_{\mathrm{o}}) > 0$,

$$\int U_g\left[g(x)\right]\frac{\partial g}{\partial \varepsilon}(x)\,dN(x) = -\int\int U_g\left[g(x')\right]\gamma(x,x')d\mu(x')\,dn(x) \\ -\frac{n(X_{\mathrm{o}})}{2}\int\int\left\{U_g\left[g(x')\right]-U_g\left[g(x'')\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x'') \quad (39)$$

where $d\mu(x) = dN(x)/N(X_{\mathrm{o}})$. Substituting (39) into (38), we arrive at:

Lemma 21. *For any $(\gamma,\varphi)\in\Lambda$:*

$$\begin{aligned}\frac{\partial L}{\partial\varepsilon}(\gamma,\varphi,0) = &\int U\left[g(x,\gamma,\varphi,0)\right]\,dn(x)\\ &-\int U_g\left[g(x',\gamma,\varphi,0)\right]\gamma(x,x')d\mu(x')\,dn(x)\\ &-\frac{n(X_{\mathrm{o}})}{2}\int\int\left\{U_g\left[g(x',\gamma,\varphi,0)\right]-U_g\left[g(x'',\gamma,\varphi,0)\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x'')\\ &-\int U_g\left(\frac{1}{2}\right)\varphi(x)\,dn(x),\end{aligned}$$

where

$$d\mu \equiv \frac{dN}{N(X_{\mathrm{o}})} \text{ if } N(X_{\mathrm{o}})>0 \text{ and } d\mu \equiv \frac{dn}{n(X_{\mathrm{o}})} \text{ otherwise.}$$

While our calculations so far have assumed that $N(X_{\mathrm{o}}) > 0$, one sees that the formula of the Lemma is valid when $N(X_{\mathrm{o}}) = 0$ as well. Indeed, in that case one sees from its definition that $g(x,\gamma,\varphi,\varepsilon)$ does not depend on ε, hence $\partial g/\partial\varepsilon = 0$ and $\int U_g\left[g(x)\right]\frac{\partial g}{\partial\varepsilon}(x)\,dN(x) = 0$ as well, so the second and third term of the formula should be equal to zero. One easily check that this is the case when setting $d\mu(x) = dn(x)/n(X_{\mathrm{o}})$.

Notice that, in the formula, the measure used to calculate average depend on whether there is, under N, positive participation in the OTC market. If $N(X_{\mathrm{o}}) > 0$, then the average is calculated based on the conditional distribution of incumbent in the OTC market. If $N(X_{\mathrm{o}}) = 0$, then the average is calculated based on the conditional distribution of entrant.

To proceed we assume that $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$.[5] We show:

Lemma 22. *Equilibrium trades solve the problem of maximizing $\partial L/\partial\varepsilon$ with respect to $(\gamma^\star,\varphi^\star)\in\Lambda^\star(0)$.*

[5] While the proposition restricts attention to the case of strictly positive participation in both market, it can be extended to the case of $N(X_{\mathrm{o}}) = 0$ or $N(X_{\mathrm{c}}) = 0$, after redefining the equilibrium in an appropriate way. Consider for example participation patterns such that $N(X_{\mathrm{o}}) = 0$ and $N(X_{\mathrm{c}}) > 0$, with a perturbation such that $n(X_{\mathrm{o}}) > 0$. Then, one needs to define equilibrium post-trade exposures when there is a large group of investor, of size $N(X_{\mathrm{c}}\setminus X_{\mathrm{o}})$, participating in the centralized market, and an "infinitesimal" group of investors, with type distribution $n(x)/n(X_{\mathrm{o}})$, participating in the OTC and possibly simultaneously in the centralized market.

Consider any socially optimal $(\gamma, \varphi) \in \Lambda^\star(0)$. Because the first-order conditions hold almost everywhere according to $N(\,\cdot\,|\,\mathrm{o}) \times N(\,\cdot\,|\,\mathrm{o})$, it follows that the integrand of the last term in the formula of Lemma 21 is equal to

$$\left| U_g\left[g(x')\right] - U_g\left[g(x'')\right] \right| \Gamma(x', x'')$$

almost everywhere according to $N(\,\cdot\,|\,\mathrm{o}) \times N(\,\cdot\,|\,\mathrm{o})$. Therefore, the second to last term is constant and equal to

$$-n(X_\mathrm{o})\frac{\bar{B}}{2},$$

for any socially optimal $(\gamma, \varphi) \in \Lambda^\star(0)$.

Now let (γ, φ) denote a collection of equilibrium trades with the associated post-trade exposures g, and let $(\hat{\gamma}, \hat{\varphi})$ denote any collection of socially optimal trades with the associated post-trade exposures $\hat{g}$. We calculate:

$$\begin{aligned}
& U\left[g(x)\right] - \int U_g\left[g(x')\right]\gamma(x,x')\,dN(x'\,|\,\mathrm{o}) - U_g\left(\frac{1}{2}\right)\varphi(x) \\
& - U\left[\hat{g}(x)\right] + \int U_g\left[\hat{g}(x')\right]\hat{\gamma}(x,x')\,dN(x'\,|\,\mathrm{o}) + U_g\left(\frac{1}{2}\right)\hat{\varphi}(x) \\
\geq & U_g\left[g(x)\right]\left[g(x) - \hat{g}(x)\right] - U_g\left(\frac{1}{2}\right)\varphi(x) + U_g\left(\frac{1}{2}\right)\hat{\varphi}(x) \\
& - \int U_g\left[g(x')\right]\left\{\gamma(x,x') - \hat{\gamma}(x,x')\right\}\,dN(x'\,|\,\mathrm{o}) \\
\geq & \left[U_g\left[g(x)\right] - U_g\left(\frac{1}{2}\right)\right]\left[\varphi(x) - \hat{\varphi}(x)\right] \\
& + \int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\left\{\gamma(x,x') - \hat{\gamma}(x,x')\right\}\,dN(x'\,|\,\mathrm{o}),
\end{aligned}$$

where: the first inequality follows by concavity, and because optimality implies that $g(x') = \hat{g}(x')$ almost everywhere according to N; the second inequality follows using the explicit expression of $g(x)$ in terms of $\gamma(x, x')$.

Both the first and the second term of the last inequality is positive because in equilibrium, the first-order conditions hold everywhere. Hence we have shown that the integrand corresponding to the first three terms in Lemma 21 is greatest when evaluated at equilibrium bilateral exposures, and the result follows.

B.2 Proof of Lemma 6

Direct calculations show that:

$$\frac{d}{d\omega}S(\omega,k,\mathrm{o}) = |U_{gg}|\,(g_{\mathrm{o}}-\omega)\left(\frac{dg_{\mathrm{o}}}{d\omega}-1\right) + |U_{gg}|k\,[2N(\omega\,|\,\mathrm{o})-1]\,\frac{dg_{\mathrm{o}}}{d\omega}$$
$$\frac{d}{d\omega}B(\omega,k,\mathrm{o}) = |U_{gg}|k\,[2N(\omega\,|\,\mathrm{o})-1]\,\frac{dg_{\mathrm{o}}}{d\omega},$$

where g_{o} and $dg_{\mathrm{o}}/d\omega$ denote, respectively, the post-trade exposure and its right-derivative for an ω-bank who participates in the OTC market. Using that $g_{\mathrm{o}}-\omega = k\,[1-2N(\omega\,|\,\mathrm{o})]$, we obtain:

$$\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{o}) = -|U_{gg}|k\,[1-2N(\omega\,|\,\mathrm{o})]\left[\frac{1}{2}\frac{dg_{\mathrm{o}}}{d\omega}-1\right].$$

Now use that $\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{c}) = -|U_{gg}|\left[\frac{1}{2}-\omega-\left(\frac{1}{2}-g_{\mathrm{c}}\right)\frac{dg_{\mathrm{c}}}{d\omega}\right]$, and obtain:

$$\begin{aligned}&\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right]\\ &\qquad = -|U_{gg}|\left\{\frac{1}{2}-\omega-\left(\frac{1}{2}-g_{\mathrm{c}}\right)\frac{dg_{\mathrm{c}}}{d\omega}+k\,[1-2N(\omega\,|\,\mathrm{o})]\left[\frac{1}{2}\frac{dg_{\mathrm{o}}}{d\omega}-1\right]\right\},\end{aligned}$$

where g_{c} and $dg_{\mathrm{c}}/d\omega$ denote, respectively, the post-trade exposure and its right-derivative for an ω-bank who participates in the centralized market. For $\omega<\omega^\star$, then $g_{\mathrm{c}}=\omega+k$, $dg_{\mathrm{c}}/d\omega = dg_{\mathrm{o}}/d\omega = 1$, $N(\omega\,|\,\mathrm{o})=0$, so that:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1}{2}-\omega-\left(\frac{1}{2}-\omega-k\right)-\frac{k}{2}\right\} = -|U_{gg}|\frac{k}{2}<0.$$

Next, for $\omega\in\left[\omega^\star,\frac{1}{2}-k\right]$, $g_{\mathrm{c}}=\omega+k$, $dg_{\mathrm{c}}/d\omega=1$, $N(\omega\,|\,\mathrm{o})=\frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg_{\mathrm{o}}}{d\omega}=1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{-(1-2\omega)\left(\frac{1}{4}+\left(\frac{k}{1-2\omega^\star}\right)^2\right)+k\right\}<0$$

because $\frac{k}{1-2\omega^\star}<\frac{1}{2}$. Finally, for $\omega\in\left[\frac{1}{2}-k,\frac{1}{2}\right]$, $dg_{\mathrm{c}}/d\omega=0$, $N(\omega\,|\,\mathrm{o})=\frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg_{\mathrm{o}}}{d\omega}=1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1-2\omega}{1-2\omega^\star}\left(\frac{1}{2}-\omega^\star-k\right)\left(\frac{1}{2}+\frac{k}{1-2\omega^\star}\right)\right\}\le 0,$$

with equality if $\omega=\frac{1}{2}$.

Using the symmetry, i.e.,

$$MPV(1-\omega,k,\mathrm{c}) = MPV(\omega,k,\mathrm{c})$$
$$MPV(1-\omega,k,\mathrm{o}) = MPV(\omega,k,\mathrm{o}),$$

one easily sees that, for $\omega \in \left[\frac{1}{2},1\right]$

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c}) - \mathrm{MPV}(\omega,k,\mathrm{o})\right] \geq 0,$$

with equality if $\omega = \frac{1}{2}$.

B.3 Proof of Lemma 7

Let w_0 denote the maximized value of the planner's problem and we consider the sets

$$A \equiv \left\{(w,y) \in \mathbb{R}^2 \,:\, W(\varphi,\gamma) \geq w \text{ and } \int \varphi(x)\,dN(x) = y \text{ for some } (\varphi,\gamma) \in \Lambda\right\}$$
$$B^+ \equiv \left\{(w,y) \in \mathbb{R}^2 \,:\, w \geq w_0 \text{ and } y \geq 0\right\}$$
$$B^- \equiv \left\{(w,y) \in \mathbb{R}^2 \,:\, w \geq w_0 \text{ and } y \leq 0\right\}.$$

It is clear that A, B^+ and B^- are all convex sets. We claim that either $A \cap \mathring{B}^+ = \emptyset$ or $A \cap \mathring{B}^- = \emptyset$. Otherwise, there would be some trades (φ^+,γ^+) and (φ^-,γ^-) such that

$$W(\varphi^+,\gamma^+) > w_0 \text{ and } \int \varphi^+(x)\,dN(x) > 0,$$

and

$$W(\varphi^-,\gamma^-) > w_0 \text{ and } \int \varphi^-(x)\,dN(x) < 0.$$

Clearly, if one forms the convex combination of (φ^+,γ^+) and (φ^-,γ^-) such that (40) holds, one obtains by concavity of W and convexity of Λ a feasible trade that attains a value larger than w_0, and we have reached a contradiction. Suppose for now without loss of generality that $A \cap \mathring{B}^- = \emptyset$. The separating hyperplane theorem then implies that there exists $(w^\star,y^\star) \in \mathbb{R}^2$, $(w^\star,y^\star) \neq (0,0)$, such that:

$$w^\star\, w_A + y^\star\, y_A \leq w^\star\, w_B + y^\star\, y_B$$

for all $(w_A,y_A) \in A$ and $(w_B,y_B) \in B^-$. It is clear that $w^\star \geq 0$ otherwise the inequality would be violated for w_B large enough. If $w^\star = 0$ then using that $(w_0,0) \in B^-$, we obtain that $y^\star\, y_A \leq 0$ for all $(w_A,y_A) \in A$. But since one can choose $(\varphi,\gamma) \in \Lambda$ such that $\int \varphi(x)\,dN(x) > 0$ or < 0, this implies that $y^\star = 0$, in contradiction with the fact that $(w^\star,y^\star) \neq (0,0)$. Hence, $w^\star > 0$. Normalizing it to one without loss of generality, we obtain:

$$w^\star w_A + y^\star y_A \leq w_0,$$

for all $(w_A, y_A) \in A$. But this upper bound is achieves at the optimum, which implies the stated result with $g_c \equiv U_g^{-1}(-y^\star)$.

B.4 Proof of Lemma 8

Define the following two functions. First:

$$\begin{aligned}\overline{V}(g) =& \omega(x) + \Phi(x)\mathbb{I}_{\{g \leq h_c\}} - \Phi(x)\mathbb{I}_{\{g < h_c\}} \\ &+ \int \Gamma(x, x')\mathbb{I}_{\{g \leq h(x')\}}\, dN(x' \,|\, \mathrm{o}) - \int \Gamma(x', x)\mathbb{I}_{\{g > h(x')\}}\, dN(x' \,|\, \mathrm{o}).\end{aligned}$$

The function $\overline{V}(g)$ represents the maximum post-trade exposure of the bank of type x, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\overline{V}(g)$ is decreasing and left-continuous. Similarly, we let:

$$\begin{aligned}\underline{V}(g) =& \omega(x) + \Phi(x)\mathbb{I}_{\{g < h_c\}} - \Phi(x)\mathbb{I}_{\{g \geq h_c\}} \\ &+ \int \Gamma(x, x')\mathbb{I}_{\{g < h(x')\}}\, dN(x' \,|\, \mathrm{o}) - \int \Gamma(x', x)\mathbb{I}_{\{g \geq h(x')\}}\, dN(x' \,|\, \mathrm{o}).\end{aligned}$$

The function $\underline{V}(g)$ represents the minimum post-trade exposure of a bank, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\underline{V}(g)$ is decreasing and right-continuous. Moreover:

$$\overline{V}(g+) = \underline{V}(g) \text{ and } \overline{V}(g) = \underline{V}(g-)$$

where $g+$ and $g-$ denote right- and left-limits. Finally, given that pre-trade exposures, $\omega(x)$, are bounded, and given that $\Gamma(x, x')$ is bounded over the support of N, there exists $a < b$ such that $\underline{V}(g) \in [a, b]$ and $\overline{V}(g) \in [a, b]$ for all g. We then have:

Step 1. We show that a post-trade exposure g solves (17)-(19) if and only if the fixed-point problem $g \in [\underline{V}(g), \overline{V}(g)]$. For the "only if" part, take a solution of (17)-(19) and use the optimality conditions (18) and (19) to show that it belongs to $[\underline{V}(g), \overline{V}(g)]$. For the "if" part, take some $g \in [\underline{V}(g), \overline{V}(g)]$, let

$$\begin{aligned}\varphi(x) &= \Phi(x)\mathbb{I}_{\{g < h_c\}} + \left[(1-\alpha)\Phi(x) - \alpha\Phi(x)\right]\mathbb{I}_{\{g = h_c\}} - \Phi(x)\mathbb{I}_{\{g > h_c\}} \\ \gamma(x, x') &= \Gamma(x, x')\mathbb{I}_{\{g < h(x')\}} + \left[(1-\alpha)\Gamma(x, x') - \alpha\Gamma(x, x')\right]\mathbb{I}_{\{g = h(x')\}} - \Gamma(x', x)\mathbb{I}_{\{g > h(x')\}}\end{aligned}$$

where α is chosen such that:

$$g = (1-\alpha)\overline{V}(g) + \alpha \underline{V}(g).$$

One can then directly verify that (φ, γ) solve the problem (17)-(19).

Step 2. Next we show that the fixed point problem $g \in [\underline{V}(g), \overline{V}(g)]$ has a unique solution. To show that a solution exists, we apply Kakutani's Fixed Point Theorem (see, e.g., Theorem 7 in Nachbar, 2017) to the correspondence

$$g \rightrightarrows V(g) \equiv [\underline{V}(g), \overline{V}(g)].$$

It is clear that $V(g)$ takes values that are convex sets included in $[a, b]$. To see that $V(g)$ has a closed graph consider any converging sequence $(g_n, v_n) \to (g, v)$ such that $v_n \in V(g_n)$ for all n. Then we can extract a subsequence of g_n such that either $g_n \leq g$ for all n or $g_n \geq g$ for all n. Suppose that we are in the former case (the latter case is symmetric). Then, since $\underline{V}(g)$ is decreasing, it follows that $v_n \geq \underline{V}(g_n) \geq \underline{V}(g)$. Going to the limit, we obtain $v \geq \underline{V}(g)$. Since $v_n \leq \overline{V}(g_n)$ and since $\overline{V}(g)$ is left-continuous, it follows that $v \leq \overline{V}(g)$. Therefore, $v \in V(g)$.

For uniqueness consider any $g \in V(g)$. Then $g \geq \underline{V}(g) = \overline{V}(g+)$. But since $\overline{V}(g)$ is decreasing, it follows that $g' > \overline{V}(g')$ for all $g' > g$. Hence, $g' \notin V(g')$. A similar argument applies to $g' < g$.

The second part of the Lemma follows directly because the proposed changes do not impact the functions $\underline{V}(g)$ and $\overline{V}(g)$, hence the fixed point remains the same.

B.5 Proof of Lemma 9

Step 1: modify trades in the centralized market. Let $U_g(g_c)$ denote the Lagrange multiplier on the constraint $\int \varphi(x)\, dN(x) = 0$ in the planner's problem. Then for all x such that (8) and (10) do not hold, we pick $\varphi(x)$ and $\gamma(x, x')$ that solve the partial equilibrium problem (17)-(19), given $h_c = g_c$ and $h(x') = g(x')$. By construction, these trades are both feasible and optimal for x (except perhaps for the bilateral feasibility constraint, which we address in the following paragraph). Moreover, because trades change for a measure zero set of x, they keep the post-trade exposures function the same N almost everywhere, they do not impact the post-trade exposures of any other x', nor do they impact the market-clearing condition in the centralized market.

Step 2: modify trades in the OTC market. First, set $\gamma(x,x')=0$ for all $(x,x')\notin X_{\rm o}$. Define the following sets:

$$\begin{aligned}
\Theta(x) &= \left\{x' \in X_{\rm o}\,\text{s.t.}\,(2)\text{ and }(8)\text{ hold for }(x,x')\right\}\\
\Theta &= \cup_{x\in X_{\rm o}}\{x\}\times\Theta(x) = \left\{(x,x')\in X_{\rm o}^2\,\text{s.t.}\,(2)\text{ and }(8)\text{ hold for }(x,x')\right\}\\
\Psi &= \{x\in X_{\rm o}\,\text{s.t.}\,N(\Theta(x)\,|\,{\rm o})=1\}\,.
\end{aligned}$$

One can show that, unsurprisingly, $N(\Psi\,|\,{\rm o})=1$. Indeed, since the first-order conditions hold almost everywhere:

$$\begin{aligned}
N(\Theta\,|\,{\rm o})=1 \Leftrightarrow 1 &= \int_{x\in\Psi} N(\Theta(x)\,|\,{\rm o})\,dN(x\,|\,{\rm o}) + \int_{x\notin\Psi} N(\Theta(x)\,|\,{\rm o})\,dN(x\,|\,{\rm o})\\
1 &= N(\Psi\,|\,{\rm o}) + \int_{X_{\rm o}\backslash\Psi} N(\Theta(x)\,|\,{\rm o})\,dN(x\,|\,{\rm o})\\
0 &= \int_{X_{\rm o}\backslash\Psi}\left[1-N(\Theta(x)\,|\,{\rm o})\right]\,dN(x\,|\,{\rm o}),
\end{aligned}$$

where the second line follows because $N(\Theta(x)\,|\,{\rm o})=1$ for all $x\in\Psi$, and the third line because $N(\Psi)=1-N(X_{\rm o}\backslash\Psi)$. But since $N(\Theta(x)\,|\,{\rm o})<1$ for all $x\notin\Psi$, the integrand in the last integral is strictly positive, so the only way that integral is zero is if $N(\Psi\,|\,{\rm o})=1$

Then, we define:

$$A=\Theta\cap\Psi^2.$$

The set A has measure one because it is the intersection of two sets of measure one. It contains pairs (x,x') with the following properties. First, since they belong to Θ, they together satisfy the feasibility condition (2) and the optimality condition (8). Second, since they belong to Ψ^2, they each satisfy the feasibility condition (2) and the optimality condition (8) with almost every other $\hat{x}$. Next we define:

$$\begin{aligned}
B &= \left\{x\in X_{\rm o}\,:\,(x,x')\in A\text{ for some }x'\in X_{\rm o}\right\}\\
C &= X_{\rm o}\backslash B.
\end{aligned}$$

Figure 8 illustrates these sets. The set B has also measure one because $A\subseteq B^2$ and, correspondingly, C has measure zero. Notice that for any $x\in B$, there exists (x,x') such that $(x,x')\in A$. But since $A\subseteq\Psi^2$, this implies that $x\in\Psi$ and so that $N(\Theta(x)\,|\,{\rm o})=1$. With these observations in mind, our modification goes as follows:

- For all $(x,x')\in A$, the feasibility condition (2) and the optimality condition (8) hold and so we keep $\gamma(x,x')$ the same.

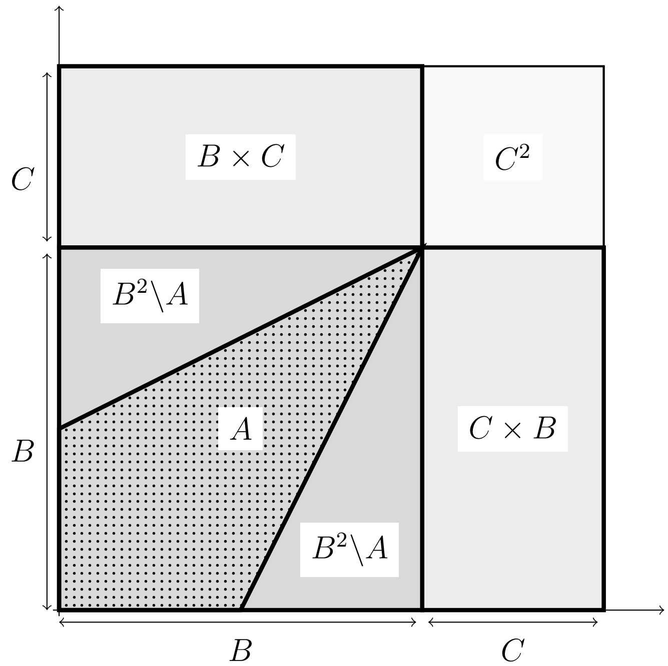

Figure 8: An illustration of the sets A, B, and C.

- For all $(x, x') \in B^2 \backslash A$, we modify $\gamma(x, x')$ so that it satisfies (2) and (8). Notice that since $N(\Theta(x) \,|\, \mathrm{o}) = N(\Theta(x') \,|\, \mathrm{o}) = 1$, these modifications concern a measure zero sets of counterparties for both x and x', and so they do not change the post-trade exposures $g(x)$ or $g(x')$.

- For all $(x, x') \in C \times B$, we pick $\gamma(x, x')$ and $g(x)$ that solves the fixed point problem of Section A.1.2. For any $x' \in B$, this changes the bilateral trades for a measure zero set of counterparties and so does not change $g(x')$.

- For $(x, x') \in C^2$, then we change the bilateral trades so that they satisfy (2) and (8). For either x or x', this changes the bilateral trades for a measure zero of counterparties, and so does not change $g(x)$ or $g(x')$.

These modification imply that the trades (φ, γ) now satisfy (2), (8), (10) everywhere in X^2, and

$$\int \varphi(x) \, dN(x) = 0. \tag{40}$$

The optimality conditions imply the capacity constraints (3) and (5). Thus, these trades are the basis of an equilibrium.

B.6 Proof of Lemma 10

Given that the third component of x, π, is discrete, we only need to prove that such a bound can be found conditional on π. Suppose for this proof that $\pi(x_1) = \pi(x_2) = \mathrm{oc}$ (the other cases are similar) and that $g(x_1) < g(x_2)$. Consider first the trades in the centralized market. If $g(x_2) \le g_c$, then $g(x_1) < g_c$, so (10) imply that $\varphi(x_1) = \Phi(x_1)$ and $\varphi(x_2) \le \Phi(x_2)$. Hence, $\varphi(x_2) - \varphi(x_1) \le \Phi(x_2) - \Phi(x_1)$. If $g(x_2) > g_c$, then $\varphi(x_2) = -\Phi(x_2)$ and $\varphi(x_1) \le \Phi(x_1)$. Therefore, $\varphi(x_2) - \varphi(x_1) \le \Phi(x_1) - \Phi(x_2)$. Hence, we obtain that, in both cases:

$$\varphi(x_2) - \varphi(x_1) \le |\Phi(x_2) - \Phi(x_1)|.$$

Next, let us turn to the trades in the OTC market. The first-order condition for $\gamma(x, x')$ implies that, if $g(x') \le g(x_1)$:

$$\gamma(x_1, x') \ge -\Gamma(x', x_1) \text{ and } \gamma(x_2, x') = -\Gamma(x', x_2).$$

On the other hand, if $g(x') > g(x_1)$:

$$\gamma(x_1, x') = \Gamma(x_1, x') \text{ and } \gamma(x_2, x') \le \Gamma(x_2, x').$$

Therefore:

$$\begin{aligned}
&\int \gamma(x_2, x')\, dN(x' \,|\, \mathrm{o}) - \int \gamma(x_1, x')\, dN(x' \,|\, \mathrm{o}) \\
&\le \int_{g(x') \le g(x_1)} \left[\Gamma(x', x_1) - \Gamma(x', x_2)\right]\, dN(x' \,|\, X_\mathrm{o}) + \int_{g(x') > g(x_1)} \left[\Gamma(x_2, x') - \Gamma(x_1, x')\right]\, dN(x' \,|\, X_\mathrm{o}) \\
&\le \sup_{x'} |\Gamma(x', x_2) - \Gamma(x', x_1)| + \sup_{x'} |\Gamma(x_2, x') - \Gamma(x_1, x')|.
\end{aligned}$$

Putting the upper bound for the difference in centralized and OTC market trades, and using that $g(x_2) - g(x_1) \ge 0$, we obtain:

$$\begin{aligned}
0 \le g(x_2) - g(x_1) \le \quad & |\omega(x_2) - \omega(x_1)| + |\Phi(x_2) - \Phi(x_1)| \\
& + \sup_{x'} |\Gamma(x', x_2) - \Gamma(x', x_1)| + \sup_{x'} |\Gamma(x_2, x') - \Gamma(x_1, x')|.
\end{aligned}$$

The inequality holds evidently if $g(x_1) = g(x_2)$ and symmetrically if $g(x_1) > g(x_2)$. This establishes the claim with $G(x_1, x_2)$ defined to be the function of the right-hand side inequality above.

B.7 Proof of Lemma 11

Proof that post-trade exposures are symmetric. Fix (ω, k, π) such that $\omega \le \frac{1}{2}$. Consider some equilibrium collection of OTC trades $\gamma\left(\omega, k, \pi; \widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right)$ and centralized trades $\varphi(\omega, k, \pi)$ and the associated post-trade exposures $g(\omega, k, \pi)$. The alternative collection of OTC trades $\widehat{\gamma}\left(\omega, k, \pi; \widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right) = -\gamma\left(1 - \omega, k, \pi; 1 - \widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right)$ and the alternative collection of centralized trades $\widehat{\varphi}(\omega, k, \pi) = -\varphi(1 - \omega, k, \pi)$

are feasible and generate post-trade exposures:

$$
\begin{aligned}
\widehat{g}(\omega,k,\pi) &= \omega+\widehat{\varphi}(\omega,k,\pi)+\int \widehat{\gamma}\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= \omega-\varphi(1-\omega,k,\pi)-\int \gamma\left(1-\omega,k,\pi;1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= \omega-\varphi(1-\omega,k,\pi)-\int \gamma\left(1-\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= 1-\left(1-\omega+\varphi(1-\omega,k,\pi)+\int \gamma\left(1-\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\right)\\
&= 1-g(1-\omega,k,\pi)\,.
\end{aligned}
$$

Now it is easy to see that $\widehat{\gamma}\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)$ satisfies bilateral optimality since $\widehat{g}(\omega,k,\pi)<\widehat{g}\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)$ is equivalent to $g\left(1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)<g(1-\omega,k,\pi)$. Similarly, the alternative centralized trades $\widehat{\varphi}(\omega,k,\pi)$ are optimal with our assumed market-clearing price $U_g\left(\frac{1}{2}\right)$, because $\widehat{g}(\omega,k,\pi)<\frac{1}{2}$ is equivalent to $\frac{1}{2}<g(1-\omega,k,\pi)$. Since equilibrium post-trade exposures are uniquely determined, we conclude from this that $\widehat{g}(\omega,k,\pi)=1-g(1-\omega,k,\pi)=g(\omega,k,\pi)$.

Proof that post-trade exposures of banks are increasing in endowment. Consider two banks with identical trading capacities and participation choices but different endowments, (ω,k,π) and (ω',k,π) with $\omega'>\omega$. Let $x=(\omega,k,\pi)$ be the type of the first bank, and $x'=(\omega',k,\pi)$ be the type of second bank. Assume, towards a contradiction, that $g(\omega,k,\pi)>g(\omega',k,\pi)$. Then, for all $y\in X_\mathrm{o}$ such that $g(y)\geq g(x)$, we have that:

$$
\begin{aligned}
&g(x)\leq g(y)\Rightarrow g(x')<g(y)\\
&\qquad\qquad\qquad\Rightarrow \gamma(x',y)=\mathbb{I}_{\{x'\in X_\mathrm{o}\}}\Gamma\left(k(x'),k(y)\right)=\mathbb{I}_{\{x\in X_\mathrm{o}\}}\Gamma\left(k(x),k(y)\right)\geq\gamma(x,y),
\end{aligned}
$$

where the first equality follows by optimality, the second equality follows by our maintained assumptions that $k(x')=k(x)$ and $\pi(x')=\pi(x)$, and the third inequality follows because of the trading capacity constraint. Likewise:

$$
g(x)\geq g(y)\Rightarrow\gamma(x,y)=-\mathbb{I}_{\{x\in X_\mathrm{o}\}}\Gamma\left(k(x),k(y)\right)=-\mathbb{I}_{\{x'\in X_\mathrm{o}\}}\Gamma\left(k(x'),k(y)\right)\leq\gamma(x',y),
$$

where the last inequality follows because $\gamma(x',y)$ is bounded by trading capacity. In all cases, $\gamma(x',y)\geq\gamma(x,y)$ for all $y\in X_\mathrm{o}$. The optimality of centralized market trades also implies $\varphi(x')\geq\varphi(x)$. Since $\omega(x')\geq\omega(x)$ by assumption, this implies that $g(x')\geq g(x)$, a contradiction.

Proof that $g(\omega, k, \pi) \leq 1/2$ **for** $\omega \leq 1/2$**.** By symmetry, $g(\omega, k, \pi) = 1 - g(1-\omega, k, \pi)$, and so, $g(\omega, k, \pi) + g(1-\omega, k, \pi) = 1$. Moreover, since we have just shown that post-trade exposures are increasing in endowment, we have $g(\omega, k, \pi) \leq g(1-\omega, k, \pi)$ and the result follows.

Proof that $1/2 \leq g(\omega, k, \pi)$ **for** $\omega > 1/2$**.** By symmetry, $g(\omega, k, \pi) = 1 - g(1-\omega, k, \pi)$. Since $1 - \omega \leq 1/2$, we know from the previous paragraph that $g(1-\omega, k, \pi) \leq 1/2$ and the result follows.

Proof that post-trade exposures are weakly increasing in k **for** $\omega \leq \frac{1}{2}$**.** Let $\mu(\omega, k \,|\, \mathrm{o}) \equiv \int \left[\mathbb{I}_{\{\omega(x)\leq\omega \text{ and } k(x)\leq k \text{ and } \pi(x)\in\{\mathrm{o},\mathrm{oc}\}\}}\, dN(x)\right] / N(X_{\mathrm{o}})$ denote the fraction of OTC market traders with endowment less than ω and capacity less than k. Note that our assumptions $G\left(\omega \,|\, k\right) = 1 - G\left(1-\omega \,|\, k\right)$ and that participation patterns are symmetric in endowment imply $d\mu(\omega, k \,|\, \mathrm{o}) = d\mu(1-\omega, k \,|\, \mathrm{o})$. Now take any $k < k'$. If $g(\omega, k', \pi) = 1/2$, then since we have shown that $g(\omega, k, \pi)$ is bounded by $1/2$, it follows that $g(\omega, k', \pi) \geq g(\omega, k, \pi)$. If $g(\omega, k', \pi) < 1/2$, then we write:

$$
\begin{aligned}
&g(\omega, k', \pi) = \omega + \mathbb{I}_{\{\pi\in\{\mathrm{c},\mathrm{oc}\}\}}\Phi(k') \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})<g(\omega,k',\pi)\}} \left[-\Gamma(k', \tilde{k}) + \Gamma(k', \tilde{k})\right]\, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}) \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})=g(\omega,k',\pi)\}} \left[\gamma(\omega, k', \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) + \Gamma(k', \tilde{k})\right]\, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}) \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})>g(\omega,k',\pi)\}} \left[\Gamma(k', \tilde{k}) + \Gamma(k', \tilde{k})\right]\, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}).
\end{aligned}
$$

The second line considers all meetings with traders with endowment $\tilde{\omega}$ or $1-\tilde{\omega}$ and trading capacity $\tilde{k}$, who trade in the OTC market, and such that $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) < g(\omega, k', \pi)$. By symmetry, half of these traders have endowment $\tilde{\omega}$, and post trade exposure $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) < g(\omega, k', \pi)$, hence the bilateral trade is $-\Gamma(k', \tilde{k})$. The other half have endowment $1-\tilde{\omega}$ and post-trade exposure $g(1-\tilde{\omega}, \tilde{k}, \tilde{\pi}) = 1 - g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > 1/2$, hence the bilateral trade is $\Gamma(k', \tilde{k})$. The third and the fourth line have similar interpretations. Keeping in mind that $\gamma(\tilde{\omega}, \tilde{k}, \tilde{\pi}; \omega, k', \pi) \geq -\Gamma(k', \tilde{k})$, we obtain that:

$$
\begin{aligned}
g(\omega, k', \pi) \geq\, &\omega + \mathbb{I}_{\{\pi\in\{\mathrm{c},\mathrm{oc}\}\}}\Phi(k') \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})>g(\omega,k',\pi)\}}\Gamma(k', \tilde{k})\, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}).
\end{aligned}
\tag{41}
$$

Now assume that for some $k < k'$ we have that $g(\omega, k, \pi) > g(\omega, k', \pi)$. We then obtain:

$$
\begin{aligned}
&g(\omega, k, \pi) = \omega + \mathbb{I}_{\{\pi \in \{\text{c,oc}\}\}} \Phi(k) \\
&+ \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\text{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)\}} \left[-\Gamma(k, \tilde{k}) + \Gamma(k, \tilde{k}) \right] d\mu(\tilde{\omega}, \tilde{k} \,|\, \text{o}) \\
&\quad + \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\text{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > g(\omega, k', \pi)\}} \left[\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) + \gamma(\omega, k, \pi; 1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) \right] \mu(d\tilde{\omega}, d\tilde{k} \,|\, \text{o}).
\end{aligned}
$$

The second line considers all meetings with traders with endowment $\tilde{\omega}$ or $1 - \tilde{\omega}$ and trading capacity $\tilde{k}$, who trade in the OTC market, and such that $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)$. By symmetry, half of these traders have endowment $\tilde{\omega}$, and post trade exposure $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)$. But $g(\omega, k, \pi) > g(\omega, k', \pi)$, hence the bilateral trade is $-\Gamma(k, \tilde{k})$. The other half have endowment $1 - \tilde{\omega}$ and post-trade exposure $g(1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) = 1 - g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > 1/2$, hence the bilateral trade is $\Gamma(k, \tilde{k})$. The third line has a similar interpretation. Simplifying we obtain that:

$$
\begin{aligned}
&g(\omega, k, \pi) = \omega + \mathbb{I}_{\{\pi \in \{\text{c,oc}\}\}} \Phi(k) \\
&+ \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\text{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > g(\omega, k', \pi)\}} \left[\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) + \gamma(\omega, k, \pi; 1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) \right] d\mu(\tilde{\omega}, \tilde{k} \,|\, \text{o}).
\end{aligned} \tag{42}
$$

Now recall that the trading capacity of k' is greater than that of k. Hence, in the centralized market, $\Phi(k) \leq \Phi(k')$. Moreover, in the OTC market, for any given counterparty, $\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq \Gamma(k, \tilde{k}) \leq \Gamma(k', \tilde{k})$. Comparing (41) with (42), one then obtains that $g(\omega, k, \pi) \leq g(\omega, k', \pi)$, which is a contradiction.

Proof that post-trade exposures are weakly decreasing in k for $\omega > \frac{1}{2}$. By symmetry, $g(\omega, k, \pi) = 1 - g(1 - \omega, k, \pi)$. Then, the result follows from the earlier result that $g(1 - \omega, k, \pi)$ is increasing in k.

Proof that post-trade exposures are weakly increasing in centralized market participation for $\omega \leq \frac{1}{2}$. Pick some (ω, k) such that $\omega \leq 1/2$. If $g(\omega, k, \text{oc}) = 1/2$ then we are done. If $g(\omega, k, \text{oc}) < 1/2$ assume, towards a contradiction, that $g(\omega, k, \text{o}) > g(\omega, k, \text{oc})$. Then, for any counterparty in the OTC market, the optimality conditions imply that:

$$
\gamma(\omega, k, \text{o}; \tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq \gamma(\omega, k, \text{oc}; \tilde{\omega}, \tilde{k}, \tilde{\pi}).
$$

Moreover, since $g(\omega, k, \text{oc}) < 1/2$, it follows that $\varphi(\omega, k, \text{oc}) = \Phi(k) \geq 0$. Sine the two bank start with the same endowment, and since the net trade of the oc bank is larger, we obtain that $g(\omega, k, \text{oc}) \geq g(\omega, k, \text{o})$, which is a contradiction.

Proof that post-trade exposures are weakly decreasing in centralized market participation for $\omega > \frac{1}{2}$. This follows by symmetry.

B.8 Proof of Lemma 12

First, let us rewrite the full appropriation surplus as follows

$$
\begin{aligned}
S(x) &= U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - \int U_g\left[g(x')\right]\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&= U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - \int U_g\left[g(x)\right]\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&+ \int \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right)\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&= \overbrace{U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - U_g\left[g(x)\right]\left(g(x) - \mathbb{I}_{\{x\in X_{\mathrm{c}}\}}\varphi\left(x\right) - \omega(x)\right)}^{=Q(x)} \\
&+ \underbrace{\int \left|U_g\left[g(x)\right] - U_g\left[g(x')\right]\right|\Gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o})}_{=B(x)},
\end{aligned}
$$

where $B(x)$ is the bargaining surplus, and $Q(x)$ a new function labeled *quasi-competitive surplus*. Hence, the marginal private value can rewrite as

$$
\mathrm{MPV}(x) = Q(x) + \frac{B(x)}{2}.
$$

The quasi-competitive surplus can rewrite as follows

$$
\begin{aligned}
Q(x) &= U[g(x)] - U[\omega(x)] - U_g\left[\frac{1}{2}\right]\varphi(x) - U_g\left[g(x)\right]\left(g(x) - \varphi(x) - \omega(x)\right) \\
&= \frac{|U_{gg}|}{2}\left((g(x)-\omega(x))^2 + \varphi(x)(1-2g(x))\right) \\
\text{or alternatively,}\quad &= \frac{|U_{gg}|}{2}\left((g(x)-\varphi(x)-\omega(x))^2 + \varphi(x)(1-\varphi(x)-2\omega(x))\right)
\end{aligned}
$$

with $\varphi(x) = 0$ if the bank does not participate in the centralized market, and $g(x) = \varphi(x) + \omega(x)$ if the bank does not participate in the OTC market.

Second, define the net OTC trading volume of a bank $x = (\omega, k, \pi)$

$$
\overline{\gamma}\left(\omega,k,\pi\right) \equiv g\left(\omega,k,\pi\right) - \varphi\left(\omega,k,\pi\right) - \omega = \int \gamma\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right).
$$

Then, if $g(\omega,\pi,k) \neq 1/2$, the net OTC trading is decreasing in ω. Indeed, take $\omega < \omega'$ and fix some (k,π). If $g(\omega,k,\pi) < g(\omega',k,\pi)$, then the result is obvious because the ω bank will buy weakly more, from any counterparty in the OTC market, than the ω' bank. If $g(\omega,k,\pi) = g(\omega',k,\pi)$ then by our maintained assumption $g(\omega, k, \pi) \neq 1/2$ which implies that the centralized market trades of both banks are the same and equal to capacity. The results then follows from the definition of $\bar{\gamma}$.

Proof that the quasi-competitive surplus is V-shaped in ω. The symmetry of the problem makes obvious the symmetry of function $Q(x)$ with respect to $\omega = 1/2$. Indeed take $x = (\omega, k, \pi)$ and $x' = (1-\omega, k, \pi)$. Then $\varphi(x') = -\varphi(x)$, $g(x') = 1 - g(x)$ according to Proposition 1, and therefore $Q(x') = Q(x)$.

Then let us show that for $\omega < 1/2$, the quasi-competitive surplus weakly decreases with ω. If $g(x) = 1/2$, we have $Q(x) = |U_{gg}|\,(1/2-\omega)^2/2$ and as the post trade exposure remains flat when ω increases, the result follows. If $g(x) < 1/2$, then we necessarily have $\varphi(x) = \Phi(x)$ if the bank participates in the centralized, and 0 otherwise, which does not depend on ω. As shown above, we have that the net OTC trading volume, $g(x) - \varphi(x) - \omega$, decreases with ω, and that $1 - 2g(x)$ decreases with ω as $g(x)$ increases with ω. It implies that $Q(x)$ decreases.

Proof that the bargaining surplus is V-shaped in ω. For any $x = (\omega, k, \pi)$, let $y = (1-\omega, k, \pi)$, and for any x' similarly define y'. Keeping in mind that $U_g(g) - U_g(g') = |U_{gg}|\,(g' - g)$ we can write the bargaining surplus of y as:

$$\begin{aligned}
B(y) &= |U_{gg}| \int \big|g(x') - g(y)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}) = |U_{gg}| \int \big|1 - g(y') - g(y)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}) \\
&= |U_{gg}| \int \big|g(y') - g(x)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}).
\end{aligned}$$

As Γ is a function of trading capacities only, we have $\Gamma(y, x') = \Gamma(x, x') = \Gamma(x, y')$. And thanks to symmetry, one can proceed to the change of variable, from x' to y', without changing the integral formula. This establishes that $B(y) = B(x)$, hence the bargaining surplus is symmetric.

Then, let us show that for $x_1 = (\omega_1, k, \pi)$ and $x_2 = (\omega_2, k, \pi)$ with $\omega_1 \leq \omega_2 \leq 1/2$, then $B(x_1) \geq B(x_2)$. One already knows that $g(x_1) \leq g(x_2)$. Denote $\lambda(x') = \Gamma(x_1, x') = \Gamma(x_2, x')$ Assuming that $\pi \in \{\mathrm{o}, \mathrm{oc}\}$, one can decompose the bargaining surplus as follows

$$\begin{aligned}
B(x_1) = |U_{gg}| &\int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \big(g(x_1) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
+ |U_{gg}| &\int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(g(x') - g(x_1)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
+ |U_{gg}| &\int \mathbb{I}_{\{g(x') \geq g(x_2)\}} \big(g(x') - g(x_1)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}$$

and

$$
\begin{aligned}
B(x_2) = |U_{gg}| &\int \mathbb{I}_{\{g(x') \le g(x_1)\}} \big(g(x_2) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(g(x_2) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x') \ge g(x_2)\}} \big(g(x') - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

Then

$$
\begin{aligned}
B(x_1) - B(x_2) = |U_{gg}| &\int \mathbb{I}_{\{g(x') \le g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(2g(x') - g(x_1) - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x') \ge g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

One further decomposes as

$$
\begin{aligned}
B(x_1) - B(x_2) = |U_{gg}| &\int \mathbb{I}_{\{g(x') \le g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(2g(x') - g(x_1) - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \ge g(x') \ge g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{1-g(x_1) > g(x') > 1-g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
&+ |U_{gg}| \int \mathbb{I}_{\{g(x') \ge 1-g(x_1))\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

By a symmetry argument, one has

$$
\begin{aligned}
\int_{X_\mathrm{o}} \mathbb{I}_{\{1-g(x_1) > g(x') > 1-g(x_2)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}) &= \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}), \\
\text{and } \int \mathbb{I}_{\{g(x') \ge 1-g(x_1))\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}) &= \int \mathbb{I}_{\{g(x') \le g(x_1)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

Therefore, one obtains

$$
\begin{aligned}
B(x_1) - B(x_2) = |U_{gg}| & \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(2g(x') - g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \geq g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

and then

$$
\begin{aligned}
B(x_1) - B(x_2) = |U_{gg}| & \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(2g(x') - 2g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \geq g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& \geq 0
\end{aligned}
$$

Proof that the MPV is increasing in k. Consider the case of a bank with $\omega < 1/2$. First, if the bank belongs to an atom so that $g(x) \leq 1/2$ is locally constant with k, then the MPV is obviously increasing. Indeed on the one hand the quasi-competitive surplus increases when the bank participates in the centralized market and buys more at a better price ($U_g\,[1/2]$ vs. $U_g\,[g(x)]$). This effect is captured by the term $\varphi(x)(1 - 2g(x))$ which increases (weakly). On the other hand, the bargaining surplus increases because the per-unit surplus is unchanged (as $g(x)$ is constant), and the gross trading volume increases with k.

Now, let us consider the case when the bank does not belong to an atom, that is $g(x) \leq 1/2$ is locally strictly increasing with k. Define the sets $\underline{X}(g) = \{x', g(x') < g \text{ or } g(x') > 1 - g\}$ and $\overline{X}(g) = \{x', g \leq g(x') \leq 1 - g\}$. First notice that if $x' = (\omega', k', \pi)$ belongs to one set, then the bank with symmetric endowment $\tilde{x}' \equiv (1 - \omega', k', \pi)$ belongs to the same set.

One can see the bank x engages in "pure intermediation" with banks in the set $\underline{X}(g(x))$, in the sense that its net trading volume is zero. Indeed, consider any counterparty x' such that $g(x') < g(x)$: then x sells to banks with $g(x')$ and buys an equal amount from banks with symmetric endowment, whose post-trade exposure is $1 - g(x')$. The combined surpluses obtained from trading with two banks with post-trade exposures of $g(x')$ and $1 - g(x')$ is $|U_{gg}|\,(1 - 2g(x'))\Gamma(x, x')$. The overall *intermediation* surplus is therefore

$$
|U_{gg}| \int_{\underline{X}(g(x))} \left(\frac{1}{2} - g(x')\right) \Gamma(x, x') dN(x' \,|\, \mathrm{o}).
$$

It is increasing in k as the size of the set $\underline{X}(g(x))$ increases, as well as $\Gamma(x, x')$.[6]

[6]The size of the set $\underline{X}(g(x))$ is increasing in k because $g(x)$ is increasing in k as implied by Proposition 1.

One can see the bank x does "pure directional trading" with the banks in the set $\overline{X}(g(x))$, that is x only buys from the banks in the set $\overline{X}(g)$, and so, its net trading volume corresponds to its OTC post-trade exposure, $g(x) - \varphi(x) - \omega$. For $g(x') \geq g(x)$, the combined surplus per-unit obtained from trading with a bank with $g(x')$ and a bank with $1 - g(x')$ is $|U_{gg}|\,(1 - 2g(x))$ which does not depend on x'. The overall *directional* surplus is therefore

$$|U_{gg}| \left(\frac{1}{2} - g(x)\right)(g(x) - \varphi(x) - \omega).$$

If one adds half this directional surplus with the quasi-competitive surplus, one obtains

$$\begin{aligned}
&\frac{|U_{gg}|}{2}\left((g(x)-\omega)^2 + \varphi(x)(1-2g(x)) + \left(\frac{1}{2} - g(x)\right)(g(x) - \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left((g(x)-\omega)^2 + \left(\frac{1}{2} - g(x)\right)(g(x) + \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left(\left(\frac{1}{2}-\omega\right)^2 - \left(\frac{1}{2} - g(x)\right)\left(\frac{1}{2} + g(x) - 2\omega\right) + \left(\frac{1}{2} - g(x)\right)(g(x) + \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left(\left(\frac{1}{2}-\omega\right)^2 - \left(\frac{1}{2} - g(x)\right)\left(\frac{1}{2} - \varphi(x) - \omega\right)\right)
\end{aligned}$$

It is increasing with respect to the terms $g(x)$ and $\varphi(x)$. Therefore it is increasing in k.

B.9 Proof of Lemma 13

Post-trade exposures for π = o. We can write the post-trade exposure of a $\pi = 0$ bank with capacity k as:

$$g_{\mathrm{o}}(k) = \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\left(\frac{1}{2}\int \gamma(k,k') f(k' \,|\, \mathrm{o})\, dk' + \frac{1}{2}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2}\right) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}, \tag{43}$$

where $m_{\mathrm{o}} = F(k^\star)$ and $m_{\mathrm{oc}} = 1 - F(k^\star)$ denote the measure of $\pi = \mathrm{o}$ and $\pi = \mathrm{oc}$ in the OTC market. Taking expectations with respect to $f(k \,|\, \mathrm{o})$, we obtain the formula (28) in the Lemma.

To check that these post-trade exposures are indeed part of an equilibrium, we need to find bilaterally feasible and optimal trades that attain them. To do so we guess that:

$$\gamma(k,k') = \alpha\frac{k'-k}{2}.$$

Plugging this into equation (43), we find that post-trade exposures are constant and equal (28) if and only if:

$$\alpha = 1 + 2\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}}.$$

This bilateral trades are optimal because all $\pi = \text{o}$ have the same exposure, so they are indifferent about any trade. They are bilaterally feasible because $\gamma(k,k') + \gamma(k',k) = 0$ by construction, and as long as:

$$\alpha \frac{k'-k}{2} \leq \frac{k'+k}{2},$$

which holds for all $(k,k') \in [\underline{k}, \overline{k}]$ if and only if it holds for $k' = \bar{k}$ and $k = \underline{k}$. This can be simplifed to:

$$\frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \overline{k} \leq \underline{k} \Leftrightarrow [1 - F(k^\star)]\, \overline{k} \leq \underline{k}.$$

Given our maintained assumption that $\underline{k} > 0$, this condition is always satisfied as long as $k^\star$ is close enough to $\overline{k}$.

Post-trade exposures of $\pi = \text{oc}$, $k \in [k^\star, \overline{k}]$. We guess and verify that the atom property holds for these banks: as long as $k^\star$ is close enough to $\overline{k}$, these banks must have the same post-trade exposure.

We first argue that, for $[k^\star, \bar{k}]$, $g_{\text{oc}}(k) > \bar{g}_{\text{o}}$. Suppose, towards a contradiction, that $g_{\text{oc}}(k) \leq g_{\text{o}}$. Then it follows that

$$g_{\text{oc}}(k) \geq \frac{k+K}{2}.$$

Indeed, the bank will trade to full capacity in the centralized market. Moreover, its net trade in the OTC market must, on net, be positive: indeed, it will buy strictly more from $\omega = 1$, who have strictly larger exposure, than it will sell to $\omega = 0$. But given that $\bar{k} + K > \mathbb{E}[k]$, we have reached a contradiction as long as $k^\star$ is close enough to $\bar{k}$.

Next, we argue that the post trade exposures must be constant. We already know from earlier work that the post-trade exposures must be weakly increasing in k and less than 1/2. Suppose, towards a contradiction, that $g_{\text{oc}}(k^\star \,|\, k^\star) < g_{\text{oc}}(\overline{k} \,|\, k^\star)$. It follows that there must be some point $\hat{k}$ such that it is strictly increasing to the right. Moreover, since the function is continuous, it can be shown that there is some sequence $k_n \downarrow \hat{k}$, $k_n \neq \hat{k}$, such that the function is strictly increasing at each k_n.[7] At each k_n, post-trade exposures must be equal to:

$$\begin{aligned} g_{\text{oc}}(k \,|\, k^\star) = {} & \frac{k+K}{2} + \frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \times 0 \\ & + \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{1}{2} \left(- \int_{k^\star}^{k} \frac{k+k'}{2} f(k' \,|\, \text{oc})\, dk' + \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \text{oc})\, dk' + \int_{k^\star}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \text{oc})\, dk' \right) \end{aligned}$$

[7] Formally one first note that the function has countably many "flat spots", defined as some interval $[k_1, k_2]$, $k_1 < k_2$ such that $g(k)$ is constant over the entire interval: indeed one can associate each flat spot to a distinct rational number." The result intuitively follows because the function must move continuously from one flat spot to the next.

The first term is the bank's trade in the centralized market. The second term is the net with $\pi = \mathrm{o}$ banks. This term is zero because the bank sells to $\pi = \mathrm{o}$, $\omega = 0$ banks, and an buys an equal amount from $\pi = 0$, $\omega = 1$ banks. The last term, on the second line, is the net trade with $\pi = \mathrm{oc}$ banks. Simplifying we obtain:

$$\begin{aligned} g_{\mathrm{oc}}(k) &= \frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \mathrm{oc})\, dk' \\ &= \frac{k+K}{2} + \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k')\, dk' \equiv \hat{g}_{\mathrm{oc}}(k) \end{aligned}$$

where we used that $m_{\mathrm{oc}}/m_{\mathrm{o}} + m_{\mathrm{oc}} = 1 - F(k^\star)$, and $f(k' \,|\, \mathrm{oc}) = f(k')/\left[1 - F(k^\star)\right]$. Hence, at each k_n, post-trade exposures are given by a function $\hat{g}_{\mathrm{oc}}(k)$ that is independent of $k^\star$. Taking derivatives, we obtain:

$$\hat{g}'_{\mathrm{oc}}(k) = \frac{1}{2} - kf(k) + \frac{1}{2}\left(1 - F(k)\right).$$

But if $\overline{k} f(\overline{k}) > 1/2$, then as long as $k^\star$ is close enough to $\overline{k}$, $\hat{g}'_{\mathrm{oc}}(k) < 0$ over $[k^\star, \bar{k}]$, which implies that $g_{\mathrm{oc}}(k_n \,|\, k^\star) > g_{\mathrm{oc}}(k_{n+1} \,|\, k^\star)$, which contradicts our earlier result that $g_{\mathrm{oc}}(k \,|\, k^\star)$ is weakly increasing.

Finally, we calculate the post-trade exposure as usual, by integrating the post-trade exposure formula across k. We find:

$$\begin{aligned} \bar{g}_{\mathrm{oc}} &= \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right] + K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2} \\ &= \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right] + K}{2} + \frac{1}{2}\int_{k^\star}^{\overline{k}} k' f(k')\, dk' = \frac{\bar{k} + K}{2} + o(1), \end{aligned}$$

as $k^\star \to \overline{k}$.

Post-trade exposures of $\pi = \mathrm{oc}$, $k \in [\underline{k}, k^\star]$. We argue that post-trade exposures must be given by the formula (29) in the Lemma. To see this, note first that $g_{\mathrm{oc}}(k) \geq \bar{g}_{\mathrm{o}}$. Otherwise, the net trade of a $\pi = \mathrm{oc}$ would be strictly larger than that of a $\pi = \mathrm{o}$ bank with the same k, a contradiction.

Note as well that, since $g_{\mathrm{oc}}(k)$ is weakly increasing and since $g_{\mathrm{oc}}(\overline{k}) = g_{\mathrm{oc}}$, it follows that $g_{\mathrm{oc}}(k) \leq g_{\mathrm{oc}}$ for all k.

If $g_{\mathrm{oc}}(k) = \bar{g}_{\mathrm{o}}$, then the net trade with $\pi = \mathrm{o}$ banks must be positive: indeed, the bank will buy to full capacity from $\pi = \mathrm{o}$, $\omega = 1$ bank, and sell less from $\pi = \mathrm{o}$, $\omega = 0$ banks. Hence, the net trade must be larger than $\frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}} \frac{k + \mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, that is, larger than the net trade obtained by buying in the centralized market, buying from all $\pi = \mathrm{oc}$ banks, and not trading with $\pi = \mathrm{o}$ banks. Hence, formula (29) holds.

If $g_{\mathrm{oc}}(k) > g_{\mathrm{o}}$ and $g_{\mathrm{oc}}(k) < g_{\mathrm{oc}}$, then one sees by direct calculation that $g_{\mathrm{oc}}(k) = \frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}} \frac{k + \mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, and formula (29) holds.

Finally, if $g_{\mathrm{oc}}(k) = \bar{g}_{\mathrm{oc}}$, then the net trades with $\pi = \mathrm{o}$ bank are zero, implying that the total net trade are less than the one obtained if buying from all $\pi = \mathrm{oc}$ banks, i.e. less than $\frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{k+\mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, and so formula (29) holds.

B.10 Proof of Lemma 14

The result is obvious for $\bar{g}_{\mathrm{o}}$. For $\bar{g}_{\mathrm{oc}}(k)$, note that it can be written:

$$g_{\mathrm{oc}}(k) = \max\left\{\bar{g}_{\mathrm{o}}^{\dagger} + o_1(1), \min\left\{\frac{\overline{k}+K}{2} + o_2(1), \frac{k+K}{2} + o_3(1)\right\}\right\},$$

because $m_{\mathrm{oc}} \to 0$ and $\mathbb{E}\left[k \,|\, \mathrm{oc}\right] \to \overline{k}$ as $k^{\star} \to \overline{k}$. Letting $\overline{o}(1) = \max\{o_1(1), o_2(1), o_3(1)\}$ and $\underline{o}(1) = \min\{o_1(1), o_2(1), o_3(1)\}$, we obtain that, for $k \in [\underline{k}, \overline{k}]$:

$$g_{\mathrm{oc}}^{\dagger}(k) + \underline{o}(1) \leq g_{\mathrm{oc}}(k) \leq g_{\mathrm{oc}}^{\dagger}(k) + \overline{o}(1),$$

and the result follows.

B.11 Proof of Lemma 15

MPV of $\pi = \mathrm{o}$. The marginal private value is calculated as follows:

$$\begin{aligned}
\mathrm{MPV}(k \,|\, \mathrm{o}) = & U(\bar{g}_{\mathrm{o}}) - U(0) - U_g(\bar{g}_{\mathrm{o}})\bar{g}_{\mathrm{o}} \\
& + \frac{1}{2}\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(1-\bar{g}_{\mathrm{o}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} \\
& + \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(\bar{g}_{\mathrm{oc}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2} \\
& + \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(1-\bar{g}_{\mathrm{oc}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}.
\end{aligned}$$

The first line is the fundamental surplus. The second, third, and fourth lines are the frictional surplus with, respectively, $\pi = \mathrm{o}$ and $\omega = 1$ banks, $\pi = \mathrm{oc}$ and $\omega = 0$ banks, $\pi = \mathrm{oc}$ and $\omega = 1$ banks. Keeping in mind that the fundamental surplus is $|U_{gg}|/2g^2$ and collecting terms, we obtain the formula shown in the Lemma.

MPV of $\pi = $ oc. The MPV is:

$$\begin{aligned}
\mathrm{MPV}(k \,|\, \mathrm{oc}) =& U(g_{\mathrm{oc}}(k)) - U(0) - U_g(g_{\mathrm{oc}}(k))g_{\mathrm{oc}}(k) \\
&+ |U_{gg}| \left(\frac{1}{2} - g_{\mathrm{oc}}(k)\right) \frac{k+K}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{2} \left(g_{\mathrm{oc}}(k) - \bar{g}_{\mathrm{o}}\right) \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{2} \left(1 - \bar{g}_{\mathrm{o}} - g_{\mathrm{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{2} \left(\bar{g}_{\mathrm{oc}} - g_{\mathrm{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{2} \left(1 - \bar{g}_{\mathrm{oc}} - g_{\mathrm{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}.
\end{aligned}$$

The first line is the fundamental surplus. The second line is the surplus (which as Semih pointed out should not be called frictional) that banks earn because, in the centralized market, they buy at price $U_g(1/2)$ and not at their marginal value. The third, fourth, fifth and sixth lines are the frictional surplus with, respectively, $\pi = $ o and $\omega = 0$ banks, $\pi = $ o and $\omega = 1$ banks, $\pi = $ oc and $\omega = 0$ banks, $\pi = $ oc and $\omega = 1$ banks. For these terms, keep in mind that $\bar{g}_{\mathrm{o}} \leq g_{\mathrm{oc}}(k) \leq \bar{g}_{\mathrm{oc}}$. Simplifying, we obtain the formula in the Lemma.

B.12 Proof of Lemma 16

The result is obvious for the MPV. It is also clear that

$$\frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{o}) = \frac{d\mathrm{MPV}^{\dagger}}{dk}(k \,|\, \mathrm{o}) + o(1),$$

so we only have to make sure that the result holds for $\mathrm{MPV}(k \,|\, \mathrm{oc})$. We have:

$$\begin{aligned}
\frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{oc}) =& |U_{gg}| \frac{dg_{\mathrm{oc}}}{dk} \left(g_{\mathrm{oc}}(k) - \frac{k+K}{2}\right) + \frac{|U_{gg}|}{4}\left(1 - 2g_{\mathrm{oc}}(k)\right) + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8}\left(1 - \bar{g}_{\mathrm{o}}\right) \\
&+ \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8}\left(1 - g_{\mathrm{oc}}(k)\right) - \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \frac{dg_{\mathrm{oc}}}{dk}(k)
\end{aligned}$$

where dg_{oc}/dk denotes either the left or right derivative of dg_{oc}/dk. The first term is positive because, wheneve the dg_{oc}/dk is not zero, then $g_{\mathrm{oc}}(k) \geq (k+K)/2$. The last term goes to zero uniformly in k. The result then follows by taking limit of the other terms.

B.13 Proof of Lemma 17

Consider a $(0, k)$-type bank that participates to the OTC markets. This bank will enter the market with a possibly non-zero position $\varphi(k)$. More precisely $\varphi(k) = k/2$ if the bank also participates in the centralized market, and $\varphi(k) = 0$ otherwise. The post-trade exposure $g_{\mathrm{o/oc}}(k)$ of such a bank is as follows:

$$
\begin{aligned}
&(i) \text{ if } \varphi(k)+\overbrace{\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right]+\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\right]}^{=\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\right]}<\bar{g}_{\mathrm{o}},\\
&\qquad \text{then } g_{\mathrm{o/oc}}(k)=\varphi(k)+\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\right];\\
&(ii) \text{ if } \varphi(k)+\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2})\middle| k'\in X_{\mathrm{oc}}\right]\leq\bar{g}_{\mathrm{o}}\leq\varphi(k)+\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\right], \text{ then}\\
&\qquad g_{\mathrm{o/oc}}(k)=\bar{g}_{\mathrm{o}};\\
&(iii) \text{ if } \bar{g}_{\mathrm{o}}<\varphi(k)+\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{oc}}\right]<\frac{1}{2}, \text{ then}\\
&\qquad g_{\mathrm{o/oc}}(k)=\varphi(k)+\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{oc}}\right];\\
&(iv) \text{ if } \frac{1}{2}\leq\varphi(k)+\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{oc}}\right], \text{ then}\\
&\qquad g_{\mathrm{o/oc}}(k)=\frac{1}{2};
\end{aligned}
$$

First, we can verify that a bank with $k=0$ and $\omega=0$ has a post-trade exposure that is at least equal to $\bar{g}_{\mathrm{o}}$ so that case (i) does not apply. Indeed when such a bank participates exclusively in the OTC market, if its post trade exposure was lower than $\bar{g}_{\mathrm{o}}$ it would be equal to $\mathbb{E}\left[k'/2\middle| k'\in X_{\mathrm{o}}\right]$. But we can check that $\mathbb{E}\left[k'/2\middle| k'\in X_{\mathrm{o}}\right]>\bar{g}_{\mathrm{o}}$. Indeed

$$
\mathbb{E}\left[\frac{k'}{2}\middle| k'\in X_{\mathrm{o}}\right]=\frac{1}{2}\left(\frac{m_{\mathrm{o}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\frac{k^{\star}}{2}+\frac{m_{\mathrm{oc}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\frac{1+k^{\star\star}}{2}\right).
$$

Hence, post-trade exposures are given by cases (ii), (iii), and (iv).

B.14 Proof of Lemma 18

The MPV of exclusive entry in the centralized exchange is obvious. In the proof of Lemma 1, we have shown that the full appropriation surplus decomposes as the sum of a quasi-competitive surplus and half the bargaining surplus. The quasi-competitive surplus of the previous bank on the OTC market can be computed as

$$
\begin{aligned}
Q(0,k,\mathrm{o/oc}) &= U(g_{\mathrm{o/oc}}(k))-U_g(1/2)\varphi(k)-U_g(g_{\mathrm{o/oc}}(k))(g_{\mathrm{o/oc}}(k)-\varphi(k))\\
&=\frac{|U_{gg}|}{2}\left[g_{\mathrm{o/oc}}(k)^2+\varphi(k)(1-2g_{\mathrm{o/oc}}(k))\right]
\end{aligned}
$$

To compute the bargaining surplus of such on the OTC market, we distinguish the different cases (*ii*), (*iii*), and (*iv*), introduced in the proof of Lemma 17. In case (*ii*)

$$
\begin{aligned}
B(0,k,\mathrm{o/oc}) &= |U_{gg}|\frac{m_\mathrm{o}/2}{m_\mathrm{o}+m_\mathrm{oc}}\left(1-2\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right] \\
&+|U_{gg}|\frac{m_\mathrm{oc}}{m_\mathrm{o}+m_\mathrm{oc}}\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{oc}\right] \\
&=|U_{gg}|\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\right].
\end{aligned}
$$

In case (*iii*), we have

$$
\begin{aligned}
B(0,k,\mathrm{o/oc}) &= |U_{gg}|\frac{m_\mathrm{o}/2}{m_\mathrm{o}+m_\mathrm{oc}}\left(g_\mathrm{o/oc}(k)-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right] \\
&+|U_{gg}|\frac{m_\mathrm{o}/2}{m_\mathrm{o}+m_\mathrm{oc}}\left(1-\bar{g}_\mathrm{o}-g_\mathrm{o/oc}(k)\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right] \\
&+|U_{gg}|\frac{m_\mathrm{oc}}{m_\mathrm{o}+m_\mathrm{oc}}\left(\frac{1}{2}-g_\mathrm{o/oc}(k)\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{oc}\right] \\
&=|U_{gg}|\frac{m_\mathrm{o}}{m_\mathrm{o}+m_\mathrm{oc}}\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right] \\
&+|U_{gg}|\left(\frac{1}{2}-g_\mathrm{o/oc}(k)\right)\left(g_\mathrm{o/oc}(k)-\varphi(k)\right)
\end{aligned}
$$

In case (*iv*), we have

$$
B(0,k,\mathrm{o/oc}) = |U_{gg}|\frac{m_\mathrm{o}}{m_\mathrm{o}+m_\mathrm{oc}}\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right].
$$

Overall, the marginal private value of such a bank is

$$
\mathrm{MPV}(0,k,\mathrm{o/oc}) = Q(0,k,\mathrm{o/oc})+\frac{1}{2}B(0,k,\mathrm{o/oc}).
$$

In case (*ii*), we have

$$
\begin{aligned}
\mathrm{MPV}(0,k,\mathrm{o/oc}) &= \frac{|U_{gg}|}{2}\left[\bar{g}_\mathrm{o}^2+\varphi(k)(1-2\bar{g}_\mathrm{o})+\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\right]\right] \\
&=\frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\left(\frac{1}{2}+\bar{g}_\mathrm{o}\right)+\varphi(k)(1-2\bar{g}_\mathrm{o})+\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\right]\right] \\
&=\frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\left(\frac{1}{2}-\varphi(k)+\overbrace{\bar{g}_\mathrm{o}-\varphi(k)-\frac{m_\mathrm{oc}}{m_\mathrm{o}+m_\mathrm{oc}}\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{oc}\right]}^{>0}\right)\right] \\
&+\frac{|U_{gg}|}{2}\frac{m_\mathrm{o}}{m_\mathrm{o}+m_\mathrm{oc}}\left(\frac{1}{2}-\bar{g}_\mathrm{o}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_\mathrm{o}\backslash X_\mathrm{oc}\right].
\end{aligned}
$$

As in case (ii), we have $g_{\mathrm{o/oc}}(k) = \bar{g}_{\mathrm{o}}$ and $\varphi(k) = 0$ for exclusive participants, or $\varphi(k) = k/2$ for non-exclusive participants, the results hold. In case (iii), we have

$$\begin{aligned}
\mathrm{MPV}(0, k, \mathrm{o/oc}) &= \frac{|U_{gg}|}{2}\left[g_{\mathrm{o/oc}}(k)^2 + \varphi(k)(1 - 2g_{\mathrm{o/oc}}(k)) + \left(\frac{1}{2} - g_{\mathrm{o/oc}}(k)\right)\left(g_{\mathrm{o/oc}}(k) - \varphi(k)\right)\right] \\
&\quad + \frac{|U_{gg}|}{2}\left[\frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k, k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right] \\
&= \frac{|U_{gg}|}{2}\left[\frac{1}{4} - \left(\frac{1}{2} - g_{\mathrm{o/oc}}(k)\right)\left(\frac{1}{2} - \varphi(k)\right) + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k, k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right]
\end{aligned}$$

As in case (ii), we have $g_{\mathrm{o/oc}}(k) = \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{oc}}\right] \geq \bar{g}_{\mathrm{o}}$ and $\varphi(k) = 0$ for exclusive participants, or $\varphi(k) = k/2$ for non-exclusive participants, the results hold. Finally in case (iv), we have

$$\mathrm{MPV}(0, k, \mathrm{o/oc}) = \frac{|U_{gg}|}{2}\left[\frac{1}{4} + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k, k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right],$$

in which case $g_{\mathrm{o/oc}}(k) = 1/2$ and the result obviously holds.

C Flat and increasing spots and the atom property

C.1 Some elementary results about flat and increasing spots

In many examples, we work with post-trade exposures that are weakly increasing and continuous functions:

$$\begin{aligned}
g\ &: X \to Y \\
&\quad x \mapsto g(x)
\end{aligned}$$

where X and $Y \equiv g(X)$ are compact intervals. The interval X indexes the heterogeneity between banks, either endowment or capacity depending on the example.

Our objective in this section is to derive some elementary mathematical properties of the flat spots of g. Formally, we define the set value function:

$$\begin{aligned}
F\ &: Y \to \mathcal{P}(X) \\
&\quad y \mapsto g^{-1}(y),
\end{aligned}$$

which maps any value in the range of g into its inverse image. We first show that the sets $F(y)$ are all compact intervals.

Lemma 23. *For all $y \in Y$, $F(y) = [\inf F(y), \sup F(y)]$.*

Proof. Because g is continuous, it follows that $F(y)$ is closed, hence $\{\inf F(y), \sup F(y)\} \subseteq F(y)$ or, equivalently:

$$g(\inf F(y)) = g(\sup F(y)) = y.$$

Since g is increasing it follows that all $g(x) = y$ for all $x \in [\inf F(y), \sup F(y)]$. $\square$

Lemma 24. *The set of flat spot of g, i.e. the set of y such that $\inf F(y) < \sup F(y)$ is countable.*

Proof. Indeed, one can associate each flat spot with a unique rational number in the interval $(\inf F(y), \sup F(y))$. $\square$

A corollary of this result is:

Lemma 25. *Take any $y < y'$ in Y. Then there exists $y'' \in (y, y')$ such that $F(y'')$ is a singleton.*

Proof. Indeed, if $y < y'$, the set (y, y') is not countable and so must contain some y'' with the desired property. $\square$

Another corollary works as follows. Consider any point x at the upper-end of a flat spot, i.e. $x = \sup F(g(x))$. Then for all $x' > x$, $g(x') > g(x)$. If we let $y = g(x)$ and $y' = g(x')$, the previous Lemma shows that there is a $x'' \in (x, x')$ such that $F(g(x''))$ is a singleton. The sequential version of this result is:

Lemma 26. *Let $x \in [\inf X, \sup X)$ such that $x = \sup F(g(x))$ and $g(x) < \sup Y$. Then there is a strictly decreasing sequence $x_n \to x$ such that $F(g(x_n))$ is a singleton.*

C.2 The atom property with exclusive participation

In this section, we explore the generality of the atom property with exclusive participation. We consider a symmetric economy with two endowment types, $\omega \in \{0, 1\}$ and heterogeneous capacities, $k \in [\underline{k}, \overline{k}]$. We only consider the banks who participate in the OTC market.

C.2.1 Some results about atoms

We let $g(k)$ denote the post-trade exposure of a bank with capacity k and endowment $\omega = 0$. We know from Lemma 11 that $g(k)$ is weakly increasing and continuous, with $g(\overline{k}) \leq 1/2$.

Consider any point k such that $g(k)$ is strictly increasing. Then, to the right of k, $g(k) < g(k') \leq 1/2$. Hence, a bank with capacity k buys up to capacity from all banks with $k' > k$ and $\omega = 0$, and from all banks with $\omega = 1$. Conversely, it will sell up to capacity to all banks with $k' < k$ and $\omega = 0$.

Hence, the sales to $k' < k$ and $\omega = 0$ cancel out with the purchase from $k' < k$ and $\omega = 1$, implying that the post-trade exposure of the bank can be written:

$$g(k) = \int_{k}^{\bar{k}} \Gamma(k, k')\, dN(k' \,|\, \mathrm{o}). \tag{44}$$

We can extend this formula to flat spots: as in Section C.1, we use the set function F to describe the flat spots of the function g: namely, for any y in the range of g, we let $F(y) \equiv g^{-1}(y)$. Take any k such that $\sup F(g(k)) < \overline{k}$. Then since g increases strictly to the right of $\sup F(g(k))$, it must be that $g(k) < 1/2$. Now from Lemma 26 it follows that there is a strictly decreasing sequence $k_n \to k$ such that $F(g(k_n)) = \{k_n\}$ and hence g is strictly increasing at k_n. Therefore, formula (44) holds at each k_n and, by continuity, it holds at $\sup F(g(k))$. This leads to:

Lemma 27. *If* $\sup F(g(k)) < \overline{k}$, *then*

$$g(k) = \int_{\sup F(g(k))}^{\bar{k}} \Gamma(\sup F(g(k)), k')\, dN(k' \,|\, \mathrm{o}).$$

Likewise, if $\inf F(g(k)) > \underline{k}$,

$$g(k) = \int_{\inf F(g(k))}^{\bar{k}} \Gamma(\inf F(g(k)), k')\, dN(k' \,|\, \mathrm{o}). \tag{45}$$

Note that a corollary of these formula is

Lemma 28. *There must be a flat spot at the top, i.e.* $F(g(\bar{k})) < \bar{k}$.

Indeed, if it were not the case, then (45) would imply that $g(\bar{k}) = 0$, which is impossible.

Let $\hat{k} = \inf F(g(\bar{k}))$ denote the lower bound of the flat spot $F(g(\bar{k}))$. The aggregate trades of $k > \hat{k}$, $\omega = 0$ banks with $k < \hat{k}$ banks net out to zero: indeed, they sell at full capacity, to all banks with $k < \hat{k}$ and $\omega = 0$, and buy, at full capacity again, from all banks with $k < \hat{k}$ and $\omega = 1$. The aggregate trades of $k > \hat{k}$, $\omega = 0$ banks with $k > \hat{k}$, $\omega = 1$ banks are less than

$$\frac{1}{2} \int_{\hat{k}}^{\overline{k}} \int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \,|\, o)\, dN(k'' \,|\, o),$$

with an equality if $g(\overline{k}) < 1/2$. Taken together, we obtain:

$$g(\overline{k}) = \min \left\{ \frac{1}{2}, \frac{1}{2} \int_{\hat{k}}^{\overline{k}} \int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \,|\, k \geq \hat{k}, \mathrm{o})\, dN(k'' \,|\, \mathrm{o}) \right\}.$$

Using (45) we obtain that, if $\hat{k} > \underline{k}$, then

$$\int_{\hat{k}}^{\bar{k}} \Gamma(\hat{k}, k')\, dN(k' \mid k \geq \hat{k}, \mathrm{o}) = \min\left\{\frac{1}{2}, \frac{1}{2}\int_{\hat{k}}^{\overline{k}}\int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \mid k \geq \hat{k}, \mathrm{o})\, dN(k'' \mid k \geq \hat{k}, \mathrm{o})\right\}.$$

This delivers a sufficient condition for $\hat{k} = \underline{k}$, that is, for the atom property to hold:

Lemma 29. *A sufficient condition for the atom property to hold is that:*

$$\mathbb{E}\left[\Gamma(\hat{k}, k) \,\middle|\, k \geq \hat{k}, \mathrm{o}\right] > \frac{1}{2}\mathbb{E}\left[\Gamma(k, k') \,\middle|\, (k, k') \geq \hat{k}, \mathrm{o}\right]. \tag{46}$$

for all $\hat{k} \in [\underline{k}, \overline{k}]$.

C.2.2 Examples

Max capacity constraint: $\Gamma(k, k') = \max\{k, k'\}$**.** We can verify that the max specification satisfies the sufficient condition. Indeed, consider any distribution N on the support $[\hat{k}, \overline{k}]$. The left-hand side of (46) is equal to

$$\int_{\hat{k}}^{\overline{k}} k\, dN(k).$$

To calculate the right-hand side, notice that the CDF of the maximum is $N(k)^2$. Hence, the right hand side writes:

$$\int_{\hat{k}}^{\overline{k}} k\, N(k) dN(k).$$

Since $N(k) < 1$ on $[\hat{k}, \overline{k})$, the result follows.

Separable capacity constraint: $\Gamma(k, k') = f(k) + f(k')$**.** Then the left-hand side of (46) is

$$f(\hat{k}) + \mathbb{E}\left[f(k)\right].$$

The right-hand side is

$$\mathbb{E}\left[f(k)\right],$$

which is greater than the left-hand side.

Submodular capacity constraint. This seems to encompass both the max and the separable cases. The definition of submodularity, according to Bach (2019) is that, for all (k_1, k_1') and (k_2, k_2'):

$$\Gamma(k_1, k_1') + \Gamma(k_2, k_2') \geq \Gamma(\max\{k_1, k_2\}, \max\{k_1', k_2'\}) + \Gamma(\min\{k_1, k_2\}, \min\{k_1', k_2'\}).$$

If Γ is twice differentiable, it is equivalent to $\partial^2\Gamma/\partial k \partial k' \leq 0$.

To see why submodularity implies the atom property, let $(k_1, k_1') = (k', \hat{k})$ and $(k_2, k_2') = (\hat{k}, k'')$, for $(k', k'') \in [\hat{k}, \overline{k}]^2$. The submodularity condition implies that:

$$\Gamma(k', \hat{k}) + \Gamma(\hat{k}, k'') \geq \Gamma(k', k'') + \Gamma(\hat{k}, \hat{k}) > \Gamma(k', k''),$$

if $\Gamma(\hat{k}, \hat{k}) > 0$. Using symmetry we have $\Gamma(k', \hat{k}) = \Gamma(\hat{k}, k')$. Taking expectations on both sides with respect to $N(k \,|\, \mathrm{o}, k \geq \hat{k})$ leads to the desired inequality.

D Exclusive participation under the "max" specification

In this section, we consider a setup very similar to the one introduced in Section 4.1. We assume that capacities are heterogeneous across banks: they are distributed according to a generic continuous density $f(k)$ over the compact interval $[0, \infty)$. We also assume that risk-sharing needs are the same for all banks: namely, the endowment distribution has just two points, $\omega = 0$, or $\omega = 1$, with equal probability, and is independent from capacities. However in this section we work under the "max" specification, that is

$$\Gamma(k, k') = \max\{k, k'\}.$$

According to Section C results, we know that the atom property holds. That is, banks with endowments $\omega = 0$, or $\omega = 1$, that participate in the OTC market end up with a post trade exposure respectively equal to g or $1 - g$, where

$$g = \frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right],$$

where X_{o} is given subset of (symmetric) banks, indexed by their trading capacities, that participate in the OTC market.

Consider a bank with endowment $\omega = 0$, and capacity k. Its quasi-competitive surplus is equal to

$$Q(0, k, \mathrm{o}) = U(g) - U(0) - U_g[g]g = \frac{|U_{gg}|}{2} g^2 \equiv Q(g).$$

Such a bank makes zero bargaining surplus with other banks with similar endowments and makes a $|U_{gg}|(1-2g)$ per unit surplus with $\omega = 1$ banks. Its bargaining surplus is therefore

$$B(0,k,\mathrm{o}) = \frac{1}{2}|U_{gg}|(1-2g)\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right].$$

By symmetry we have $Q(1,k,\mathrm{o}) = Q(0,k,\mathrm{o}) = Q(g)$ and $B(1,k,\mathrm{o}) = B(0,k,\mathrm{o})$, and we can compute the marginal private value of entry in the OTC market of a bank with capacity k as

$$\mathrm{MPV}(k,\mathrm{o}) = \frac{|U_{gg}|}{2}\left(g^2 + \left(\frac{1}{2} - g\right)\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right]\right).$$

One can also compute the marginal private value of entry in the centralized market of bank with capacity k as

$$\mathrm{MPV}(k,\mathrm{c}) = \frac{|U_{gg}|}{2}\min\left\{\max\{k,K\}, \frac{1}{2}\right\}\left(1 - \min\left\{\max\{k,K\}, \frac{1}{2}\right\}\right).$$

Lemma 30. *For any participation pattern,* $\mathrm{MPV}(k,\mathrm{o})$ *is convex and increasing on* $[0,\infty)$*, and* $\mathrm{MPV}(k,\mathrm{c})$ *is flat on* $[0,K]$*, and then increasing and concave on* $[K,\infty)$*. Therefore* $\mathrm{MPV}(k,\mathrm{o})$ *and* $\mathrm{MPV}(k,\mathrm{c})$ *cross at most twice on* $[K,\infty)$*.*

Proof. Denote $F(k|\mathrm{o})$ the conditional distribution of capacities in the OTC market. Then,

$$\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right] = kF(k|\mathrm{o}) + \int_k^{\infty} k' dF(k'|\mathrm{o}).$$

Taking the derivative of the former leads to

$$F(k|\mathrm{o}) + kF'(k|\mathrm{o}) - kF'(k|\mathrm{o}) = F(k|\mathrm{o}),$$

which is increasing in k. $\mathrm{MPV}(k,\mathrm{o})$ is therefore convex and obviously increasing.

$\mathrm{MPV}(k,\mathrm{c})$ is S-shaped and increasing because it is flat on $[0,K]$, increasing and concave on $[K,1/2]$, and flat again on $[1/2,\infty)$. □

Lemma 31. *Consider a general distribution of capacities and a general exclusive participation pattern. Then,* $\mathrm{MPV}(k,\mathrm{o}) - \mathrm{MPV}(k,\mathrm{c})$ *is negative for some* k*, in particular for* k_g *such that* $\mathrm{MPV}(k_g,\mathrm{o}) = \frac{|U_{gg}|}{2}g(1-g)$*. Moreover* $B(k_g,\mathrm{o}) = \bar{B}$*.*

Proof. Consider k_g such that $\mathbb{E}\left[\max\{k_g,k'\} \mid k' \in X_{\mathrm{o}}\right] = 2g$. Then, we have

$$\mathrm{MPV}(k_g,\mathrm{o}) = \frac{|U_{gg}|}{2}g(1-g).$$

If $k_g > g$, then we necessarily have $\mathrm{MPV}(k_g,\mathrm{o}) \leq \mathrm{MPV}(k_g,\mathrm{c})$.

Proving that $k_g \geq g$ is then equivalent to show that

$$\begin{aligned}
&\mathbb{E}\left[\max\left\{g, k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq 2g \\
\Leftrightarrow\ &\mathbb{E}\left[\max\left\{\frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq \mathbb{E}\left[\max\left\{k', k'''\right\} \mid (k', k''') \in X_{\mathrm{o}}^2\right]
\end{aligned}$$

We know that

$$\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right] \leq 2\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right],$$

thus we have

$$\mathbb{E}\left[\max\left\{\frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq \mathbb{E}\left[\max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right]$$

For a given k''', as $\max\left\{k', k'''\right\}$ is a convex function of k', thanks to Jensen's Inequality we have

$$\mathbb{E}\left[\max\left\{k', k'''\right\} \mid k' \in X_{\mathrm{o}}\right] \geq \max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\},$$

and then by taking the expectation w.r.t k'''

$$\mathbb{E}\left[\max\left\{k', k'''\right\} \mid (k', k''') \in X_{\mathrm{o}}^2\right] \geq \mathbb{E}\left[\max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right].$$

Hence the result.

In addition one can see that taking the average MPV gives on the one hand $Q(g) + \bar{B}/2$, and on the other hand

$$\frac{|U_{gg}|}{2}\left(g^2 + \left(\frac{1}{2} - g\right)2g\right) = \frac{|U_{gg}|}{2}g(1-g) = \mathrm{MPV}(k_g, \mathrm{o}) = Q(g) + \frac{B(k_g, \mathrm{o})}{2}.$$

$\square$

Proposition 7. *When $C(\mathrm{o}) = C(\mathrm{c}) = 0$, if an equilibrium exists then the set of banks that participate in the OTC market, X_{o}, and the set of banks that participate in the centralized market, X_{c}, are defined as*

$$X_{\mathrm{o}} = [k^{\star\star\star}, k^{\star\star}] \cup [k^{\star}, \infty), \ \textit{and}\ X_{\mathrm{c}} = [0, k^{\star\star\star}) \cup (k^{\star\star}, k^{\star}),$$

where thresholds, $k^{\star\star\star}$, $k^{\star\star}$ and $k^{\star}$, are pined down by equilibrium conditions,

(a) $k^{\star\star\star} = \sup\{k \in [0,K], \mathrm{MPV}(k,\mathrm{o}) \leq \mathrm{MPV}(K,\mathrm{c})\}$, *or* $k^{\star\star\star} = 0$ *if* $\mathrm{MPV}(0,\mathrm{o}) > \mathrm{MPV}(K,\mathrm{c})$

(b) $\mathrm{MPV}(k^{\star\star},\mathrm{o}) = \mathrm{MPV}(k^{\star\star},\mathrm{c})$ *and* $\frac{1}{2} \geq k^{\star\star} \geq K$,

(c) $\mathrm{MPV}(k^{\star},\mathrm{o}) = \mathrm{MPV}(k^{\star},\mathrm{c})$ *and* $k^{\star} \geq k^{\star\star}$.

Moreover $B(k^{\star\star},\mathrm{o}) \leq \bar{B} \leq B(k^{\star},\mathrm{o})$.[8]

Proof. According to Lemma 31, the set of OTC banks cannot be convex, because the bank k_g does not participate in the OTC market but still above and below capacities of banks participating in the OTC market. So according to Lemma 30, there must be at least two crossing points $k^{\star\star}$ and $k^{\star} > k^{\star\star}$ between $\mathrm{MPV}(k,\mathrm{o})$ and $\mathrm{MPV}(k,\mathrm{c})$, so that $\mathrm{MPV}(k,\mathrm{c}) > \mathrm{MPV}(k,\mathrm{o})$ on $(k^{\star\star}, k^{\star})$, and so that $k_g \in (k^{\star\star}, k^{\star})$. These two crossing points must necessarily be on the concave branch of $\mathrm{MPV}(k,\mathrm{c})$, that is $[K,\infty)$, and $k^{\star\star}$ must be on its strictly increasing part, that is $[K, 1/2]$.

On top of those necessary conditions, it is possible that $\mathrm{MPV}(k,\mathrm{o})$ and $\mathrm{MPV}(k,\mathrm{c})$ coincide again on the segment $[0,K]$. Define $k^{\star\star\star}$ such that $\mathrm{MPV}(k,\mathrm{o}) > \mathrm{MPV}(k,\mathrm{c})$ on $(k^{\star\star\star}, K]$, and $\mathrm{MPV}(k,\mathrm{o}) \leq \mathrm{MPV}(k,\mathrm{c})$ otherwise. Notice that on $[0,k^{\star\star\star})$, $\mathrm{MPV}(k,\mathrm{o})$ must be flat as the lowest capacity in the OTC market would be $k^{\star\star\star}$. Therefore we would have $\mathrm{MPV}(k,\mathrm{o}) = \mathrm{MPV}(k,\mathrm{c}) = \mathrm{MPV}(K,\mathrm{c})$ on $[0,k^{\star\star\star})$. Banks on this segment would therefore be indifferent between participating in the OTC or the centralized. However in equilibrium they would all participate in the centralized market.

In order to restore strict preference in participation, one could introduce an arbitrarily small mass of $k = 0$-banks to participate in the OTC market. In such case banks in $[0,k^{\star\star\star})$ would strictly prefer to participate in the centralized market. We use this variation when we solve for the equilibrium existence, so that $k^{\star\star\star}$ can be pinned down as a locally unique indifference threshold.

$B(k^{\star\star},\mathrm{o}) \leq \bar{B} \leq B(k^{\star},\mathrm{o})$ is obtained according to Lemma 31. □

Lemma 32. *In equilibrium,* $k^{\star} \geq 1/2$.

Proof. Denote with $|U_{gg}|a/2$ the slope of the affine portion of $\mathrm{MPV}(.,\mathrm{o})$ between on $(k^{\star\star}, k^{\star})$, that is

$$a = \left(\frac{1}{2} - g\right) \frac{F(k^{\star\star}) - F(k^{\star\star\star})}{1 - F(k^{\star}) + F(k^{\star\star}) - F(k^{\star\star\star})} \leq \frac{1}{2} - g$$

To prove this result, it is sufficient to show that

$$h(k^{\star\star}) = k^{\star\star}(1-k^{\star\star}) + \left(\frac{1}{2} - g\right)\left(\frac{1}{2} - k^{\star\star}\right) < \frac{1}{4}.$$

[8] In this section we "inversely" rank thresholds as $k^{\star} > k^{\star\star} > k^{\star\star\star}$ because the third one may not exist, i.e $k^{\star\star\star} = 0$.

which ensures that $\text{MPV}(1/2, \text{o}) = \text{MPV}(k^{\star\star}, \text{o/c}) + |U_{gg}|a(1/2 - k^{\star\star})/2 < |U_{gg}|/8$. One can compute the derivative of h as

$$h'(k) = 1 - 2k - \frac{1}{2} + g = \frac{1}{2} + g - 2k > 0, \ \forall k \in [0, g].$$

Remember that Lemma 31 shows that there is a $k_g \geq g$ such that $\text{MPV}(k_g, \text{o}) = \frac{|U_{gg}|}{2} g(1-g) = \text{MPV}(g, \text{c}) < \text{MPV}(k_g, \text{c})$. Then we necessarily have that $k_g \in (k^{\star\star}, k^{\star})$. Since $\text{MPV}(k_g, \text{o}) \geq \text{MPV}(k^{\star\star}, \text{o}) = \text{MPV}(k^{\star\star}, \text{c})$, we have $k^{\star\star} \leq g$. Then we obtain that

$$h(k^{\star\star}) \leq h(g) = g(1-g) + \left(\frac{1}{2} - g\right)^2 = \frac{1}{4}.$$

$\square$

Example. We consider a case in which capacities are uniformly distributed over the segment $[0, 1]$, that is $f(k) = \mathbb{I}_{\{k \in [0,1]\}}$. In addition we assume that $C(\text{o}) = C(\text{c}) = 0$ and $C(\text{oc}) = \infty$ in order to induce exclusive participation. Furthermore, we consider the case in which K is low enough so that $k^{\star\star\star} = 0$.

Consider the three variables, $\{g, k^{\star}, k^{\star\star}\}$, equations system:

$$\begin{aligned}
g &= \frac{2 + 2(k^{\star\star})^3 - 3k^{\star} + (k^{\star})^3 + 3k^{\star\star}\left(1 - (k^{\star})^2\right)}{6\left(1 - k^{\star} + k^{\star\star}\right)^2} \\
\frac{1-2g}{2} \frac{k^{\star\star}(k^{\star} - k^{\star\star})}{1 - k^{\star} + k^{\star\star}} &= \frac{1}{4} - k^{\star\star}(1 - k^{\star\star}) \\
g^2 + \frac{1-2g}{4} \frac{2(k^{\star\star})^2 + 1 - (k^{\star})^2}{1 - k^{\star} + k^{\star\star}} &= k^{\star\star}(1 - k^{\star\star})
\end{aligned}$$

such that $k^{\star} > k^{\star\star}$, $k^{\star\star} > 0$, and $k^{\star\star} < 1/2$. Our aim is to show numerically (with *Mathematica*) that the former solution exists and is unique, and that if

$$0 \leq K \leq \frac{1 - \sqrt{1 - 4\left(g^2 + \frac{1-2g}{4} \frac{(k^{\star\star})^2 + 1 - (k^{\star})^2}{1 - k^{\star} + k^{\star\star}}\right)}}{2},$$

then there exists a unique equilibrium in which banks with $k \in [0, k^{\star\star}] \cup [k^{\star}, 1]$ enter exclusively the OTC market, and vice-versa, banks with $k \in (k^{\star\star}, k^{\star})$ enter exclusively the centralized market. Conditional on $\omega = 0$ and $\pi = \text{o}$, all banks obtain the same post-trade exposure g. Symmetrically, all banks obtain the same post-trade exposure $1 - g$ conditional on $\omega = 1$ and $\pi = \text{o}$. Bilateral trades depend on capacity k. Small-k banks trade like customers: they tend to buy from all banks. Large-k banks trade like dealers: they tend to buy from $\omega = 1$ banks and sell to $\omega = 0$ banks.

Conditional on $\pi = \mathrm{c}$, post-trade exposures may or may not depend on k: $g(0,k,\mathrm{c}) = \min\left\{k, \frac{1}{2}\right\}$ and $g(1,k,\mathrm{c}) = 1 - \min\left\{k, \frac{1}{2}\right\}$.

Proof. Using the fact that k is distributed uniformly on $[0,1]$,

$$\mathbb{E}\left[\max\{k,k'\} \,|\, k' \in X_{\mathrm{o}}\right] = \begin{cases} \frac{k^2+(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} & \text{if } k \le k^{\star\star}, \\ \frac{2kk^{\star\star}+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} & \text{if } k^{\star\star} < k < k^\star, \\ \frac{1+k^2-2k(k^\star-k^{\star\star})}{2(1-k^\star+k^{\star\star})} & \text{if } k^\star \le k. \end{cases}$$

Using this, and that $g = \mathbb{E}\left[\max\{k',k''\} \,|\, (k',k'') \in X_{\mathrm{o}}^2\right]/2$, one obtains the first equation in the system, where $g \equiv g(0,k,\mathrm{o})$. This is one of the equations that connect together g, $k^\star$, and $k^{\star\star}$ that are all determined in equilibrium. The remaining two equations will come from the indifference conditions of the two marginal banks with the capacities $k^\star$ and $k^{\star\star}$. Recall that an OTC bank's MPV is equal to its full appropriation surplus minus half of its bargaining surplus. With atom property, it is equal to

$$\mathrm{MPV}(k,\mathrm{o}) = \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4}(1-2g)\,\mathbb{E}\left[\max\{k,k'\} \,|\, k' \in X_{\mathrm{o}}\right].$$

In the centralized market, the MPV is equal to

$$\mathrm{MPV}(k,\mathrm{c}) = \frac{|U_{gg}|}{2} \min\left\{\max\{k,K\}, \frac{1}{2}\right\}\left(1 - \min\left\{\max\{k,K\}, \frac{1}{2}\right\}\right).$$

We focus on the case of sufficiently small K such that no bank benefits from the centralized market capacity. Thus, $\max\{k^\star, K\} = k^\star$ and $\max\{k^{\star\star}, K\} = k^{\star\star}$.

Using the uniformity of k, the indifference conditions are

$$\begin{aligned} \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4}(1-2g)\frac{2k^\star k^{\star\star}+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= \frac{|U_{gg}|}{2}\min\left\{k^\star, \frac{1}{2}\right\}\left(1-\min\left\{k^\star, \frac{1}{2}\right\}\right) \\ \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4}(1-2g)\frac{2(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= \frac{|U_{gg}|}{2}\min\left\{k^{\star\star}, \frac{1}{2}\right\}\left(1-\min\left\{k^{\star\star}, \frac{1}{2}\right\}\right). \end{aligned}$$

Because $k^{\star\star} < k^\star$, the LHS of the first equation is larger than the LHS of the second equation. Thus, these two equations can be satisfied simultaneously only if $k^{\star\star} < 1/2$. Using this fact, subtracting the second equation from the first one, and after cancellations,

$$\begin{aligned} \frac{1-2g}{2}\frac{k^{\star\star}(k^\star-k^{\star\star})}{2(1-k^\star+k^{\star\star})} &= \min\left\{k^\star, \frac{1}{2}\right\}\left(1-\min\left\{k^\star, \frac{1}{2}\right\}\right) - k^{\star\star}(1-k^{\star\star}) \\ g^2 + \frac{1-2g}{2}\frac{2(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= k^{\star\star}(1-k^{\star\star}). \end{aligned}$$

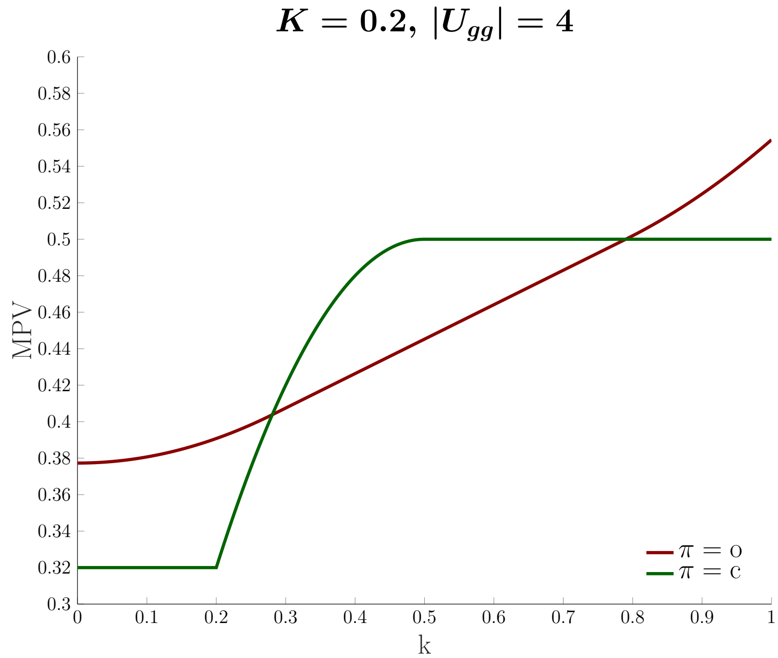

Figure 9: The MPV of centralized (green curve) and OTC market participation (red curve, as functions of capacity, k.

These two equations combined with the first equation stated in the Proposition pin down g, $k^\star$, and $k^{\star\star}$. *Mathematica* finds that under the restrictions $k^\star > k^{\star\star}$, $k^{\star\star} > 0$, and $k^{\star\star} < 1/2$, the solution is unique: $g = 0.334903$, $k^\star = 0.789951$, and $k^{\star\star} = 0.280706$. Because $k^\star > 1/2$ in the unique solution, we can get rid of the min operator in the equation system for simplicity.

What remains is to verify that the MPVs implied by g, $k^\star$, and $k^{\star\star}$ are consistent with the conjectured participation patterns, as illustrated in Figure 9:

$$MPV(k, \text{o}) - MPV(k, \text{c}) \geq 0 \text{ for } k \in [0, k^{\star\star}] \cup [k^\star, 1]$$
$$MPV(k, \text{c}) - MPV(k, \text{o}) \geq 0 \text{ for } k \in (k^{\star\star}, k^\star)$$

For $k \in [0, k^{\star\star}]$,

$$MPV(k, \text{o}) - MPV(k, \text{c}) = \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4} (1 - 2g) \frac{k^2 + (k^{\star\star})^2 + 1 - (k^\star)^2}{2(1 - k^\star + k^{\star\star})} - \frac{|U_{gg}|}{2} \max\{k, K\}(1 - \max\{k, K\}).$$

We should guarantee that this difference is weakly positive for all $k \in [0, k^{\star\star}]$. It is easy to notice that it is increasing in $k < K$ and decreasing in $k \geq K$. It is weakly positive at $k = k^{\star\star}$ by the indifference condition. Thus, it suffices to impose that it is weakly positive at $k = 0$:

$$g^2 + \frac{1-2g}{2} \frac{(k^{\star\star})^2 + 1 - (k^\star)^2}{2(1-k^\star+k^{\star\star})} - K(1-K) \geq 0,$$

which holds when

$$K \leq \frac{1 - \sqrt{1 - 4\left(g^2 + \frac{1-2g}{4} \frac{(k^{\star\star})^2+1-(k^\star)^2}{1-k^\star+k^{\star\star}}\right)}}{2} = 0.2524.$$

This is the first necessary condition stated in the Proposition. For $k \in [k^\star, 1]$,

$$\mathrm{MPV}(k, \mathrm{o}) - \mathrm{MPV}(k, \mathrm{c}) = \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4} (1-2g) \frac{1 + k^2 - 2k(k^\star - k^{\star\star})}{2(1-k^\star+k^{\star\star})} - \frac{|U_{gg}|}{8}.$$

This MPV difference is strictly increasing in $k \in [k^\star, 1]$, and so, the indifference condition guarantees that it is weakly positive for all $k \in [k^\star, 1]$. For $k \in (k^{\star\star}, k^\star)$,

$$\begin{aligned} &\mathrm{MPV}(k, \mathrm{c}) - MPV(k, \mathrm{o}) \\ &\quad = \frac{|U_{gg}|}{2} \min\left\{k, \frac{1}{2}\right\}\left(1 - \min\left\{k, \frac{1}{2}\right\}\right) - \frac{|U_{gg}|}{2} g^2 - \frac{|U_{gg}|}{4}(1-2g) \frac{2kk^{\star\star} + 1 - (k^\star)^2}{2(1-k^\star+k^{\star\star})}. \end{aligned}$$

This MPV difference is hump-shaped in $k \in (k^{\star\star}, k^\star)$, and so, the two indifference conditions guarantee that it is weakly positive for all $k \in (k^{\star\star}, k^\star)$. □