What is Missing in Asset-Pricing Factor Models?*

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Abstract

Our objective is to develop a methodology to price the cross section of asset returns. Despite the hundreds of *systematic* risk factors considered in the literature ("factor zoo"), there is still a sizable pricing error. We show that what is missing in asset-pricing factor models is not systematic risk factors but compensation for asset-specific risk. We use this insight to construct a stochastic discount factor (SDF) that prices the cross section of stock returns *exactly*, and therefore, resolves the factor zoo. Empirically, we demonstrate that more than half of the variation in this SDF is explained by an aggregate measure of asset-specific risk that is correlated with strategies reflecting market frictions and behavioral biases.

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1 Introduction

A major challenge in asset pricing is to explain the cross section of asset returns. To achieve this goal, the literature has searched for alternative systematic risk factors, leading to a factor zoo (Cochrane, 2011). However, virtually all examined factor models featuring different factors from this zoo imply sizable pricing errors in asset returns, called alpha. In our work, we develop a methodology that resolves the factor zoo by demonstrating that the alpha is compensation not just for sources of systematic risk that are missing in a candidate factor model but also for asset-specific risk. This insight allows us to price the cross-section of stock returns exactly.

The foundation for our analysis is the Arbitrage Pricing Theory (APT) of Ross (1976, 1977), with a more formal treatment in Chamberlain (1983) and Chamberlain and Rothschild (1983). The APT postulates a linear factor model for the deviations of asset returns from their means and allows for asset-specific components in expected returns. These asset-specific components are unrelated to common factors, also known as sources of systematic risk, and satisfy a no-arbitrage restriction. We exploit the APT to evaluate and correct misspecification in popular candidate linear factor models that are used by financial economists to price the cross-section of assets.

A candidate factor model may be misspecified for at least two reasons: it may omit (i) systematic sources of risk and (ii) asset-specific components in expected returns that should be present, in line, for example, with the empirical findings of Daniel and Titman (1997). To identify what is missing in a candidate asset-pricing factor model, we evaluate misspecification in the candidate model through the lens of the SDF. Our methodology shows how to address both sources of misspecification to construct an *admissible* SDF, namely an SDF that prices correctly a given set of assets.

The first main insight of our analysis is that the asset-specific components of expected returns should be interpreted not as pricing errors but as compensation for asset-specific risk represented by asset-return shocks that are orthogonal to common risk factors. This insight is a major departure from the conventional wisdom that financial markets compensate investors only for exposure to systematic sources of risk. The standard view in finance, that one should diversify away all asset-specific risk, holds only because the return for bearing asset-specific risk is zero in popular factor models. However, if asset-specific risk earns a non-zero reward then, instead of diversifying this risk, an investor will optimally adjust her portfolio to reap this compensation. To provide microfoundations for this idea, we demonstrate that the SDF implied by an equilibrium model such as Merton (1987), where investors are aware of only a subset of available securities, is one where asset-specific risk is priced even when the number of assets is large. Furthermore, we show that the equilibrium asset returns and SDF in Merton (1987) coincide with the ones implied by the APT.

To demonstrate empirical support for our insight, we examine what is missing in popular candidate factor models. To set the stage, we start our analysis with a model without any observed risk factors, so that the candidate SDF is just the inverse of the risk-free rate. After examine this model, we consider: (i) a model with the market factor, as suggested by the capital asset pricing model (CAPM) of Sharpe (1964), (ii) a model with the consumption-mimicking portfolio, as implied by the consumption capital asset pricing model (C-CAPM) of Breeden (1979), and (iii) the three-factor model (FF3) of Fama and French (1993). Using data for 202 portfolios of monthly stock returns, for each of these models we identify and characterize the required correction term to move from the candidate to admissible SDF.

The correction term consists of two terms: a measure of systematic risk and an aggregate measure of asset-specific risk. The measure of systematic risk is a function of risk factors that are common in asset returns but missing in the candidate factor model. The aggregate measure of asset-specific risk is a portfolio of asset returns orthogonal to all common risk factors, including those missing in the candidate model. Despite starting from different candidate factor models, once we include the correction term for each of the candidate models listed above, we obtain admissible SDFs that are almost perfectly correlated.

A key finding that emerges from our empirical analysis is that the asset-specific components in expected returns are non-zero and represent reward for asset-specific risk. This risk plays a major role in pricing the cross-section of asset returns, despite the risk premia associated with each individual asset-specific shock being small on average. Specifically, the aggregate measure of asset-specific risk explains more than half of the variation in the admissible SDF. The quantitative role of the aggregate measure of asset-specific risk suggests that candidate factor models with different proxies for only common risk factors will never lead to an admissible SDF. Instead, the only way to obtain an admissible SDF is to recognize that asset-specific risk is priced and we show how to do this. This insight resolves the factor zoo.

We explore the nature of the aggregate measure of asset-specific risk by examining the composition of the portfolio that represents it. We find that small stocks contribute substantially to the aggregate measure of asset-specific risk, followed by stocks with extremely low or high values of net issuances. We also measure the correlations of the aggregate measure of asset-specific risk with the returns on 472 trading strategies examined in the literature. The strategies related to behavioral biases and financial frictions exhibit the largest correlations: the R^2 in the regressions of these strategies' returns on the aggregate measure of asset-specific risk exceeds 30 percent. The most prominent examples of these strategies are those based on 5-year analyst growth forecast (La Porta, 1996), betting-against-beta factor (Frazzini and Pedersen, 2014), long-term behavioral mispricing factor (Daniel, Hirshleifer, and Sun, 2020), the ratio of book debt to market equity (Bhandari, 1988), and implied equity duration (Dechow, Sloan, and Soliman, 2004).

These high correlations imply that the large expected excess returns on these trading strategies reflect sizable compensation for asset-specific risk. We find, for instance, that the risk premium for asset-specific risk associated with the strategy of La Porta (1996) is 8.20% per annum. There are also 27 other strategies with an absolute value of risk premium earned for exposure to asset-specific risk greater than 5% per annum. Thus, one can construct trading strategies with high expected excess returns but zero exposure to systematic risk.

When traditional asset pricing models, such as the CAPM, fail to explain a cross-section of stock returns, the response has typically been to search for additional systematic factors. For instance, Value (Fama and French, 2015), Investment (Hou, Xue, and Zhang, 2015), and Momentum (Jegadeesh and Titman, 1993) have attracted attention as successful explanatory factors. We find that such factors are successful, at least partly, because they correlate more highly with the aggregate measure of asset-specific risk than with the systematic component of the SDF. These factors appear to be weak (Lettau and Pelger, 2020; Giglio, Xiu, and Zhang, 2021); that is, they affect only a small number of asset returns out of the cross-section of stock returns considered.

The implicit accounting for asset-specific risk in factor models via factors such as Value, Investment, or Momentum, suggests that perhaps adding an arbitrary combination of tradable factors to the candidate factor model can capture the aggregate-measure of asset specific risk. However, it is not the case. The aggregate measure of asset-specific risk is a weak factor in the cross-section of asset returns. Estimating reliably the risk premium associated with a weak factor is not feasible statistically. As a result, one cannot recover the admissible SDF simply by using the traditional two-pass approach, which we show formally. This constitutes the second main insight of our analysis.

Turning next to the analysis of the systematic component of the SDF, we find that 95% of its variation is explained by the market factor. At first glance, this finding may seem surprising in light of the extensive literature documenting the poor performance of the market factor in explaining the cross-sectional differences in stock returns.[1] However, even though the market factor does not explain the cross-section of expected stock returns, we find, similar to Clarke (2020), among others, that it plays an important role in determining the *level* of stock returns. In fact, the systematic component of the SDF accounts for only 44% of the variation in the admissible SDF, so that the overall contribution of the market factor to the total variation in the admissible SDF is only 42% (= 95% $\times$ 44%). We also find that the cross-sectional differences in expected stock returns are spanned by their exposures to nineteen traded factors, in addition to the aggregate measure of asset-specific risk. Among these traded factors, the Size factor of Fama and French (1993) is the most prominent, explaining 89% of the residual systematic variation in the SDF after adjusting for the market factor.

Our work is related to several strands of the literature. First, because we correct a given candidate factor model through the lens of a misspecified SDF, we contribute to the literature that studies misspecification of the SDF and develops methods to estimate the minimum-variance SDF, that is, the projection of the SDF on asset returns. The idea of misspecification of the SDF motivates the work of Hansen and Jagannathan (1991), in which they provide the minimum-variance bound that must be satisfied by any admissible SDF. Luttmer (1996) extends their analysis to economies with proportional transaction costs, short-sale constraints, and margin requirements. Korsaye, Quaini, and Trojani (2021) advance this literature substantially by allowing for more general convex pricing constraints, which then allows them to nest in a single unifying framework several asset-pricing approaches not covered by the SDF literature. In contrast to these papers, our objective is not to identify a bound on the SDF; instead, we provide the exact correction required for a proposed SDF to become admissible and we highlight the role of asset-specific risk in this correction.

A number of papers develop a non-parametric approach to correct misspecified SDF models. Hansen and Jagannathan (1997) provide the smallest additive nonparametric ad-

[1]See Fama and French (2004) for a review of this literature.

justment (in a least-squares sense) required to make a given SDF admissible. Almeida and Garcia (2012) provide an additive correction term that is based on minimum-discrepancy projections. Ghosh, Julliard, and Taylor (2017) provide a multiplicative correction using a Kullback-Leibler entropy-minimization approach. In order to get as close as possible to the true SDF, ideally one would like to estimate a projection of the SDF on a large number of assets. However, it is challenging to use these non-parametric approaches when the number of basis assets is large relative to the number of observations. Furthermore, these methodologies are silent about the role of common versus idiosyncratic shocks in the admissible SDF. In contrast, our approach, because it is founded on the APT, is designed to handle a large number of assets. Moreover, it allows us to correct both for omitted common factors in a candidate asset pricing model and to explicitly allow for priced asset-specific risk.

To handle a large number of assets, Kozak, Nagel, and Santosh (2020), Lettau and Pelger (2020), and Giglio and Xiu (2021) develop methods based on Principal Component Analysis (PCA) for estimating the SDF, identifying factors that price the cross-section of expected returns, and estimating risk premia in the presence of model misspecification, respectively. We complement this literature by developing a methodology that allows us to study the importance of asset-specific risk when correcting a candidate SDF for misspecification.

From the perspective of financial economics, our work is related to the literature on the role of idiosyncratic risk in asset pricing. Early contributions include Douglas (1969), Fama and MacBeth (1973), Levy (1978), and Lehmann (1990), with more recent work by, among others, Goyal and Santa-Clara (2003), Herskovic, Kelly, Lustig, and Van Nieuwerburgh (2016), and Mehra, Wahal, and Xie (2021). These papers study the effect on risk premia of the volatilities of asset residuals that are obtained after risk-adjusting asset returns for the market or other commonly used risk factors. We complement this literature by establishing, through the lens of the SDF, a formal risk-return relation between asset-specific risk, represented by direct shocks to systematic risk-adjusted returns rather than their volatilities, and expected asset-specific returns. Our factorization of the SDF into systematic and asset-specific components is in line with a body of theoretical work, including Levy (1978), Merton (1987), and Malkiel and Xu (2006).

Our findings also emphasize the arguments of MacKinlay (1995) and Daniel and Titman (1997) about the importance of characteristics for understanding risk premia and inability of a factor model to explain a cross section of stock returns, but with two important differences. First, our model ensures asymptotic no-arbitrage. Second, we demonstrate that in our framework the asset-specific components in expected returns represent compensation for asset-specific risk.

The rest of the paper is organized as follows. Section 2 contains our main theoretical results underpinning the methodology that we develop to construct an admissible SDF in the presence of model misspecification. Section 3 provides details of how to estimate a correction term for a candidate SDF in order to obtain the admissible SDF. Section 4 describes the data that we use to illustrate our methodology. Section 5 presents the empirical findings from applying our methodology to this data. Section 6 provides an example of the SDF in an equilibrium model in which asset-specific risk is priced. We collect our conclusions in Section 7. Proofs for all the propositions along with additional results are collected in appendices.

2 From a Candidate to an Admissible SDF

The starting point for our analysis is a candidate asset pricing linear factor model, represented by a vector of K^{can} observable risk factors f_{t+1}^{can}. Potentially, any candidate model suffers from misspecification. We evaluate misspecification by evaluating the candidate model's pricing performance measured by the covariance of the stochastic discount factor and asset returns. Because asset returns are given, effectively we evaluate misspecification in the SDF implied by the candidate factor model.

Our work is founded on the classical APT (Ross, 1976) to correct candidate misspecified SDFs.[2] Effectively, the APT is our working assumption about the true data-generating process for asset returns. There are several advantages of using the APT. First, the APT is a flexible model that does not take a stand on what constitutes a pricing factor. Second, the APT is a no-arbitrage model. The absence of arbitrage opportunities implies the existence of an SDF.

In this section, we first review the classical APT. Next, we explain through the lens of the APT how we correct a misspecified SDF. We derive the closed-form expression for an admissible SDF and identify what the candidate SDF is missing in order to be an admissible SDF. Finally, we address three empirical challenges that we face when estimating the admissible SDF: (i) nonnegativity of the SDF, (ii) econometric feasibility of the SDF, and (iii) weak factors (e.g., Lettau and Pelger (2020)) in the candidate factor model.

[2] Chamberlain (1983), and Chamberlain and Rothschild (1983) provide a formal analysis of the APT.

2.1 The Arbitrage Pricing Theory (APT)

Let the N-dimensional vector $R_{t+1} = (R_{1,t+1}, R_{2,t+1}, \dots, R_{N,t+1})'$ denotes the vector of gross returns of the N risky assets. Let R_{ft} be the gross return on the risk-free asset.[3] Let f_{t+1} be the $K \times 1$ vector of common risk factors, with $K < N$ and a $K \times K$ positive definite covariance matrix $V_f = \text{var}(f_{t+1}) > 0$.

The classical APT builds on two assumptions.

Assumption 1 (Linear Factor Model)**.** *The vector R_{t+1} of gross asset returns satisfies*

$$R_{t+1} = \mathbb{E}(R_{t+1}) + \beta\big(f_{t+1} - \mathbb{E}(f_{t+1})\big) + e_{t+1},$$

where $\mathbb{E}$ denotes an operator of mathematical expectation, $\beta = (\beta_1, \beta_2, \dots, \beta_N)'$ is the $N \times K$ full-rank matrix of loadings of asset returns on the common factors, the vector of asset-specific errors e_{t+1} has zero mean and the $N \times N$ positive definite covariance matrix $V_e = \text{var}(e_{t+1}) > 0$ with uniformly bounded eigenvalues. The asset-specific shocks e_{t+1} are uncorrelated with the K common factors f_{t+1}, implying that the covariance matrix of returns is $V_R = \text{var}(R_{t+1}) = \beta V_f \beta' + V_e$.

Assumption 2 (Asymptotic No Arbitrage)**.** *There are no arbitrage opportunities for a sufficiently large number of assets N; that is, there is no sequence of portfolios containing a large number of risky assets with the weights $w = (w_1, w_2, \dots, w_N)'$, for which:*

$$\text{var}(R'_{t+1}w) \to 0 \qquad \textit{and} \qquad (\mathbb{E}(R_{t+1}) - R_{ft}1_N)'w \geq \delta > 0 \qquad \textit{as} \quad N \to \infty,$$

where δ denotes an arbitrary positive scalar and 1_N denotes the $N \times 1$ vector of ones.[4]

Assumptions 1 and 2 imply that a model of asset excess returns is

$$R_{t+1} - R_{ft}1_N = a + \beta\lambda + \beta(f_{t+1} - \mathbb{E}(f_{t+1})) + e_{t+1}, \tag{1}$$

where the expected *excess* returns $\mathbb{E}(R_{t+1} - R_{ft}1_N) = a + \beta\lambda$ contain two components: a and $\beta\lambda$. The $K \times 1$ vector of risk premia λ represents the compensations for one unit of asset exposures to the factors f_{t+1}. Ingersoll (1984) derives the precise condition for λ to exist and shows that $\lambda = \lim_{N\to\infty} \left(\beta' V_e^{-1}\beta\right)^{-1} \beta' V_e^{-1}(\mathbb{E}(R_{t+1}) - R_f 1_N)$. The $N \times 1$ vector $a = (\mathbb{E}(R_{t+1}) - R_{ft}1_N) - \beta\lambda$, which is typically referred to as the vector of pricing errors, satisfies the following no-arbitrage restriction

$$a' V_e^{-1} a \leq \delta^*_{\text{apt}} < \infty, \tag{2}$$

[3] If a risk-free asset does not exist, one can use instead the return on the minimum-variance portfolio or the return on the zero-beta portfolio.

[4] Throughout the paper, we use δ to denote an arbitrary positive scalar, not always taking the same value.

as shown in Ross (1976), Huberman (1982), Chamberlain and Rothschild (1983), and Ingersoll (1984), where δ^*_{apt} is some arbitrary positive scalar. The main insight of our methodology, which we will describe shortly, is a different interpretation of the vector a. We show that, when viewed through the lens of an SDF, the vector a should be interpreted as compensation for asset-specific risk e_{t+1}, rather than the vector of pricing errors.

Relative to the APT, any standard candidate factor model with K^{can} observable risk factors f^{can}_{t+1} suffers from potentially two sources of misspecification. First, the candidate model may omit systematic risk factors, that is, $K^{\text{can}} < K$. Second, the candidate model typically does not allow for the asset-specific components of expected excess returns represented by the vector a in equation (1). In the data, some components of this vector a may be non-zero, and therefore the candidate factor model that assumes that $a = 0_N$, where 0_N denotes the $N \times 1$ vector of zeros, is misspecified. A popular example of a candidate model is the market model, in which $a^{\text{can}} = 0_N$ and the vector f^{can}_{t+1} includes only the market factor, with $K^{\text{can}} = 1$.

To understand the implications of model misspecification, consider a candidate model with $a^{\text{can}} = 0_N$ and $K^{\text{can}} < K$. Let β^{can} denote the $N \times K^{\text{can}}$ matrix of loadings of asset returns on the candidate factors and λ^{can} the $K^{\text{can}} \times 1$ vector of risk premia for unit exposures to these factors. The candidate factor model implies

$$R_{t+1} - R_{ft} = \alpha + \beta^{\text{can}}\lambda^{\text{can}} + \beta^{\text{can}}(f^{\text{can}}_{t+1} - \mathbb{E}[f^{\text{can}}_{t+1}]) + \varepsilon_{t+1}, \tag{3}$$

where $\alpha = (\mathbb{E}(R_{t+1}) - R_{ft}1_N) - \beta^{\text{can}}\lambda^{\text{can}}$ captures the residual variation in the expected excess returns left unexplained by compensation for asset exposures to common risk factors, and ε_{t+1} with covariance matrix V_ε captures the residual variation in asset returns that is not explained by the set of candidate factors f^{can}_{t+1}.

The proposition below shows that, just as the vector a in the classical APT satisfies the no-arbitrage restriction given in expression (2), the vector of pricing errors α in the candidate model satisfies a similar no-arbitrage restriction even if the candidate model omits some systematic risk factors.

Proposition 1 (APT in the presence of model misspecification)**.** *Suppose that the vector of asset returns R_{t+1} satisfies Assumptions 1 and 2. Given a candidate factor model with K^{can} factors, suppose the first $K^{\text{mis}} = K - K^{\text{can}}$ eigenvalues of the covariance matrix V_ε are unbounded when $N \to \infty$, the remaining eigenvalues are uniformly bounded, and the smallest eigenvalue is strictly positive. Then, the pricing error α in the misspecified can-*

didate model satisfies the following no-arbitrage restriction, for some constant $\tilde{\delta}_{apt}$ possibly different from δ^*_{apt},

$$\alpha' {V_\varepsilon}^{-1} \alpha \leq \tilde{\delta}_{\text{apt}}, \tag{4}$$

where, by no arbitrage, there exist an $N \times 1$ vector a, $K^{\text{mis}} \times 1$ vector λ^{mis}, and an $N \times K^{\text{mis}}$ matrix β^{mis} such that

$$\alpha = \beta^{\text{mis}} \lambda^{\text{mis}} + a \qquad \textit{and} \qquad V_\varepsilon = \text{var}(\varepsilon_{t+1}) = \beta^{\text{mis}} \beta^{\text{mis}\prime} + V_e. \tag{5}$$

We see from (5) that α is the sum of the vector a and the compensation for the missing systematic risk, $\beta^{\text{mis}}\lambda^{\text{mis}}$, with β^{mis} being the matrix of loadings of asset returns on the missing systematic risk factors and λ^{mis} being the vector representing the prices of the missing systematic risk factors. The covariance matrix V_ε is the variance of asset returns because of their exposure to the systematic risk factors f^{mis}_{t+1} that are missing in the candidate model and the asset-specific shocks e_{t+1}.

Without loss of generality, given that f^{mis}_{t+1} are latent factors we can rotate them freely and normalize them in an arbitrary way, we assume that the factors f^{mis}_{t+1} are mutually orthogonal to the factors f^{can}_{t+1}, and f^{mis}_{t+1} has a $K^{\text{mis}} \times K^{\text{mis}}$ identity covariance matrix $V_{f^{\text{mis}}} = \text{var}(f^{\text{mis}}_{t+1}) = I_{K^{\text{mis}} \times K^{\text{mis}}}$. Therefore, the covariance matrix of asset returns can be represented as $V_R = \text{var}(R_{t+1}) = \beta^{\text{can}} V_{f^{\text{can}}} \beta^{\text{can}\prime} + \beta^{\text{mis}} \beta^{\text{mis}\prime} + V_e$, where $V^{\text{can}}_f = \text{var}(f^{\text{can}}_{t+1})$. Because it is difficult to empirically distinguish an exact from approximate factor structure of asset returns, we assume that the covariance matrix of shocks e_{t+1}, $V_e = \text{var}(e_{t+1})$, is diagonal.

2.2 The SDF in the Presence of Misspecification

Below, we derive the closed-form expression for the SDF in the presence of misspecification. This result complements Chamberlain (1983), who shows existence and continuity of the "cost functional" (i.e., the SDF) under the classical APT, without providing its closed-form representation. More importantly, we establish a class of admissible SDFs in the presence of misspecification in a candidate factor model for asset returns. To this end, we identify and construct the correction terms that transform the misspecified SDF implied by the candidate model to an admissible SDF.

2.2.1 A Candidate Model that is Correctly Specified

To set the stage for our analysis, we start by considering the case in which the *true* APT model given in expression (1) has K factors and $a = 0_N$ with the APT restriction (2) guaranteed to hold. Suppose the candidate model coincides with the true APT model, that is, $K^{\text{can}} = K$, and thus there is no misspecification. Denote $M_{t+1}^{\beta,\text{can}}$ the SDF implied by the candidate model

$$M_{t+1}^{\beta,\text{can}} = \frac{1}{R_{ft}} - \frac{1}{R_{ft}} \lambda' V_f^{-1} (f_{t+1} - \lambda).$$

In the absence of misspecification, $M_{t+1}^{\beta,\text{can}}$ prices all the risky assets and the risk-free asset exactly:

$$\mathbb{E}[M_{t+1}^{\beta,\text{can}}(R_{it+1} - R_{ft})] = 0 \quad \text{for each asset } i \quad \text{and} \quad \mathbb{E}[M_{t+1}^{\beta,\text{can}} R_{ft}] = 1;$$

that is, $M_{t+1}^{\beta,\text{can}}$ coincides with the linear admissible SDF M_{t+1} under the APT

$$M_{t+1}^{\beta,\text{can}} = M_{t+1}.$$

Because the admissible SDF M_{t+1} is spanned only by the K-dimensional vector f_{t+1}, the asset-specific shocks e_{it+1} are not priced:

$$\text{cov}[M_{t+1} e_{it+1}] = 0 \quad \text{for each asset } i.$$

2.2.2 A Candidate Model with Omitted Factors

Consider again the case, in which the true APT model given in expression (1) has K factors and $a = 0_N$. However, the candidate model now has $K^{\text{can}} < K$ factors, that is, there are $K^{\text{mis}} = K - K^{\text{can}}$ omitted systematic risk factors in the candidate model. In this case, under the APT the admissible linear SDF M_{t+1} can be written as the sum of two components:

$$M_{t+1} = M_{t+1}^{\beta,\text{can}} + M_{t+1}^{\beta,\text{mis}}.$$

The component $M_{t+1}^{\beta,\text{can}}$, which is based on the candidate factor model, is

$$M_{t+1}^{\beta,\text{can}} = \frac{1}{R_{ft}} - \frac{1}{R_{ft}} \lambda^{\text{can}\prime} V_{f^{\text{can}}}^{-1} (f_{t+1}^{\text{can}} - \lambda^{\text{can}}),$$

and the correction term $M_{t+1}^{\beta,\text{mis}}$, which accounts for the missing systematic risk factors, is

$$M_{t+1}^{\beta,\text{mis}} = -\frac{1}{R_{ft}} \lambda^{\text{mis}\prime} V_{f^{\text{mis}}}^{-1} (f_{t+1}^{\text{mis}} - \lambda^{\text{mis}}]),$$

where, as discussed earlier, we normalize $V_{f^{\text{mis}}} = I_{K^{\text{mis}} \times K^{\text{mis}}}$.[5]

From the perspective of the candidate model, the vector α from equation (3) is the vector of pricing errors

$$\mathbb{E}[M_{t+1}^{\beta,\text{can}}(R_{t+1} - R_{ft})] \times R_f = \alpha.$$

However, from the point of view of the true model given in expression (1), the vector α represents the compensation for assets' exposures to f_{t+1}^{mis}, because it satisfies

$$-\operatorname{cov}(M_{t+1}^{\beta,\text{mis}}, (R_{t+1} - R_{ft})) \times R_f = \alpha = \beta^{\text{mis}} \lambda^{\text{mis}}.$$

The *admissible* SDF, just as in the previous case of the correctly specified candidate model, is fully spanned by the K risk factors f_{t+1}, and therefore, asset-specific shocks are not priced:

$$\operatorname{cov}[M_{t+1} e_{it+1}] = 0 \quad \text{for each asset } i.$$

2.2.3 A Candidate Model with Omitted Factors and Asset-Specific Components in Expected Excess Returns

We now consider the general case, in which the true APT model given in expression (1) has K factors and a non-zero vector $a \neq 0_N$ that satisfies the no-arbitrage restriction (2), while the candidate model features *two* sources of misspecification. First, the candidate model includes $K^{\text{can}} < K$ risk factors thereby omitting $K^{\text{mis}} = K - K^{\text{can}}$ risk factors. Second, the candidate model omits the non-zero vector a by assuming that $a^{\text{can}} = 0_N$. The admissible linear SDF is now the sum of $M_{t+1}^{\beta,\text{can}}$, which is the SDF based on the candidate factor model, and a correction term labeled the alpha-SDF M_{t+1}^{α}. The correction term has two components: $M_{t+1}^{\beta,\text{mis}}$ and M_{t+1}^{a}. The first component accounts for the omitted systematic risk factors f_{t+1}^{mis}, whereas the second component accounts for omitted asset-specific components a in expected returns.

Proposition 2 (SDF: Linear Case)**.** *Under Assumptions 1 and 2, there exists an admissible SDF M_{t+1},*

$$M_{t+1} = M_{t+1}^{\beta,\text{can}} + M_{t+1}^{\alpha} = M_{t+1}^{\beta,\text{can}} + \underbrace{(M_{t+1}^{a} + M_{t+1}^{\beta,\text{mis}})}_{=M_{t+1}^{\alpha}}, \quad \textit{where} \tag{6}$$

$$M_{t+1}^{\beta,\text{can}} = \frac{1}{R_{ft}} - \frac{(\lambda^{\text{can}})' V_{f^{\text{can}}}^{-1}}{R_{ft}} (f_{t+1}^{\text{can}} - \lambda^{\text{can}}),$$

[5]Our normalization assumption about mutual orthogonality of the candidate risk factors f_{t+1}^{can} with the missing risk factors f_{t+1}^{mis} has no bearing for the admissible SDF because the SDF is rotation invariant.

$$M_{t+1}^{\beta,\text{mis}} = -\frac{(\lambda^{\text{mis}})' V_{f^{\text{mis}}}^{-1}}{R_{ft}} (f_{t+1}^{\text{mis}} - \lambda^{\text{mis}})),$$

$$M_{t+1}^{a} = -\frac{a' V_e^{-1}}{R_{ft}} e_{t+1}, \tag{7}$$

with $\text{cov}(M_{t+1}^{\beta,\text{can}}, M_{t+1}^{a}) = 0$*,* $\text{cov}(M_{t+1}^{a}, M_{t+1}^{\beta,\text{mis}}) = 0$*, and* $\text{cov}(M_{t+1}^{\beta,\text{can}}, M_{t+1}^{\beta,\text{mis}}) = 0$*, and where, without loss of generality, as indicated earlier,* $V_{f^{\text{mis}}} = I_{K^{\text{mis}} \times K^{\text{mis}}}$*.*

The SDF component M_{t+1}^{a} is a linear function (scaled by the risk-free rate) of asset-specific shocks e_{t+1}. Because M_{t+1}^{a} constitutes part of the admissible SDF, we refer to it as an *aggregate* measure of asset-specific risk to emphasize that it represents an aggregate source of risk even though it consists of asset-specific shocks.

The correction components M_{t+1}^{a} and $M_{t+1}^{\beta,\text{mis}}$ are latent quantities, and therefore, the issue of econometric identification arises. At the estimation stage, we resolve this identification challenge by imposing the no-arbitrage restriction given in expression (4) that gives us precisely the condition required to identify M_{t+1}^{a} and $M_{t+1}^{\beta,\text{mis}}$.

A central insight from the above proposition is that the interpretation of α is different in the candidate and corrected models. From the perspective of the candidate model, α represents the vector of pricing errors

$$\alpha = \mathbb{E}[M_{t+1}^{\beta,\text{can}} (R_{t+1} - R_{ft})] \times R_f,$$

whereas in the corrected model α represents the compensation for assets' exposures to the risk factors f_{t+1}^{mis} and asset-specific risk e_{t+1}:

$$\alpha = -\,\text{cov}(M_{t+1}^{\alpha}, (R_{t+1} - R_{ft})) \times R_f.$$

In particular, in the corrected model the elements of the vector a represent the compensation for exposure to the asset-specific shocks e_{t+1}:

$$a = -\,\text{cov}(M_{t+1}, e_{t+1}) \times R_f = -\,\text{cov}(M_{t+1}^{a}, e_{t+1}) \times R_f.$$

This result paves the way towards a quantitative assessment of asset-specific risk in financial markets that we will explore in our empirical analysis.

The dependence of M_{t+1}^{a} on e_{t+1} implies that expanding a candidate model to include an increasing number of observable variables proxying for common risk factors is not a fruitful avenue to build an admissible SDF. In Appendix A, we show that M_{t+1}^{a} is a weak factor in the cross-section of asset returns. Therefore, even if it were possible to add to a candidate factor model an observable variable that was perfectly correlated with M^a_{t+1}, it would not lead to an admissible SDF. The risk premia associated with a weak factor cannot be estimated accurately (Anatolyev and Mikusheva, 2021). We show below how to account for asset-specific risk and construct an accurate estimator of M^a_{t+1}, and thus, of an admissible SDF.

2.3 Nonnegative Feasible SDF

There are three problems in constructing an admissible SDF in practice. First, the linear SDF characterized in the previous section may not always be strictly positive, thus possibly leading to negative asset prices. Second, the components $M^{\beta,\text{mis}}_{t+1}$ and M^a_{t+1} depend on unobservable quantities, such as, f^{mis}_{t+1} and e_{t+1}, respectively. Finally, a candidate factor model may omit not only strong but also weak factors. We explain below how to address these three challenges.

2.3.1 Exponential SDF

There are at least two approaches for ensuring that the SDF is positive. The first approach is to express the SDF as the payoff to an option (Hansen and Jagannathan, 1997, Eq. (24)). The second approach is to specify the SDF as an *exponential* function of the payoffs (Ghosh, Julliard, and Taylor, 2017; Gourieroux and Monfort, 2007). For obtaining closed-form solutions, we assume that asset returns are Gaussian and follow the second approach.

Proposition 3 (SDF: Exponential Case)**.** *Under Assumptions 1 and 2 and the assumption that returns R_{t+1} are Gaussian, there exists an admissible SDF $M_{\text{exp},t+1}$*

$$
\begin{aligned}
M_{\text{exp},t+1} &= M^{\beta,\text{can}}_{\text{exp},t+1} \times M^a_{\text{exp},t+1} \times M^{\beta,\text{mis}}_{\text{exp},t+1}, \quad \textit{where} \\
M^{\beta,\text{can}}_{\text{exp},t+1} &= \frac{1}{R_{ft}} \exp\Big[-(\lambda^{\text{can}})' V^{-1}_{f^{\text{can}}} (f^{\text{can}}_{t+1} - \lambda^{\text{can}})) - \frac{1}{2} (\lambda^{\text{can}})' V^{-1}_{f^{\text{can}}} \lambda^{\text{can}} \Big], \\
M^{\beta,\text{mis}}_{\text{exp},t+1} &= \exp\Big[-(\lambda^{\text{mis}})' V^{-1}_{f^{\text{mis}}} (f^{\text{mis}}_{t+1} - \lambda^{\text{mis}}) - \frac{1}{2} (\lambda^{\text{mis}})' V^{-1}_{f^{\text{mis}}} \lambda^{\text{mis}} \Big], \\
M^a_{\text{exp},t+1} &= \exp\Big[-a' V_e^{-1} e_{t+1} - \frac{1}{2} a' V_e^{-1} a \Big],
\end{aligned}
$$

where $\text{cov}(M^a_{\text{exp},t+1}, M^{\beta,\text{can}}_{\text{exp},t+1}) = 0$, $\text{cov}(M^{\beta,\text{mis}}_{\text{exp},t+1}, M^{\beta,\text{can}}_{\text{exp},t+1}) = 0$, *and* $V_{f^{\text{mis}}} = I_{K^{\text{mis}} \times K^{\text{mis}}}$.

2.3.2 Projection SDF

Even if the values of the parameters of the data-generating process (3) are known, the admissible SDF M_{t+1} depends on the unobservable quantities f_{t+1}^{mis} and e_{t+1}. As a result, M_{t+1} is not feasible empirically. To overcome this challenge, we rely on a projection version of the SDF, $\hat{M}_{t+1}$, with $\hat{\cdot}$ used to indicate that we use a projection. In particular, we take the exponential function of the linear projections of M_{t+1}^{a} and $M_{t+1}^{\beta,\text{mis}}$ on the set of the risk-free and risky assets and obtain[6]

$$\hat{M}_{\exp,t+1}^{a} = \exp\Big[-a'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}a'V_R^{-1}a\Big], \tag{8}$$

$$\hat{M}_{\exp,t+1}^{\beta,\text{mis}} = \exp\Big[-(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}(\beta^{\text{mis}}\lambda^{\text{mis}})'V_R^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}\Big], \tag{9}$$

where

$$\mathbb{E}[R_{t+1}] - R_{ft} = a + \beta^{\text{mis}}\lambda^{\text{mis}} + \beta^{\text{can}}\lambda^{\text{can}},$$

$$V_R = \beta^{\text{can}}V_{f^{\text{can}}}\beta^{\text{can}\prime} + \beta^{\text{mis}\prime}\beta^{\text{mis}} + V_e.$$

The component $M_{\exp,t+1}^{\beta,\text{can}}$ depends on observable quantities, so that the projection nonnegative admissible SDF takes the form

$$\hat{M}_{\exp,t+1} = M_{\exp,t+1}^{\beta,\text{can}} \times \hat{M}_{\exp,t+1}^{\beta,\text{mis}} \times \hat{M}_{\exp,t+1}^{a}.$$

The next proposition shows that, as $N \to \infty$, the SDF components, $M_{\exp,t+1}^{\beta,\text{mis}}$ and $M_{\exp,t+1}^{a}$, and their corresponding projection versions, $\hat{M}_{\exp,t+1}^{\beta,\text{mis}}$, and $\hat{M}_{\exp,t+1}^{a}$, have the same limits. We denote the matrix of the loadings of returns on the candidate and missing factors by $\beta = (\beta^{\text{can}}\ \beta^{\text{mis}})$ and an arbitrary $K \times K$ positive-definite matrix by $A > 0$. We use $\xrightarrow{p}$ to denote convergence in probability and $a = o(b)$ with $b > 0$ to denote that $|a|/b \to 0$, as $N \to \infty$, where the dependence of a and b on N is implicit.[7]

Proposition 4 (Limiting properties of SDF projections)**.** *Under Assumptions 1 and 2 and the conditions $N^{-1}\beta'{V_\varepsilon}^{-1}\beta \xrightarrow{p} A$ and $\beta'{V_\varepsilon}^{-1}a = o(N^{\frac{1}{2}})$, as $N \to \infty$,*

$$\hat{M}_{\exp,t+1}^{a} - M_{\exp,t+1}^{a} \xrightarrow{p} 0, \quad \hat{M}_{\exp,t+1}^{\beta,\text{mis}} - M_{\exp,t+1}^{\beta,\text{mis}} \xrightarrow{p} 0,$$

$$\text{cov}(M_{\exp,t+1}^{\beta,\text{can}}, \hat{M}_{\exp,t+1}^{\beta,\text{mis}}) \to 0, \quad \text{cov}(M_{\exp,t+1}^{\beta,\text{can}}, \hat{M}_{\exp,t+1}^{a}) \to 0, \ \text{cov}(\hat{M}_{\exp,t+1}^{\beta,\text{mis}}, \hat{M}_{\exp,t+1}^{a}) \to 0.$$

[6]The formulae (8) and (9) indicate that the assumption that asset returns are Gaussian is mild in practice. By the arguments of the Central Limit Theorem, the projection version of our feasible SDF, being an exponential function of the sum of N asset returns that are non-Gaussian, is approximately log-normal as $N \to \infty$.

[7]Strictly speaking, the matrix of loadings β from the data-generating process given in expression (1) can be different from the matrix $\beta = (\beta^{\text{can}}\ \beta^{\text{mis}})$ because missing factors are identified only up to a rotation. This difference, however, does not have any economic bearing.

The above proposition implies that to construct the admissible SDF we do not need to pre-estimate the missing factors omitted in the candidate factor model and asset-specific risk omitted in the candidate SDF model.

2.4 Weak Factors

Finally, we consider the case, in which the candidate model includes all strong systematic risk factors but omits weak factors collected in a vector f_{t+1}^{mis}. Following Lettau and Pelger (2020), we define weak factors as factors that affect only a subset of the underlying assets or affect all assets only marginally. Mathematically, the loadings of asset returns on weak factors satisfy

$$\beta^{\text{mis}\prime}\beta^{\text{mis}} \xrightarrow{p} A,$$

where $A > 0$ is some arbitrary positive-definite matrix. This implies that $\beta^{\text{mis}\prime}\beta^{\text{mis}}/N \xrightarrow{p} 0_{K^{\text{mis}}\times K^{\text{mis}}}$, where $0_{K^{\text{mis}}\times K^{\text{mis}}}$ is a $K^{\text{mis}} \times K^{\text{mis}}$ matrix of zeros. The proposition below shows that also in this case we can correct the candidate SDF to obtain the admissible SDF, even though it is not possible to estimate consistently missing weak factors (Lettau and Pelger (2020, Proposition 2)).

Proposition 5 (Properties of $\hat{M}_{\exp,t+1}^{\alpha}$, when missing factors are weak and $N \to \infty$)**.** *Under Assumptions 1 and 2 and conditions* $N^{-1}\beta' V_{\varepsilon}{}^{-1}\beta \xrightarrow{p} A$, $\beta^{\text{mis}\prime} V_e{}^{-1}\beta^{\text{mis}} = O(1)$, $\beta^{\text{mis}\prime} V_{\varepsilon}{}^{-1}\beta = o(N^{\frac{1}{2}})$, $\beta' V_e{}^{-1}a = o(N^{\frac{1}{2}})$, $\beta^{\text{mis}\prime} V_{\varepsilon}{}^{-1}a = O(1)$, *and* β^{mis} *and* β *are not asymptotically collinear, then as* $N \to \infty$,

$$\hat{M}_{\exp,t+1}^{\alpha} - M_{\exp,t+1}^{\alpha} \xrightarrow{p} 0 \quad \textit{with} \quad M_{\exp,t+1}^{\alpha} = \exp\Big[-\alpha' V_{\varepsilon}^{-1}\varepsilon_{t+1} - \frac{1}{2}\alpha' V_{\varepsilon}^{-1}\alpha\Big].$$

This proposition highlights that weak factors and pure asset-specific risk (shocks uncorrelated across asset returns) cannot be identified separately. The presence of weak factors is compatible with the approximate factor structure in the APT of Chamberlain and Rothschild (1983). The approximate factor structure requires that the covariance matrix of returns adjusted for strong factors has uniformly bounded eigenvalues, limiting the degree of cross-sectional dependence. Proposition 5 clarifies that, although latent weak factors cannot be estimated, let alone their corresponding risk premia, we can still estimate the component M_{t+1}^{a}, which is a function of the priced weak factors and pure asset-specific risk.

3 Estimation Details

In this section, we describe our approach for estimating the admissible SDF, explain the role of the no-arbitrage restriction (4), and discuss how to identify the number of missing factors in a candidate factor model and choose the no-arbitrage bound δ_{apt}. Also, we provide a diagnostic tool for detecting missing factors in the candidate factor model.

We recover the admissible SDF in two steps. In the first step, we use a (pseudo) Gaussian maximum-likelihood estimator (MLE) subject to the no-arbitrage restriction formulated in expression (4) to estimate the model of asset returns given in expression (3).[8] We consider the case of a factor model for asset returns that includes only tradable factors represented by either factor returns (for example, the market factor) in excess of the risk-free rate or long-minus-short strategies. In the second step, we use the results in Propositions 3 and 4 to recover the nonnegative feasible admissible SDF. The Online Appendix contains a more general case, in which (i) the candidate model for asset returns includes both tradable and non-tradable factors and (ii) the risk factors in the candidate model are allowed to be correlated with the missing systematic risk factors.

3.1 Formulating the Likelihood

For a generic vector Θ that collects all the elements of the matrices β^{can}, β^{mis}, V_e, $V_{f^{\text{can}}}$, and vectors λ^{can}, λ^{mis}, and a, the (up to a constant) Gaussian joint log-likelihood of the vectors of asset returns in excess of the risk-free rate, $R_{t+1} - R_{ft}$, and observable factors f^{can}_{t+1} is

$$\begin{aligned}\log(L(\Theta)) = &-\frac{T}{2}\log(|V_\varepsilon|) - \frac{T}{2}\log(|V_f^{\text{can}}|)\\ &-\frac{1}{2}\sum_{t=0}^{T-1}\varepsilon_{t+1}' V_\varepsilon^{-1}\varepsilon_{t+1} - \frac{1}{2}\sum_{t=0}^{T-1}(f^{\text{can}}_{t+1} - \lambda^{\text{can}})' V_{f^{\text{can}}}^{-1}(f^{\text{can}}_{t+1} - \lambda^{\text{can}}),\end{aligned}$$

where $\varepsilon_{t+1} = (R_{t+1} - R_{ft}) - a - \beta^{\text{mis}}\lambda^{\text{mis}} - \beta^{\text{can}} f^{\text{can}}_{t+1}$.

We maximize this log-likelihood function subject to the no-arbitrage restriction. We substitute the no-arbitrage restriction given in expression (4) with

$$a' V_\varepsilon^{-1} a \leq \delta_{\text{apt}}, \tag{10}$$

[8] For the asymptotic analysis of the MLE for large dimensional latent factor models see Bai and Li (2012).

which is computationally simpler to handle and that satisfies the no-arbitrage restriction (4), when $N \to \infty$.

We use the the Karush-Kuhn-Tucker multiplier method to solve the resulting constrained optimization problem:

$$\hat{\Theta} = \underset{\Theta}{\text{argmax}}\{\log(L(\Theta)) - \kappa(a'V_\epsilon^{-1}a - \delta_{\text{apt}})\}.$$

In the above expression, the parameter κ is the Karush-Kuhn-Tucker multiplier on the APT restriction and δ_{apt} is obtained using a cross-validation procedure that is explained below in Section 3.3. Appendix B1 provides the solution of the optimization problem.

3.2 The Role of the No-Arbitrage Restriction

The no-arbitrage restriction on the vector a serves several purposes. Economically, it rules out asymptotic arbitrage. For example, if the elements of the vector a are left unconstrained, the Hansen and Jagannathan (1997) (HJ) distance explodes, as we demonstrate in Section 5. Moreover, the APT restriction constrains the Sharpe ratio of the so-called alpha portfolio of Raponi, Uppal, and Zaffaroni (2021). This alpha portfolio is an inefficient portfolio that, when combined with a portfolio invested in the candidate factors (the so-called beta portfolio), delivers a portfolio on the efficient frontier. In the same spirit, Kozak, Nagel, and Santosh (2020) rules out near-arbitrage opportunities by restricting the maximum squared Sharpe ratio implied by the entire SDF.

Statistically, the APT restriction (when binding) leads to identification of the vectors a and λ^{mis}. Specifically, at the estimation stage the APT restriction provides N conditions that allow one to split the estimate of α into the estimates of a and $\beta^{\text{mis}}\lambda^{\text{mis}}$. Identification of a and λ^{mis} is a necessary step for constructing the missing systematic and asset-specific components of the admissible SDF, $M_{t+1}^{\beta,\text{mis}}$ and M_{t+1}^{a}, respectively. Even in population, the no-arbitrage restriction binds and is further influenced by the presence of financial frictions (Korsaye, Quaini, and Trojani (2019, section 2)).

The estimator of a under the APT restriction has the form of a ridge estimator, as can be seen from Proposition B1. The ridge estimator has the appealing property of mitigating the estimation noise that in our case affects the estimates of the asset-specific risk premia. The estimation noise can be significant because the vector a is an N-dimensional object.

3.3 Identifying the Number of Missing Systematic Risk Factors

Given that the candidate factor model for asset returns may feature $K^{\text{can}} < K$ risk factors, we need to determine the number K^{mis} of missing systematic risk factors f_{t+1}^{mis}. We estimate K^{mis} together with the bound δ_{apt} on the no-arbitrage restriction (10), using cross-validation with the HJ distance as a selection metric. The choice of the HJ distance is natural, given our objective of identifying the correction required to obtain an admissible SDF from the candidate factor model.

Our cross-validation procedure uses 20 folds. We split the entire sample into 20 folds and estimate the model on all but one fold. We repeat this procedure 20 times and compute the HJ distance on the validation folds. We fix a grid of δ_{apt} from 0 to 0.1 that corresponds to Sharpe ratios ranging from 0 to 0.32 per month for the portfolio associated with purely asset-specific risk.[9] We vary the number of systematic factors missing in the candidate model, K^{mis}, from 0 to 10. We pick K^{mis} and the value of δ_{apt} that deliver the smallest HJ distance in the validation step. Our procedure never selects the binding values of K^{mis} and δ_{apt}. Finally, using the optimal K^{mis} and δ_{apt}, we reestimate the model on the entire sample.

In the literature, other methods have been used for selecting the number of systematic risk factors in SDF models. For example, Giglio and Xiu (2021) use an information criterion similar to Bai and Ng (2002). Alternatively, Lettau and Pelger (2020) and Kozak, Nagel, and Santosh (2020) use economic restrictions relating expected returns to the covariance of returns with factors in addition to time series information on variation of asset returns.[10] Because none of these approaches directly applies to a model with asset-specific components in expected returns, we face a choice: either use a two-stage estimation that pins down K^{mis} in the first step and δ_{apt} in the second step or design our own method. We choose the latter and optimize the objective function that explicitly incorporates a no-arbitrage restriction and select simultaneously K^{mis} and δ_{apt} that deliver the minimal value of the HJ-distance.

[9] Ross (1977) suggests using a bound that is a multiple of the Sharpe ratio for the market portfolio, which is about 0.4 per annum.

[10] Even though our objective function is similar to that of Lettau and Pelger (2020), there are several important differences in the two approaches. First and foremost, our goal is not to compress α as much as possible, but rather to ensure that the no-arbitrage restriction holds. From the perspective of the corrected model, α is not a pricing error, and therefore does not need to be the null vector. Relatedly, our objective function, in contrast to that of Lettau and Pelger (2020), does not explicitly include a pricing metric measuring goodness of fit. If we were to include such a pricing metric into the objective function, we would have to augment our log-likelihood function with an additional penalty term represented by the HJ distance.

3.4 Detecting the Missing Factors

Propositions 3 and 4 imply that, as $N \to \infty$, $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ converges to a linear function of the missing factors. As a result, a simple time-series regression approach applied to $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ provides a diagnostic tool for detecting the missing factors in the candidate factor model for asset returns. Our approach is robust with respect to weak factors and also to factors with asset exposures that are highly correlated. The latter would lead to multicollinearity in the second pass of the traditional two-step methodology. This is because we do not need to estimate the exposures of basis assets to these factors, but only to verify that these factors have explanatory power for the SDF.

Define g_t as the vector of some observable variables that may represent missing factors in the candidate model and collect its values for each t in a matrix $G = (g_1 \cdots g_T)'$. Similarly for each t collect the values of the systematic component $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ of the admissible SDF in a vector $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp}}) = (\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},1}) \cdots \log(\hat{M}^{\beta,\text{mis}}_{\text{exp},T}))'$. The R^2 of the regression of $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$ on an intercept and the vector g_{t+1},

$$\hat{M}^{\beta,\text{mis}}_{t+1} = \gamma_0 + \gamma_1' g_{t+1} + u_{t+1},$$

is

$$R^2_g = \frac{\hat{\gamma}_1' G'(I_T - 1_T 1_T'/T)G\hat{\gamma}_1}{(\hat{M}^{\beta,\text{mis}})'(I_T - 1_T 1_T'/T)\hat{M}^{\beta,\text{mis}}},$$

where $\hat{\gamma}_1 = (G'(I_T - 1_T 1_T'/T)G)^{-1} G'(I_T - 1_T 1_T'/T)\log(\hat{M}^{\beta,\text{mis}}_{exp})$.

The following proposition confirms that a simple time-series regression approach reveals if a set of observable variables explains the variation in asset returns that is left unexplained by a candidate factor model, and if so, delivers the prices of risk associated with these missing factors.

Proposition 6 (Detecting the missing factors). *Under the assumptions of Proposition 4 and if $g_{t+1} = Q f^{\text{mis}}_{t+1}$ for some non-singular Q, as $N \to \infty$, we have*

$$\hat{\gamma}_1 \xrightarrow{p} -\frac{(Q')^{-1}\lambda^{\text{mis}}}{R_f}, \quad \textit{and} \quad R^2_g \xrightarrow{p} 1.$$

If g_{t+1} is orthogonal to f^{mis}_{t+1}, that is $G'(I_T - 1_T 1_T'/T)F^{\text{mis}} = 0_{K^{\text{mis}} \times K^{\text{mis}}}$ then

$$\hat{\gamma}_1 \xrightarrow{p} 0_{K^{\text{mis}}} \quad \textit{and} \quad R^2_g \xrightarrow{p} 0.$$

Proposition 6 does not require large T but only large N, that is $R^2_g \xrightarrow{p} 1$ as long as T exceeds the number of factors in the vector g_t.

4 Data

In this section, we describe the data that we use in our empirical analysis. In the first subsection, we describe the set of basis assets that we use to estimate the SDF. In the second subsection, we describe the set of factors that could potentially be related to the estimated SDF and its components.

4.1 Basis Assets

We construct a projection of the SDF on a large set of standard characteristics-based portfolios of U.S. stocks. We collect monthly data on 202 portfolios from July 1963 to August 2019 from Kenneth French's website. The dataset includes 25 portfolios sorted by size and book-to-market ratio (ME & BM), 17 industry portfolios (Ind), 25 portfolios sorted by operating profitability and investment (OP & INV), 25 portfolios sorted by size and variance (ME & VAR), 35 portfolios sorted by size and net issuance (ME & NetISS), 25 portfolios sorted by size and accruals (ME & ACCR), 25 portfolios sorted by size and beta (ME & BETA), and 25 portfolios sorted by size and momentum (ME & MOM).

We use portfolios instead of individual assets because we posit a data generating process for asset returns with constant exposures to factors and prices of risk. Giglio and Xiu (2021) and Lettau and Pelger (2020) argue that constant factor loadings is a reasonable modeling assumption in the case of portfolio returns. We view the case of a model with constant factor loadings and prices of risk as a natural setting to illustrate how our methodology works in practice.

4.2 Factors Potentially Spanning the SDF

To examine which economic variables may explain variation in the SDF, we collect a comprehensive set of factors available at monthly frequency. We include factors used in Chen and Zimmermann (2021), Jensen, Kelly, and Pedersen (2021), and Kozak, Nagel, and Santosh (2020). We also include factors from Bryzgalova, Huang, and Julliard (2020); the Online Appendix of Bryzgalova, Huang, and Julliard (2020) lists the sources of these factors.

We augment the dataset of factors with vector-autoregressive (VAR) residuals of the first three principal components of 279 macro variables from Jurado, Ludvigson, and Ng (2015). We also include VAR residuals of the first eight principal components of 128 macro variables (FRED-MD dataset of McCracken and Ng, 2015) and log differences in consumer sentiment.[11] We use the data vintage from February 2021.

We collect real per capita consumption data on nondurables and services and the corresponding price index from the Bureau of Economic Analysis (BEA). We use a one-month and 3-year log consumption growth and residuals from an autoregressive process of order 1 (AR(1)) for log inflation as factors. Our dataset also includes the U.S. business confidence index, U.S. consumer confidence index and U.S. composite leading indicator from the OECD library, confidence economic activity index, NBER recession index, TED spread, effective federal funds rate, real federal funds rate, and Chicago Fed National Financial Condition Index from FRED, and credit spread index of Gilchrist and Zakrajšek (2012).

We add the market-dislocations index of Pasquariello (2014), and the disagreement index of Huang, Li, and Wang (2021). We include industry-adjusted value, momentum, and profitability factors (Novy-Marx, 2013), intra-industry value, momentum, and profitability factors, and basic profitable-minus-unprofitable factor. We complement our dataset with the expected growth factor of Hou, Mo, Xue, and Zhang (2021) and the momentum Up minus Down (UMD) factor. Finally, we include the Chicago Board Options Exchange (CBOE) volatility index (VIX) available on the website of the CBOE, the U.S. economic policy uncertainty index (EPU) of Baker, Bloom, and Davis (2016), and the equity market volatility (EMV) tracker of Baker, Bloom, Davis, and Kost (2019) (with the last two available on `www.policyuncertainty.com`). For highly persistent variables, e.g., the disagreement index of Huang, Li, and Wang (2021), VIX, etc., we define factors as the first-order log differences and residuals from the corresponding univariate autoregressive processes of order one.

5 Empirical Analysis

In this section, we demonstrate how our methodology can be used to correct popular candidate factor models with the purpose of constructing an admissible SDF. To set the stage, we start by studying a candidate model with zero risk factors so that $M^{\beta,\text{can}}_{\exp,t+1}$ is just the inverse of the risk-free rate. We use this model to establish the relative quantitative importance of systematic versus asset-specific risk. This exercise is a clean experiment that is free of concerns that candidate factors may include unpriced sources of common variation

[11] We download macroeconomic variables from `https://research.stlouisfed.org/econ/mccracken/fred-databases`. We exclude four variables, ACOGNO, ANDENOx, TWEXAFEGSMTHx, UMCSENT, which have missing observations at the start of the sample.

in asset returns (Daniel, Mota, Rottke, and Santos, 2020) or spurious factors (Kan and Zhang, 1999; Kleibergen, 2009). Furthermore, we use this experiment to examine which popular factors correlate with the systematic component of the estimated admissible SDF and which trading strategies correlate with the aggregate measure of asset-specific risk.

After our analysis of a candidate model with zero risk factors, we examine commonly used candidate factor models of asset returns, such as the market model, the model with a consumption-mimicking portfolio, and the model of Fama and French (1993) and characterize the missing systematic and asset-specific components in each of the candidate SDFs.

5.1 A Candidate Model with Zero Risk Factors

We start our analysis of a candidate model with zero risk factors ($K^{\text{can}} = 0$) by identifying how many systematic factors explain the cross-section of asset returns. That is, we estimate K^{mis} given that the candidate model does not include any risk factors and determine the no-arbitrage bound δ_{apt}. Figure 1 illustrates how the HJ distance changes as we vary K^{mis} and δ_{apt}. The top two panels show the estimation results based on cross validation, while the bottom two panels show the in-sample results.

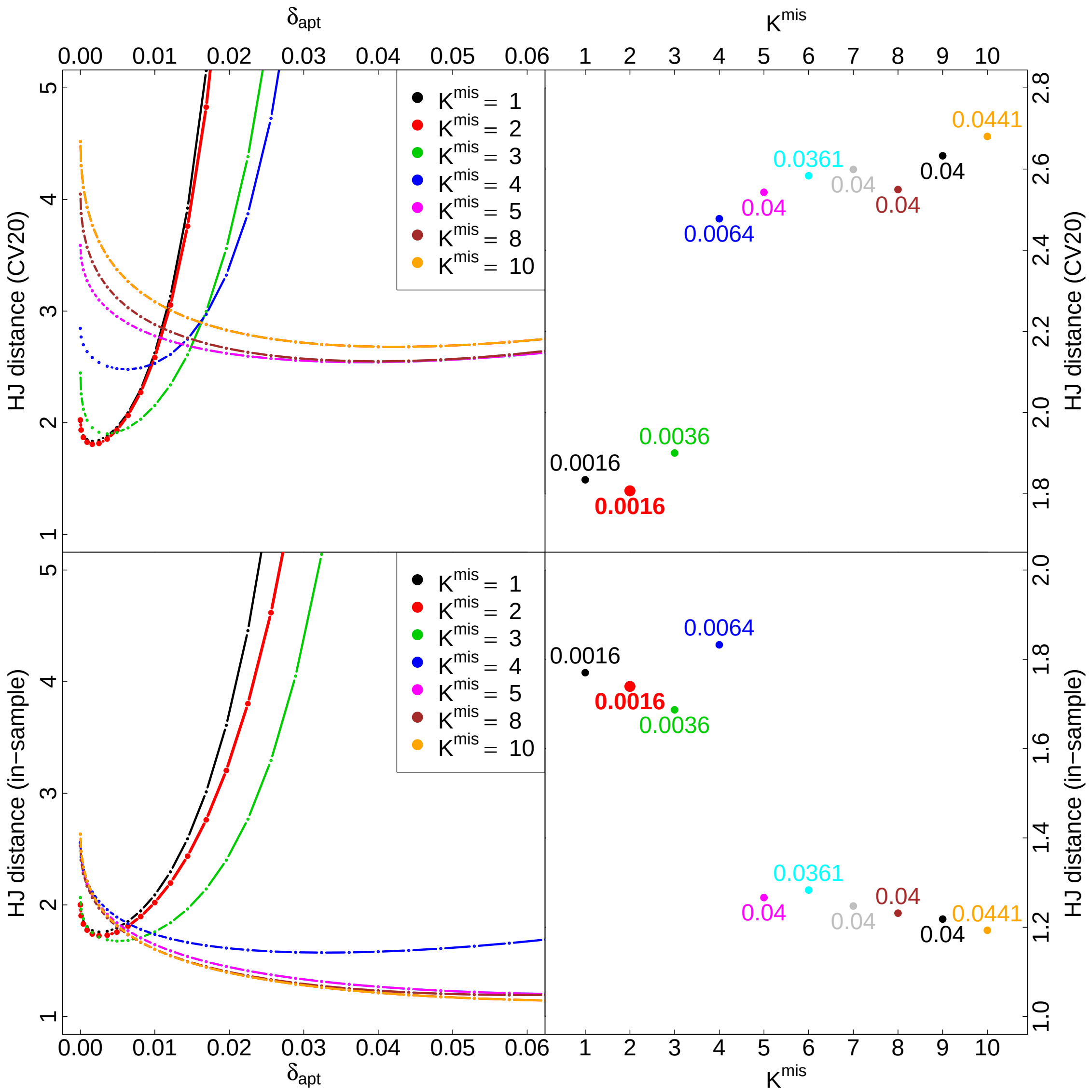
Using the HJ distance as the selection criteria, we see from Figure 1 that our estimation procedure selects $K^{\text{mis}} = 2$ systematic factors and $\delta_{\text{apt}} = 0.0016$. The two top panels demonstrate that this combination of K^{mis} and δ_{apt} achieves the smallest HJ distance, consistent with the evidence on low-dimensional factor pricing models in Kozak, Nagel, and Santosh (2018). The two bottom panels show that a naive in-sample analysis would lead to a choice of $K^{\text{mis}} = 10$ and $\delta_{\text{apt}} = 0.04$. The top right-hand panel shows that the combination of $K^{\text{mis}} = 10$ and $\delta_{\text{apt}} = 0.04$ performs poorly in the cross-validation exercise, with the poor fit in the out-of-sample analysis suggesting evidence of overfitting.

The non-zero value for the optimal δ_{apt} indicates that asset-specific risk is priced. This result constitutes our first main finding because it challenges the conventional view in financial economics that only systematic (common) risk factors are compensated in the market. Below we will shed light on what constitutes the priced asset-specific risk.

Figure 2 illustrates the estimated elements of the vector a and diagonal matrix V_e for the 202 basis assets. The top panel shows that these assets have different asset-specific volatilities, so our assumption that V_e is a diagonal rather than spherical matrix is war-

Figure 1: Model selection using the HJ distance

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}. The top two panels show the estimation results based on cross validation, while the bottom two panels show the in-sample results. The panels on the left plot the HJ distance for a given choice of K^{mis} as one varies δ_{apt}. The panels on the right display the δ_{apt} (numbers inside the boxes) that minimizes the HJ distance for a given choice of K^{mis}.

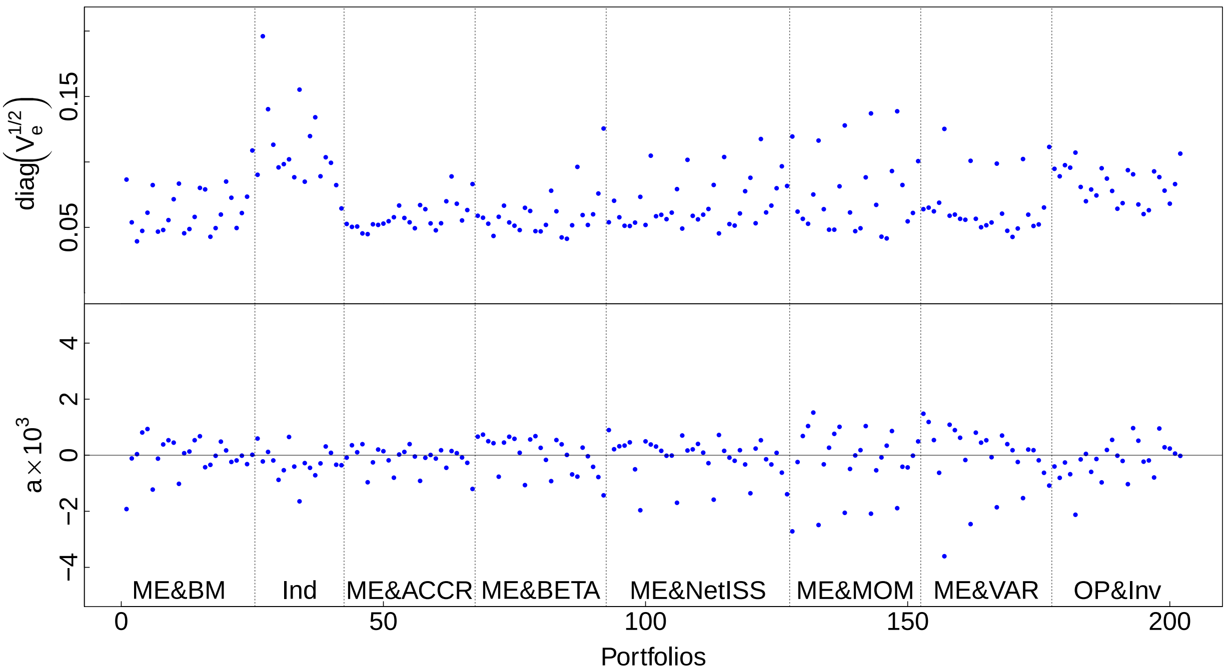


ranted. The bottom panel indicates that the compensation for exposures of basis assets to individual asset-specific shocks is small relative to the premia of conventional risk factors.

To examine the economic importance of asset-specific risk, we compare the pricing performance of our model with that of a model that ignores that asset-specific risk is priced. To do this, we use an approach based on principal component analysis (PCA). We choose

Figure 2: Estimated asset-specific risk and its compensation

This figure illustrates the estimated elements of the vector a and diagonal matrix V_e for the 202 basis assets, which we split into eight groups based on characteristics by which stocks are sorted into portfolios. The top panel shows the asset-specific volatilities $diag(V_e^{1/2})$ and the bottom panel shows the compensation a for asset-specific risk.



a different number of principal components to include in the SDF, ranging from one to five (increasing the number of principal components (PC) to ten does not change the results).[12]

Four empirical observations stand out from our specification analysis and Table 1. First, we find that ignoring the compensation for asset-specific risk, that is, setting $a = 0_N$, leads to a sizable increase in pricing errors, $\mathbb{E}[\hat{M}_{\exp,t+1}(R_{t+1} - R_{ft})]$, across the different assets. This increase in pricing errors translates to a statistically significant increase in the HJ distance by 14.79% relative to the model with $a \neq 0_N$.[13] Second, we observe that the largest benefit in accounting for the compensation for asset-specific risk in reducing the pricing error is for the portfolios sorted by size and variance.[14] Third, we find that the PCA-based model with two principal components is similar to our model with $a = 0_N$ and $K^{\text{mis}} = 2$, which is expected because the maximum-likelihood estimator corresponds

[12]In unreported results, we also consider a model with $K = 2$ systematic risk factors estimated from our methodology but where the asset-specific components of expected returns represented by the vector a are set equal to zero. The pricing performance of this model is similar to that based on the SDF with the first two PCs.

[13]For statistical inference, we run bootstrap.

[14]The corresponding results are available upon request.

Table 1: Relative HJ-distance of various models

This table reports the HJ distances of alternative candidate models and their corrected versions relative to the corrected version of the model with zero candidate risk factors, $(\text{HJ}^Y/\text{HJ}^X - 1) \times 100\%$, where X indicates the corrected version of the candidate model with zero risk factors and Y indicates any of the other considered models.

Class of models	Candidate model	Relative HJ (%)
Traditional	CAPM	14.72
	CCAPM	14.83
	FF3	15.45
PCA-based	PCA1	14.76
	PCA2	14.91
	PCA3	17.42
	PCA4	35.11
	PCA5	34.98
Fully corrected	Zero risk factors	0.00
	CAPM	0.77
	CCAPM	−0.16
	FF3	−3.37

to a weighted-PCA estimator (Bai and Li, 2012). Finally, consistent with Lettau and Pelger (2020) and Kozak, Nagel, and Santosh (2020), we notice that a naive strategy of including a larger number of principal components leads to inferior pricing performance of the corresponding model because of overfitting.

A common approach for evaluating asset-pricing factor models is to overlay the average excess returns on the test assets with the model-implied risk premia. We follow this approach in Figure 3, which shows that, as expected, our model exhibits a perfect fit. This is a consequence of our main insight, namely to consider the elements of the vector a as compensation for exposure to individual asset-specific shocks, as opposed to simply pricing errors.[15]

Next, we study the time-series properties of the estimated SDF, $\hat{M}_{\exp,t+1}$ and its components, $\hat{M}^{a}_{\exp,t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$. Figure 4 shows that both $\hat{M}^{a}_{\exp,t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$ exhibit sizable volatility during recessions and also during normal times. We see that different components of the SDF dominate variation in the SDF in different time periods. For example, in the Fall of 1987, common systematic shocks in asset returns are responsible for a dra-

[15] Recall that the estimated vector a is a ridge estimator of the asset-specific risk premia (see formula given in expression (B8) in Appendix B). For this exercise, we undo the shrinkage by multiplying each element of the estimated vector a by the estimated value of the Karush-Kuhn-Tacker multiplier κ plus 1. The estimated value of κ is 19.16.

Figure 3: Comparing average excess returns with model-implied risk premia
This figure overlays the average excess returns on the test assets with the model-implied risk premia after undoing the effect of shrinkage resulting from the ridge estimator for a.

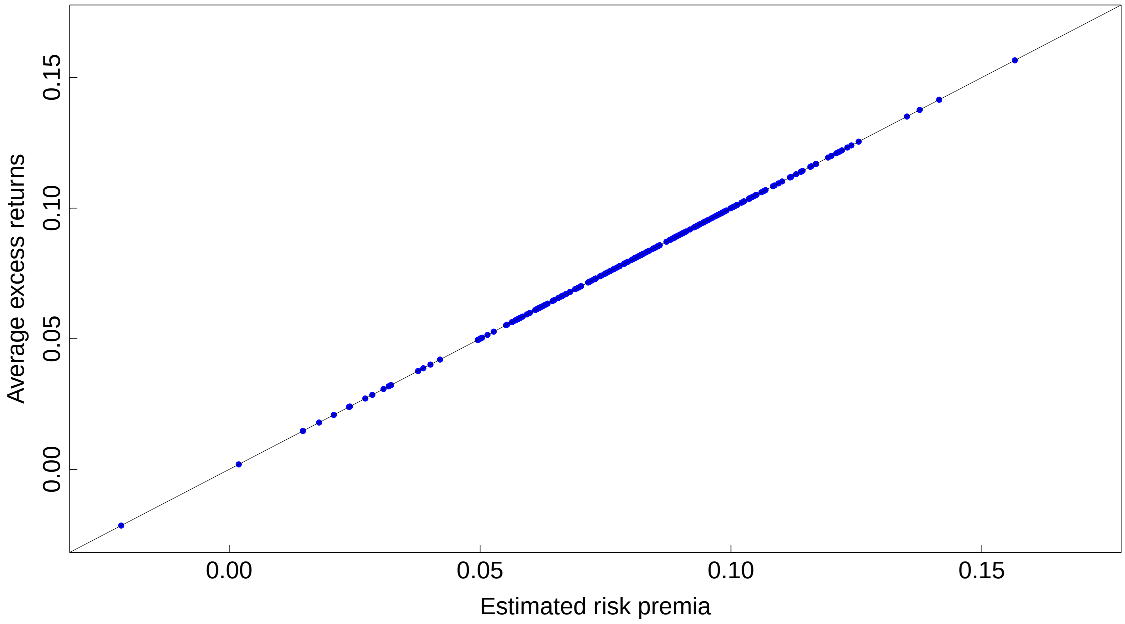
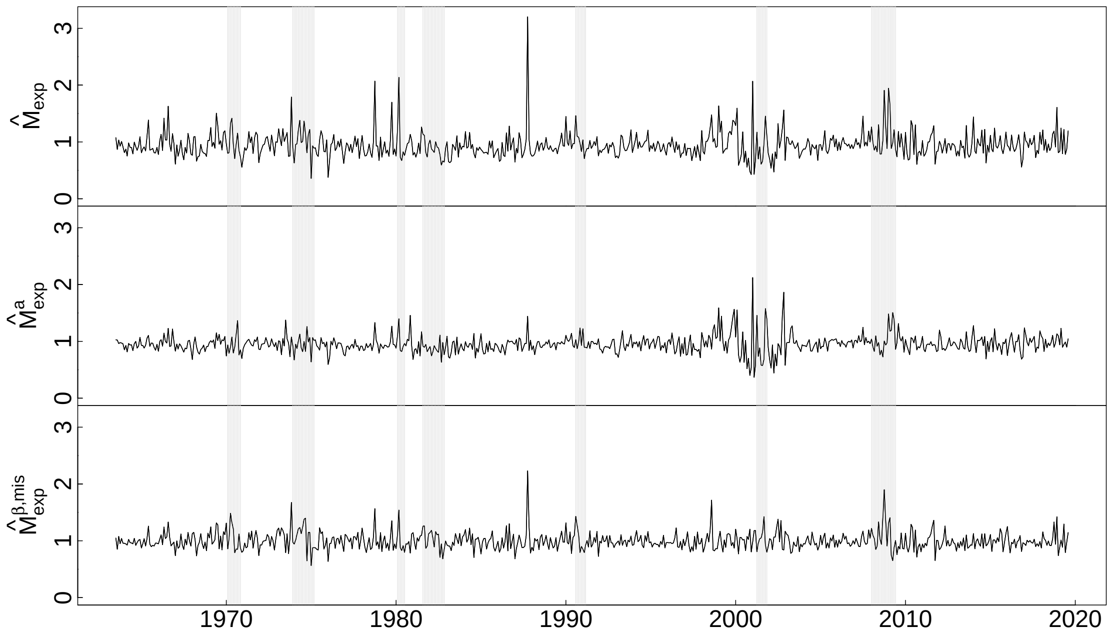


Figure 4: Time series behavior of the SDF and its components
This figure has three panels. The top, middle, and bottom panels of this figure show the dynamics of the SDF $\hat{M}_{\text{exp},t+1}$, its asset-specific component $\hat{M}^{a}_{\text{exp},t+1}$, and its systematic component $M^{\beta,\text{mis}}_{\text{exp},t+1}$, respectively.



matic increase in the level and volatility of the SDF. On the other hand, in the beginning of the 2000s, it is the aggregate measure of asset-specific risk that generates a spike in the volatility of the SDF. Thus both common and asset-specific shocks contribute meaningfully to explaining asset valuations.

We perform a formal variance decomposition of the SDF and report its results in Table 2. The standard deviation of $\log \hat{M}^a_{\exp,t+1}$ implies an annual Sharpe ratio for the aggregate measure of asset-specific risk (that is, the premium for the assets' exposure per unit of volatility of the aggregate measure of asset-specific risk), is $\mathrm{SR} = 0.59$. This result is interesting for two reasons. First, the compensation for one unit of aggregate measure of asset-specific risk is sizable because it is in the same ballpark as the market Sharpe ratio. Second, given that the elements of the vector a are very small, it becomes immediately clear that what matters for pricing assets is not individual asset-specific shocks but a specific combination of these shocks that represent the component $\hat{M}^a_{\exp,t+1}$ of the admissible SDF.

Relatedly, 56% of variation of the SDF can be attributed to the aggregate measure of asset-specific risk with the rest due to systematic sources of risk. These results are consistent with Daniel and Titman (1997) and Chaieb, Langlois, and Scaillet (2021), among others, who document that a substantial portion of expected excess returns is left unexplained by factor risk premia. Our finding also speaks to the puzzling evidence reported in Herskovic, Moreira, and Muir (2019). The authors document that the portfolios of stock returns that hedge factor risk exposure exhibit high positive expected returns. These high expected returns could simply reflect compensation for the aggregate measure of asset-specific risk.

Furthermore, an unreported, but available upon request, regression analysis indicates that $\hat{M}^a_{\exp,t+1}$ is acyclical: $\log(\hat{M}^a_{\exp,t+1})$ does not significantly correlate with any business-cycle indicator. In contrast to $\log(\hat{M}^a_{\exp,t+1})$, the systematic component $\log(\hat{M}^{\beta,\mathrm{mis}}_{\exp,t+1})$ has a significant correlation with the NBER recession indicators.[16] Given that $\hat{M}^{\beta,\mathrm{mis}}_{\exp,t+1}$ and $\hat{M}^a_{\exp,t+1}$ reflect the composition of the systematic and asset-specific components of the SDF, our approach provides a procedure for constructing trading strategies with and without exposures to systematic risk.

Having established the quantitative importance of the aggregate measure of asset-specific risk $\hat{M}^a_{\exp,t+1}$, we examine which trading strategies reflect exposures to this component of the SDF. As a first step, we run individual regressions of $\log(\hat{M}^a_{\exp,t+1})$ on the excess returns

[16] In regression analysis, we use the log SDF because the log SDF is linear in common risk factors and asset-specific shocks.

Table 2: Analysis of the SDF if candidate model has zero risk factors

This table reports the Sharpe ratios implied by the estimated admissible SDFs and its components along with the percentage of the variation of the SDF that is explained by each component.

Corrected models	$\hat{M}_{\exp,t+1}$	$\hat{M}^{a}_{\exp,t+1}$	$\hat{M}^{\beta,\text{can}}_{\exp,t+1}$	$\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$
Zero risk factors				
Sharpe Ratio	0.84	0.59	0.00	0.55
Variance decomp (%)	100.00	56.21	0.05	43.74
CAPM				
Sharpe Ratio	0.81	0.63	0.44	0.28
Variance decomp (%)	100.00	57.60	31.37	11.03
C-CAPM				
Sharpe Ratio	0.91	0.71	0.36	0.44
Variance decomp (%)	100.00	58.69	21.45	19.85
FF3				
Sharpe Ratio	0.98	0.69	0.70	0.00
Variance decomp (%)	100.00	42.16	57.84	0.00

of 457 strategies. We find 27 strategies with an R^2 larger than 30%. As a second step, we compute the risk premia associated with the exposures of these 457 trading strategies to the aggregate measure of asset-specific risk as the negative covariance of the return on the strategy and $M^{a}_{\exp,t+1}$:

$$\text{RP}^{a}_{strategy} = -\operatorname{cov}(R_{strategy,t+1}, \hat{M}^{a}_{\exp,t+1}) \times R_f.$$

As expected, we find that many of the strategies that are highly correlated with $\log(M^{a}_{\exp,t+1})$ are associated with large risk premia. However, there are also some strategies that are not as highly correlated with $\log(M^{a}_{\exp,t+1})$ but that still command sizable risk premia; for example, momentum strategies. Table 3 list strategies that have high correlations with and high compensation for exposure to the aggregate measure of asset-specific risk.

Examining the strategies from Table 3 closely, we find that there is large overlap across these strategies and that they fall into the following clusters, adopting the classification of Jensen, Kelly, and Pedersen (2021): Investment, Leverage, Low Risk, Profitability, and Value. In the literature, some of these strategies have been interpreted as being behavioral (for example, the Management factor of Stambaugh and Yuan (2017) and the Long-Horizon Financial factor of Daniel, Hirshleifer, and Sun (2020)), while others as reflecting market frictions (for example, the Betting-Against-Beta factor of Frazzini and Pedersen (2014) and constraints-return relation among high R&D firms in Li (2011)).

Table 3: Strategies highly correlated with $\log(\hat{M}^a_{\exp,t+1})$ and with high asset-specific risk premium RP_a

This table reports trading strategies that are highly correlated with $\log(\hat{M}^a_{\exp,t+1})$ and that have a large asset-specific risk premium. The first column gives the name of the cluster to which the strategy belongs, using the classification scheme in Jensen, Kelly, and Pedersen (2021). If a strategy is not in the list of Jensen, Kelly, and Pedersen (2021), we assign it to the cluster Unclassified. The second column gives the source. The third column gives the name of the variable, as in Chen and Zimmermann (2021) or Jensen, Kelly, and Pedersen (2021). The penultimate column reports the R^2 of the univariate regressions of $\log(\hat{M}^a_{\exp,t+1})$ on the return of each individual strategy. The last column reports the risk premium. The clusters, and within each cluster the sources, are listed in alphabetical order.

Cluster name	Source	Variable name	R^2(%)	RP_a(%)
Investment	Daniel, Hirshleifer, and Sun (2019)	beh_fin	44.85	5.33
	Fama and French (2015)	cma	30.21	2.08
	Hou, Xue, and Zhang (2015)	r_ia	30.48	2.04
	Ortiz-Molina and Phillips (2014)	aliq_at	32.21	−3.89
	Ritter (1991)	ageipo	29.31	5.83
	Stambaugh and Yuan (2016)	mgmt	33.59	3.22
	Xing (2008)	invcap	39.26	6.06
Leverage	Bhandari (1988)	leverage	43.21	5.54
	Fama and French (1992)	am	38.42	5.16
	Fama and French (1992)	bookleverage	39.58	−4.25
	Palazzo (2012)	cash	32.26	−5.98
	Palazzo (2012)	cash_at	36.66	−3.72
	Penman Richardson and Tuna (2007)	netdebt_me	34.61	3.84
Low risk	Ali, Hwang, and Trombley (2003)	idiovolaht	15.68	5.88
	Ang Chen and Xing (2006)	betadown_252d	32.11	−5.03
	Ang, Hodrick, Xing, Zhang (2006)	rvol_21d	25.92	−5.25
	Ang, Hodrick, Xing, Zhang (2006)	ivol	18.14	6.14
	Bali, Cakici, and Whitelaw (2010)	maxret	20.67	6.73
	Bradshaw, Richardson, Sloan (2006)	netequityfinance	37.50	5.24
	Bradshaw, Richardson, Sloan (2006)	xfin	39.68	6.54
	Fama and MacBeth (1973)	beta	17.37	−6.05
	Frazzini and Pedersen (2014)	betafp	17.60	−6.93
	Frazzini and Pedersen (2014)	bab	46.65	4.48
	Pontiff and Woodgate (2008)	shareiss1y	37.48	2.90
Momentum	Jegadeesh and Titman (1993)	mom12m	9.78	5.36
	Jegadeesh and Titman (1993)	mom6m	11.98	5.51
Profitability	Chen, Novy-Marx, Zhang (2011)	rome	25.22	5.11
	Diether, Malloy and Scherbina (2002)	forecastdispersion	26.83	5.05
	Frankel and Lee (1998)	predictedfe	39.73	5.46
	Frankel and Lee (1998)	analystvalue	38.00	4.86
	La Porta (1996)	fgr5yrlag	50.71	8.19
Value	Barbee, Mukherji and Raines (1996)	sp	33.09	4.46
	Basu (1977)	ep	32.65	5.33
	Boudoukh, Michaely, Richardson, Roberts (2007)	eqnpo_me	34.34	4.74
	Daniel and Titman (2006)	eqnpo_12m	30.31	3.39
	Dechow, Sloan and Soliman (2004)	equityduration	40.83	5.11
	Fama and French (1992)	hml	37.32	3.30
	Litzenberger and Ramaswamy (1979)	div12m_me	31.50	4.61
Unclassified	Cen, Wei, and Zhang (2006)	feps	24.87	6.15
	Cooper, Gulen, Schill (2008)	betaarb	24.90	6.19
	Datar, Naik, Radcliffe (1998)	shvol	25.10	5.88
	Easley, Hvidkjaer and O'Hara (2002)	probinformedtrading	21.36	6.20
	Elgers, Lo and Pfeiffer (2001)	sfe	29.24	7.00
	Li (2011)	rdcap	33.24	−4.47
	Ritter (1991)	indipo	36.42	4.30

Examining the composition of $\hat{M}^{a}_{\exp,t+1}$, we identify a substantial contribution of small stocks.[17] Specifically, of the 34 basis assets with the highest contribution to the variation of $\hat{M}^{a}_{\exp,t+1}$, fifteen represent various portfolios of small stocks, such as small stocks with low and high book-to-market, small stocks with high accruals, and small stocks with high prior returns. In addition, sixteen basis assets represent a range of portfolios of stocks sorted by Size and Market Beta or Size and Variance. Finally, seven of these 34 basis assets are portfolios of stocks with extremely high or low values of net issuances.

The special role of small stocks in the aggregate measure of asset-specific risk is our estimation result, not an assumption hardwired into the corrected candidate model. Proposition 3 shows that the relative weight of a basis asset in $M^{a}_{\exp,t+1}$ depends on the ratio of the asset's compensation for asset-specific risk represented by the corresponding element of the vector a and asset-specific volatility. While it is natural to expect that small stocks have larger firm-specific components of expected returns, it is also known that small stocks have higher volatility. Therefore, ex-ante it is not clear if small stocks will feature prominently in the aggregate measure of asset-specific risk.

Our finding regarding the role of small stocks in the aggregate measure of asset-specific risk complements the literature on granular origins of aggregate fluctuations (Gabaix, 2011). In this literature, idiosyncratic shocks to fundamentals of *large* firms can lead to nontrivial aggregate effects, that is, these shocks explain a substantial part of variation in aggregate fundamentals, or equivalently, in $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$.[18] In contrast, our central result is about the importance of idiosyncratic shocks to the returns of small companies that drive the acyclical component of the SDF, $\hat{M}^{a}_{\exp,t+1}$.

We now turn our attention to $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$, the component of the SDF related to systematic risk factors. We find that the market factor of Sharpe (1964) exhibits the highest explanatory power for $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$ with $R^2 = 0.95$. It is remarkable that, despite all the criticism of the market model, when we consider only the systematic component of the SDF, the market factor explains such a large proportion of its variation. Besides the market factor, we find that there are 23 trading strategies and 3 non-traded factors (shocks in VIX, intermediary capital (He, Kelly, and Manela, 2017), and dividend yield) that each individually explain more than 30% of variation in $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$. Because the market factor already

[17]Note that expression (7) provides composition of M^{a}_{t+1} in terms of asset returns neutral to common factors, e_{t+1}.

[18]Recall that the candidate model has zero risk factors, so that $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$ reflects all systematic risk factors.

explains a large proportion of the variation in $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$, the other factors explain only a small proportion of the variation not explained by the market factor. A combination of nineteen trading strategies is needed to explain 99% of variation in $\log(\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1})$.[19] These results are in line with the findings of Kozak, Nagel, and Santosh (2020) and Bryzgalova, Huang, and Julliard (2020) about the non-sparsity of the SDF in characteristics and the existence of several combinations of trading strategies that deliver a similar cross-sectional fit.

5.2 Popular Candidate Factor Models

To illustrate how our methodology brings new insights about factor models, we now consider three classic candidate models—those implied by the CAPM of Sharpe (1964), the Consumption-CAPM (C-CAPM) of Breeden (1979), and the three-factor model of Fama and French (1993). For the SDF $M^{\beta,\text{can}}_{\text{exp},t+1}$ implied by each of these three candidate factor models, we estimate the required correction terms $\hat{M}^{a}_{\text{exp},t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\text{exp},t+1}$ and characterize their properties.

5.2.1 The Market Model

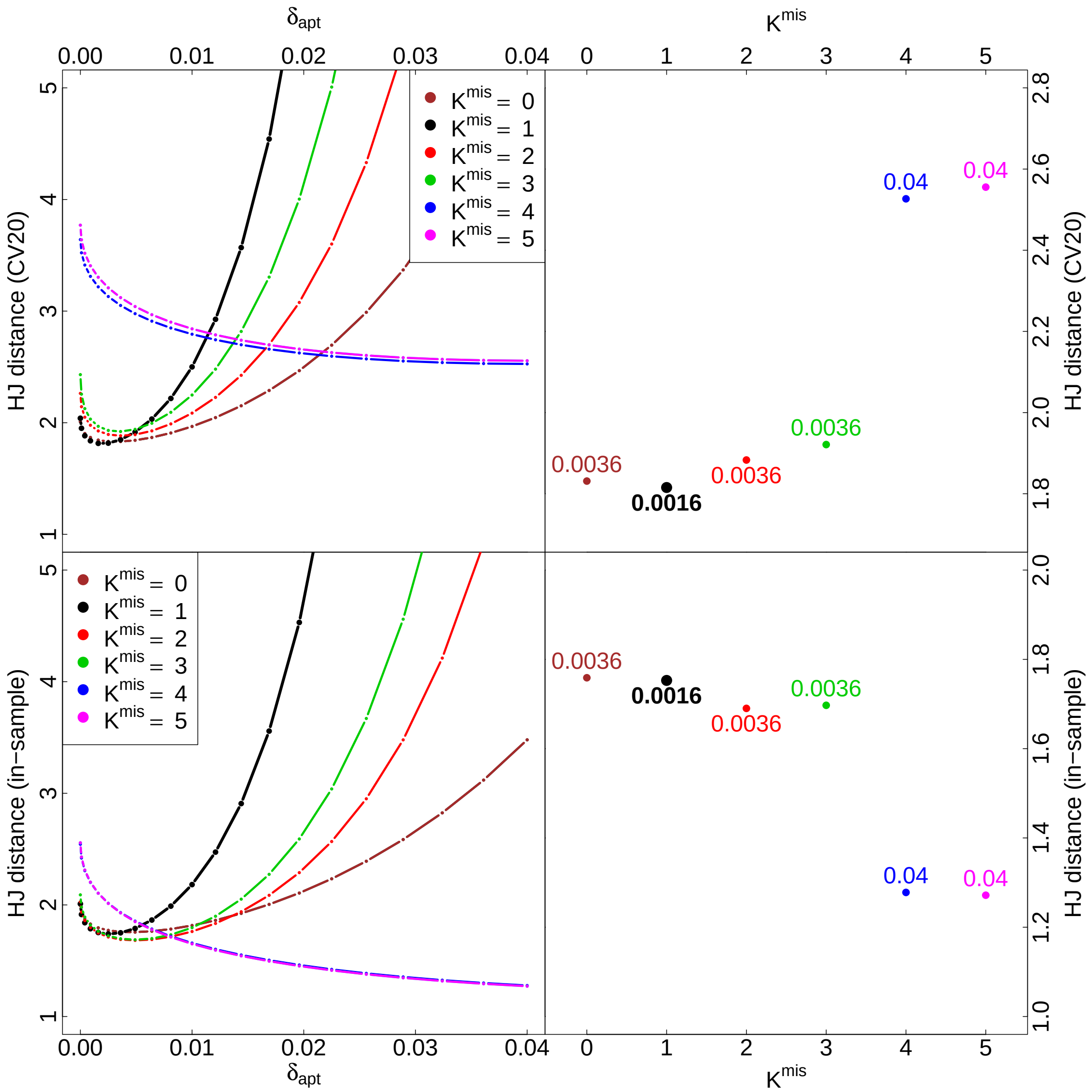
Figure 5 shows that for the candidate model with the market as a sole factor, the estimation procedure selects $K^{\text{mis}} = 1$ and $\delta_{\text{apt}} = 0.0016$. The obtained number of missing factors to correct the market model is in line with our earlier finding that two latent factors summarize the common variation in asset returns, with one factor being a proxy for the market factor. Furthermore, Table 4 shows that the admissible SDF obtained from correcting the market model is almost perfectly correlated with that previously obtained from correcting the candidate model with zero risk factors.

Table 1 shows that the pricing errors under the candidate market model are substantially larger than those under the corrected market model, in which the market factor is augmented by one latent factor and the vector a of the asset-specific components in expected returns. Once the candidate model is corrected, the HJ distance drops by almost 14%, with the drop

[19]These strategies include: market, size, betting-against-beta, sales-to-market (Barbee Jr, Mukherji, and Raines, 1996), change in current operating working capital and change in noncurrent operating liabilities (Richardson, Sloan, Soliman, and Tuna, 2005), Kaplan-Zingales index (Lamont, Polk, and Saa-Requejo, 2001), cash-to-assets (Palazzo, 2012), dollar trading volume Brennan, Chordia, and Subrahmanyam (1998), highest 5 days of return scaled by volatility (Asness, Frazzini, Gormsen, and Pedersen, 2020), quality minus junk growth (Asness, Frazzini, Israel, Moskowitz, and Pedersen, 2018), and short interest (Dechow, Hutton, Meulbroek, and Sloan, 2001).

Figure 5: Correction of the market model using the HJ distance

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model includes only the market factor. The top two panels show the estimation results based on cross validation, while the bottom two panels show the in-sample results. The panels on the left plot the HJ distance for a given choice of K^{mis} as one varies δ_{apt} (numbers inside the boxes). The panels on the right display the optimal values of the HJ distance for a given choice of K^{mis}.



being statistically significant. We analyze the pricing errors for each individual asset and record the largest improvement in pricing for the portfolios formed by sorting stocks by Size and Value, Size and Beta, Size and Net Issuance, and Size and Variance.[20]

[20]These results are unreported to save space but available upon request.

Table 4: Correlation matrix of corrected SDFs
This table reports the correlation matrix of admissible SDFs obtained from correcting different candidate models: the model with zero risk factors, CAPM, C-CAPM, and FF3.

	Zero risk factors corrected	CAPM corrected	C-CAPM corrected	FF3 corrected
Zero risk factors	1.00			
CAPM corrected	0.99	1.00		
C-CAPM corrected	0.94	0.94	1.00	
FF3 corrected	0.94	0.94	0.89	1.00

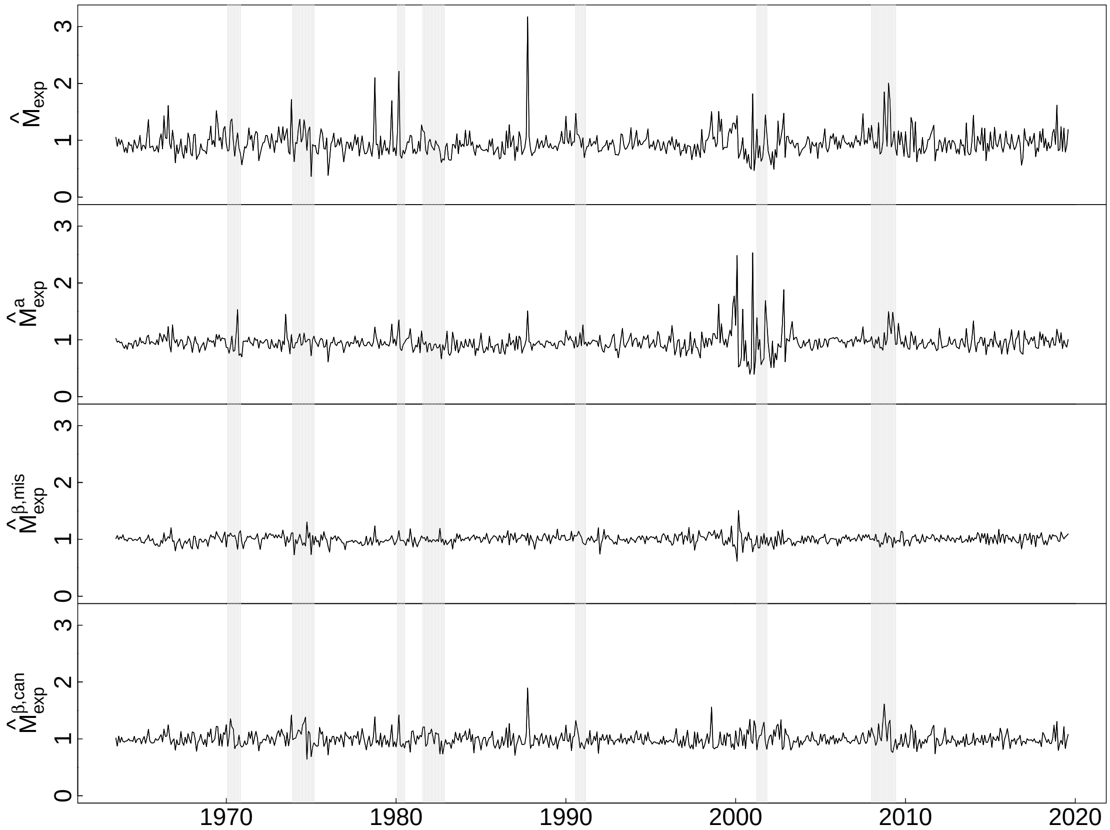
The relative importance of asset-specific risk in pricing assets is evident from Figure 6 and Table 2. Figure 6 shows that $\hat{M}^{a}_{\exp,t+1}$ exhibits higher volatility than $\hat{M}^{\beta,\text{can}}_{\exp,t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$, and Table 2 confirms this finding by showing that $\log(\hat{M}^{a}_{\exp,t+1})$ explains 58% of the variation in the admissible SDF. The high standard deviation of $\log(\hat{M}^{a}_{t+1})$ translates into an annual Sharpe ratio for the aggregate measure of asset-specific risk of SR $= 0.59$.

The remaining 42% of the variation in the log SDF is due to the combination of $\hat{M}^{\beta,\text{can}}_{\exp,t+1}$ and $\hat{M}^{\beta,\text{mis}}_{\exp,t+1}$, with $31.37/(31.37 + 11.03) \approx 73.99\%$ of this variation attributable to the market factor. Recall that when starting from the model with zero risk factors as a candidate model, we find that the market factor explains 95% of the systematic component of the admissible SDF. The quantitative difference in the role of the observable market factor is because the market factor is only a proxy for the latent risk factor recovered when considering the case of the candidate model with no observable risk factors. Specifically, the ratio of the standard deviations of the systematic component of the SDF explained by the first systematic component in the case of the candidate model with zero risk factors to that explained by the market factor in the case of the CAPM candidate model is 1.2.

In contrast to the case of a candidate model with zero risk factors, we find that the systematic component, $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$, of the correction term $\log(\hat{M}^{\alpha}_{\exp,t+1})$ is uncorrelated with nontraded factors. The reason for this zero correlation is that the candidate SDF $\log(M^{\beta,\text{can}}_{\exp,t+1})$ based on the market model subsumes the explanatory power of innovations in VIX, intermediary capital, and dividend yield. We find 27 trading strategies that individually explain more than 30% of variation in $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$. The Size factor (Fama and French, 1993) is one of the most prominent among them with an explanatory power of about 89%, which explains the success of the models developed in Fama and French (1993, 2015).

Figure 6: Time series of the SDF and its components when the market model is the candidate factor model

This figure has four panels, which show the dynamics of the SDF $M_{\exp,t+1}$ and its three components: the asset-specific component $M^{a}_{\exp,t+1}$, the component $M^{\beta,\text{can}}_{\exp,t+1}$ corresponding to the candidate model with the market factor, and the missing systematic component $M^{\beta,\text{mis}}_{\exp,t+1}$.



5.2.2 The Model with a Consumption-Mimicking Portfolio

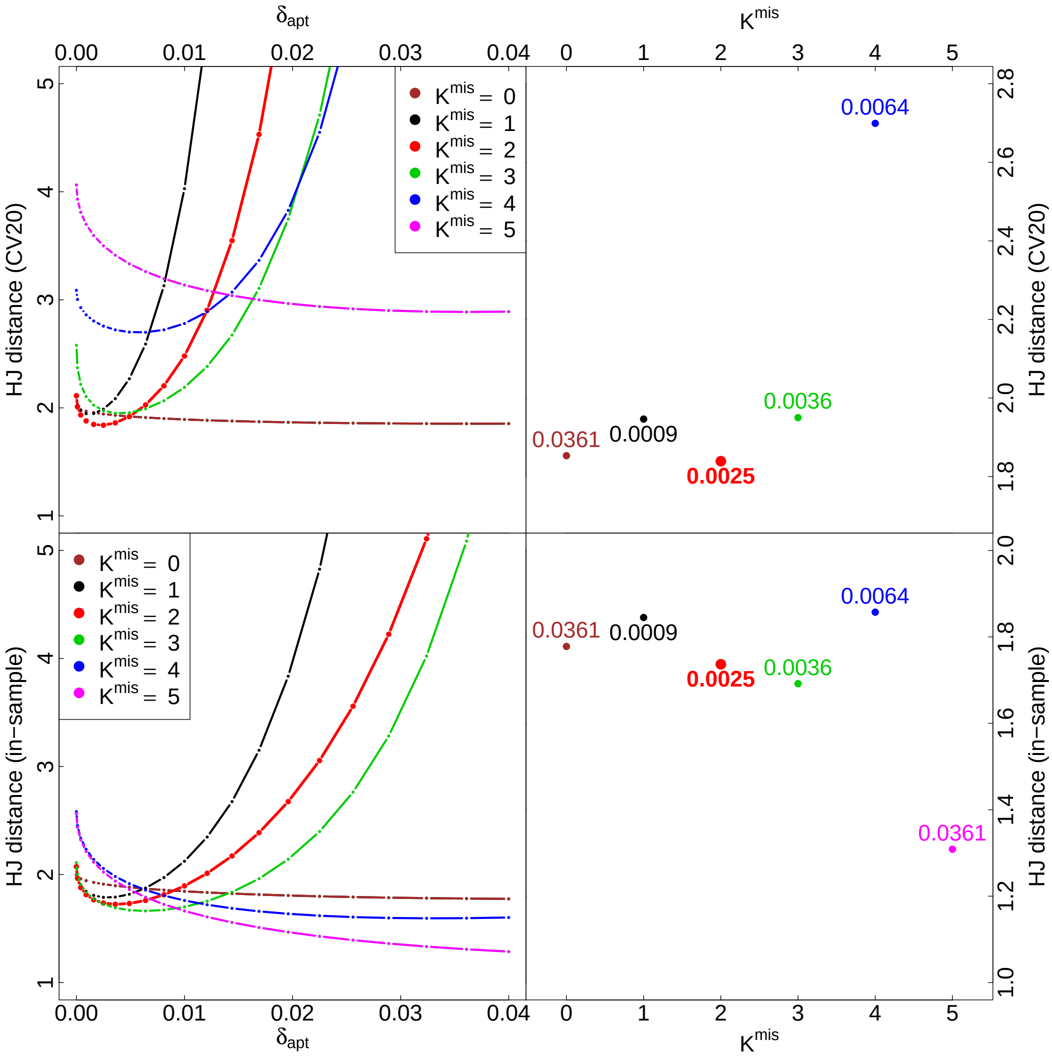
We now consider the case in which the candidate factor model is one with a consumption-mimicking portfolio. We follow the standard approach of Breeden, Gibbons, and Litzenberger (1989) to construct the consumption-mimicking portfolio.[21]

Figure 7 shows that if one starts from a candidate model with the consumption-mimicking portfolio as a single factor, then the estimation procedure selects $K^{\text{mis}} = 2$ and $\delta_{\text{apt}} = 0.0025$. The consumption-mimicking portfolio does not highly correlate with either of the latent factors estimated when correcting the candidate factor model with zero risk factors—

[21] As outlined in Giglio and Xiu (2021), construction of factor mimicking portfolios can be sensitive to the choice of basis assets. They propose a three-stage procedure, which is insensitive to the choice of basis assets. However, their procedure does not allow for asset-specific risk, which we document plays a major role in the risk-return trade-off.

Figure 7: Correction of the model with a consumption-mimicking portfolio using the HJ distance

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model includes only the consumption-mimicking portfolio of Breeden, Gibbons, and Litzenberger (1989). The top two panels show the estimation results based on cross validation, while the bottom two panels show the in-sample results. The panels on the left plot the HJ distance for a given choice of K^{mis} as one varies δ_{apt} (numbers inside the boxes). The panels on the right display the optimal values of the HJ distance for a given choice of K^{mis}.



the correlations are 0.3 and 0 — and therefore, two additional factors are still required to capture the common variation in asset returns.

The results reported in Table 1 imply that the pricing errors for the factor model based on C-CAPM are much larger than those for the corrected factor model, in which the consumption-mimicking portfolio factor is augmented by two latent factors and the vector of the asset-specific components in expected returns, a. Correcting the C-CAPM for misspecification leads to a substantial and statistically significant drop in the HJ distance by 14.99%. Given the estimated annual Sharpe ratio for the aggregate measure of asset-specific risk of SR $= 0.72$ reported in Table 2, the degree of misspecification resulting from omitting the vector a in the candidate model is large. The second main source of misspecification is in missing a level factor in the candidate C-CAPM. The pricing errors in the candidate model are centred around 0.06, whereas the pricing errors in the the corrected model are centred around zero.[22]

Figure 8 reports the estimated time series of the admissible SDF and its components obtained after correcting the candidate C-CAPM and paints a consistent and robust picture. The admissible SDF obtained when correcting the candidate model with the consumption-mimicking portfolio highly correlates with those obtained when correcting the candidate model with zero risk factors and the candidate model with the market factor. Table 4 reports the corresponding correlations.

5.2.3 The Three-Factor Model of Fama and French (1993)

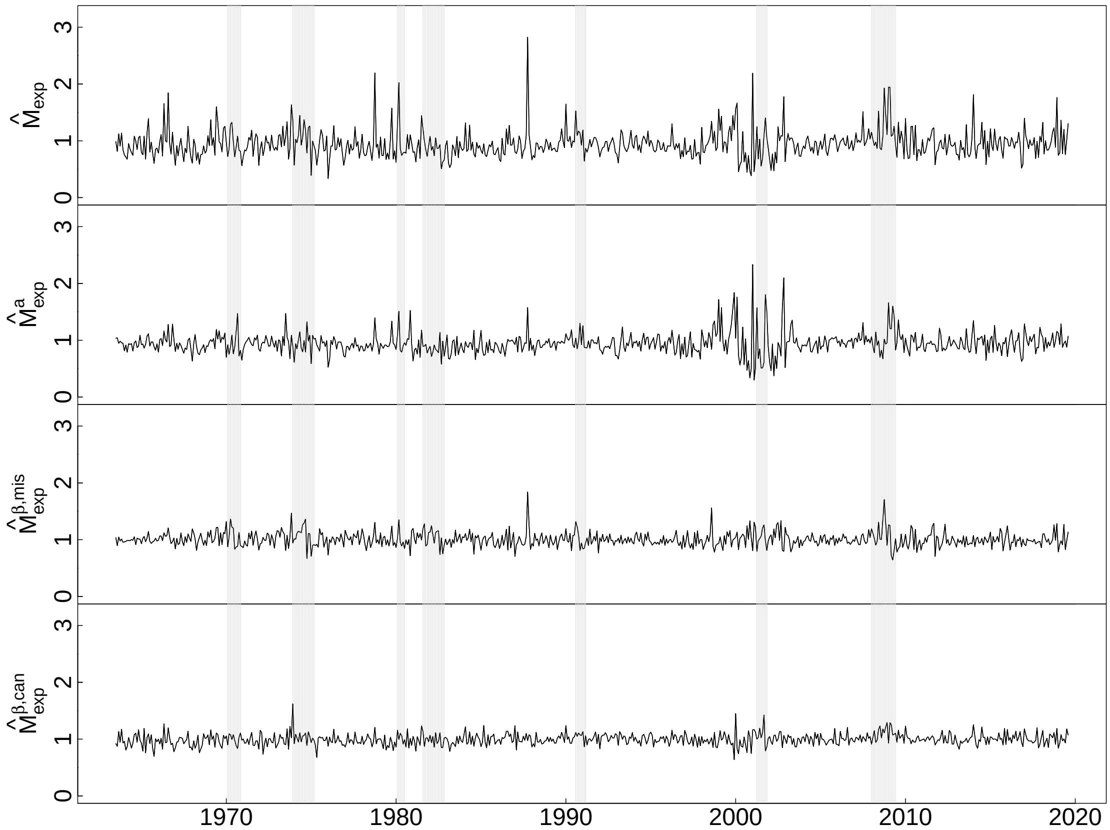
Figure 9 shows that if we use the three-factor model of Fama and French (1993) as the candidate model, our methodology selects *zero* missing sources of systematic risk and optimal $\delta_{\text{apt}} = 0.0036$. At first glance, it may seem surprising that this candidate factor model incorporates all systematic variation in asset returns given that about 19 strategies are necessary to capture the systematic SDF, as explained in the case of the candidate model with zero risk factors in Section 5.1. However, the FF3 model already includes the market and size factors that jointly explain more than 96% of the variation in the systematic component of the SDF. The contribution of each individual remaining factor is so small that it is indistinguishable from asset-specific risk.

Differentiating between the remaining common factors and asset-specific risk is especially challenging given that observable traded factors are noisy versions of the true risk factors that span $\log(\hat{M}^{\beta,\text{mis}}_{\exp,t+1})$ obtained in the case of the model with zero risk factors. As a result, when starting from FF3 as the candidate model, the estimated asset-specific shocks appear confounded with weak latent factors. Moreover, the FF3 model includes the Value factor that correlates more strongly with the reference aggregate measure of asset-specific risk (the correlation is -0.61) than with the systematic component of the SDF (the correlation is 0.14). Thus, FF3 implicitly incorporates some asset-specific risk. Despite these challenges,

[22] To save space, these results are not reported, but they are available upon request.

Figure 8: Time series of the SDF and its components when the model with consumption-mimicking portfolio is the candidate factor model

This figure has four panels, which show the dynamics of the SDF $M_{\exp,t+1}$ and its three components: the asset-specific component $M^{a}_{\exp,t+1}$, the component $M^{\beta,\text{can}}_{\exp,t+1}$ corresponding to the candidate model with the consumption-mimicking portfolio, and the missing systematic component $M^{\beta,\text{mis}}_{\exp,t+1}$.



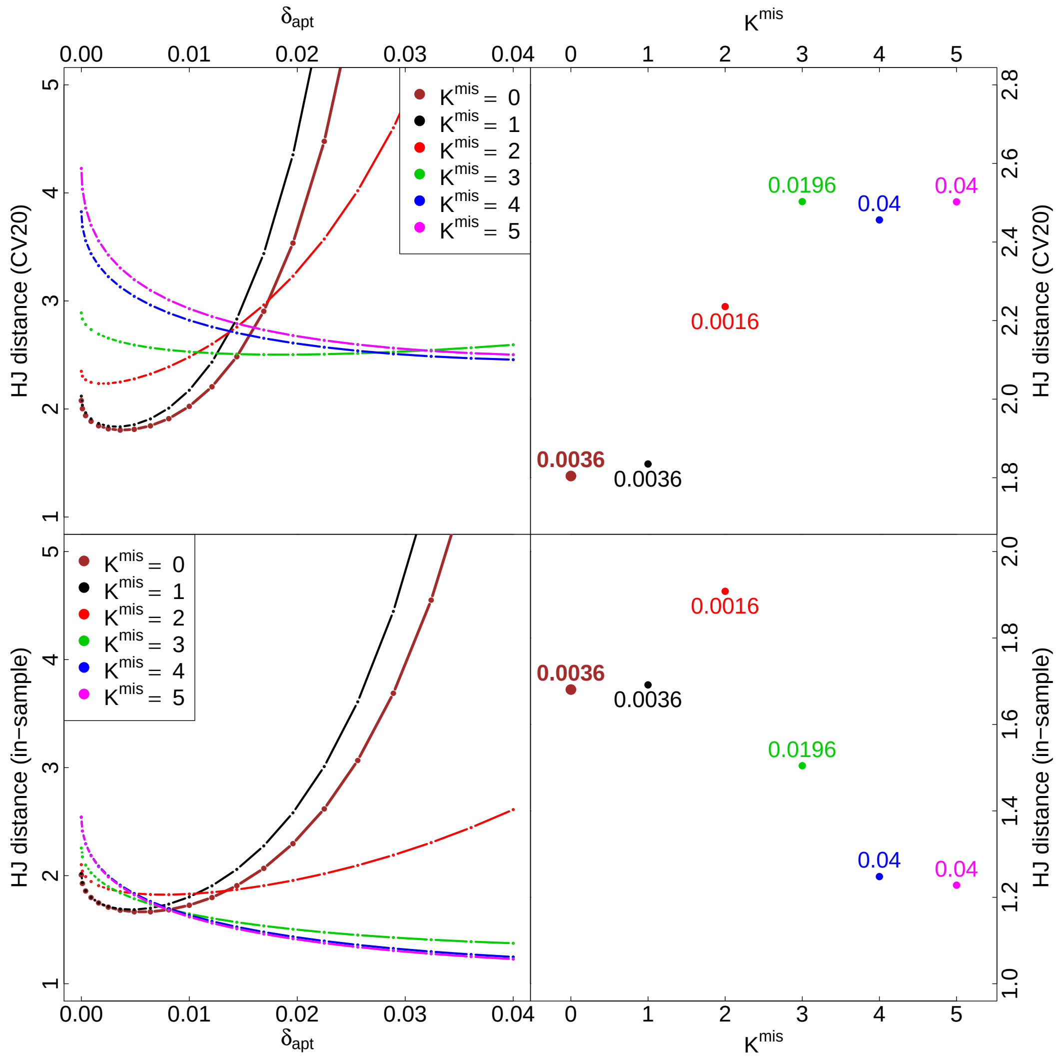
the admissible SDF obtained after correcting the FF3 model is highly correlated with those obtained after correcting the other candidate factor models. Table 4 and Figure 10 support this statement.

Table 2 shows that the estimated admissible SDF implies a higher Sharpe ratio relative to that of the SDFs obtained after correcting the other candidate models. Because the Sharpe ratio is the volatility of the SDF divided by its mean, this increase is due to the noise that observable factors implicitly introduce in the candidate factor model, consistent with Daniel, Mota, Rottke, and Santos (2020).

From the perspective of pricing, accounting for missing asset-specific risk improves the pricing performance of the model, especially with respect to the Size-Momentum and Size-

Figure 9: Correction of FF3 model using HJ distance

This figure illustrates how the HJ distance changes with K^{mis} and δ_{apt}, when the candidate model is the three-factor model (Fama and French, 1993). The top two panels show the estimation results based on cross validation, while the bottom two panels show the in-sample results. The panels on the left plot the HJ distance for a given choice of K^{mis} as one varies δ_{apt}. The panels on the right display the optimal values of the HJ distance for a given choice of K^{mis}.

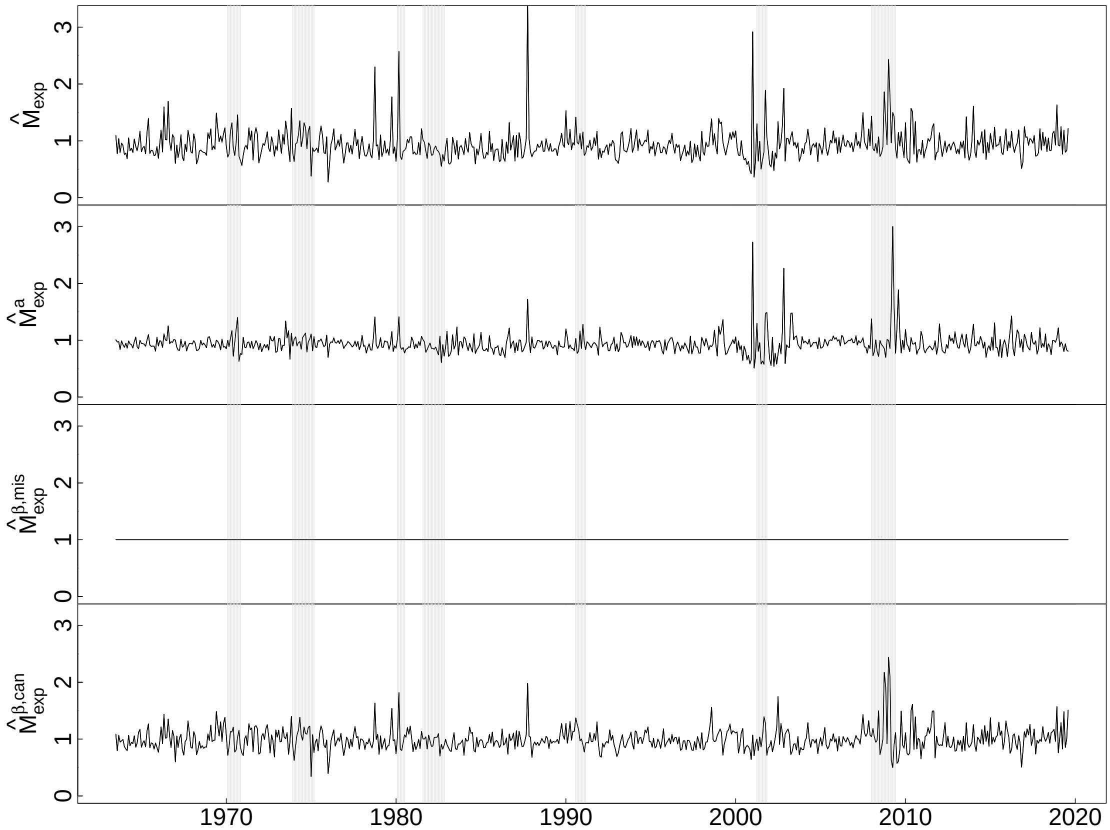


Variance portfolios.[23] The HJ distance drops by a significant 18.82% once we account for compensation for asset-specific risk. The large drop in the HJ distance is not surprising giving the quantitative role of the asset-specific risk for pricing: the aggregate measure of asset-specific risk has the Sharpe ratio of 0.69. Thus, similar to Stambaugh and Yuan (2017), Bryzgalova, Huang, and Julliard (2020), and Clarke (2020) among others, we document sizable misspecification in the FF3 model.

[23]To save space, here we do not report the pricing errors for each portfolio.

Figure 10: Time series of the SDF and its components when the model with consumption-mimicking portfolio is the candidate factor model

This figure has four panels, which show the dynamics of the SDF $M_{\text{exp},t+1}$ and its three components: the asset-specific component $M^{a}_{\text{exp},t+1}$, the component $M^{\beta,\text{can}}_{\text{exp},t+1}$ corresponding to the candidate FF3 model, and the missing systematic component $M^{\beta,\text{mis}}_{\text{exp},t+1}$.



5.3 Out-of-Sample Analysis

To illustrate the robustness of our approach, we run out-of-sample analyses: we evaluate how the candidate and corrected models price two additional cross-section of stock returns. The first dataset, also used in Korsaye, Quaini, and Trojani (2021), includes 100 portfolios sorted by size and book-to-market, 25 portfolios sorted by size and long-term reversal, 25 portfolios sorted by size and short-term reversal, and 49 industry portfolios.[24] The second dataset includes 100 portfolios sorted by size and book-to-market, 100 portfolios sorted by size and operating profitability, 100 portfolios sorted by size and investment, and 49 industry portfolios.

[24]The dataset of Korsaye, Quaini, and Trojani (2021) also includes twenty five momentum portfolios that we exclude because they are present in our 202 basis assets.

Table 5: Cross-sectional Out-of-Sample Analysis

This table reports the relative improvement in the HJ distance when using the admissible SDF constructed by correcting a model with zero risk factors relative to the SDFs of the alternative original and corrected candidate factor models. We analyze the performance of these models on the set of 202 basis assets (column Benchmark) and on two additional datasets not used at the estimation. The improvement is tabulated in %.

		Relative HJ (%)		
Class of models	Candidate model	Benchmark	Dataset 1	Dataset 2
Traditional	CAPM	14.72	9.68	8.92
	C-CAPM	14.83	10.07	8.57
	FF3	15.45	11.57	11.03
PCA-based	PCA1	14.76	9.94	9.16
	PCA2	14.91	10.13	9.34
	PCA3	17.42	15.10	13.46
	PCA4	35.11	33.64	29.36
	PCA5	34.98	33.48	29.27
Corrected SDFs	Zero risk factors	0.00	0.00	0.00
	CAPM	0.77	0.34	0.42
	C-CAPM	−0.16	1.06	0.60
	FF3	−3.37	−1.81	−1.65

Table 5 reports the relative *improvement* in the HJ distance when using the admissible SDF constructed by correcting a model with zero risk factors relative to the SDFs of the various traditional candidate models—both before and after they have been corrected for misspecification. This table shows that the admissible SDFs constructed by correcting the three candidate asset-pricing models that we have considered (CAPM, C-CAPM, FF3) have superior pricing performance compared to the candidate models that are based on just the observable traditional factors or on principal components.

6 Microfoundations for Priced Asset-Specific Risk

In the previous section, we have shown empirically the need to include an aggregate measure of asset-specific risk in the SDF. We could repeat our empirical analysis for other candidate factor models. However, our main conclusion is not going to change—the aggregate measure of asset-specific risk accounts for the lion's share of pricing of the cross-section of asset returns.[25] This is consistent with the empirical finding in Bryzgalova, Huang, and Julliard

[25] If by chance a candidate factor model contains a factor that is correlated with the aggregate measure of asset-specific risk, then one may find that the role of missing asset-specific risk is biased down, as we saw in the case of the corrected FF3 model.

(2020), who undertake a large-scale search for a factor model that prices a cross-section of asset returns but find none. We have also shown that, given a candidate asset-pricing model, adding extra common risk factors to this model cannot proxy for the aggregate measure of asset-specific risk. The aggregate measure of asset-specific risk is a weak factor in the cross-section of asset returns, and therefore, its risk premia cannot be estimated accurately. We show this result explicitly in Appendix A1.

At this point, one may wonder in what kind of economic environment asset-specific risk will be priced. Below we present an example of an equilibrium model that provides microfoundations for the notion that asset-specific risk is priced. Our example relies on the well-known static model of Merton (1987). We show that the equilibrium asset returns and SDF in this model have the same functional forms as those we have specified for our APT model.

In Merton (1987), investors are aware about only a subset of the available securities. This type of "incomplete information" then implies that not only common risk factor but also shocks specific to each security are priced. While this kind of incomplete information may not be the only reason why the aggregate measure of asset-specific risk plays a dominant role in the pricing assets, it is an appealing argument given the large empirical evidence documenting that both retail (Polkovnichenko, 2005; Campbell, 2006; Goetzmann and Kumar, 2008) and institutional investors (Koijen and Yogo, 2019, table 2) invest in only a small number of available stocks.[26]

Below we summarize the main assumptions of the model and then analyze its equilibrium implications for the SDF. For details of the model, we refer the reader to Merton (1987).

Assume that there are N firms in the economy whose end-of-period cash flows are technologically given by[27]:

$$C_i = I_i \left[\mu_i + \eta_i Y + s_i \epsilon_i\right],$$

where, for simplicity, it has been assumed that there is only a single random common factor Y with $E(Y) = 0$ and $E(Y^2) = 1$, with $E(\epsilon_i) = E\left(\epsilon_i \mid \epsilon_1, \ldots, \epsilon_N, Y\right) = 0$, for $i = \{1, \ldots, N\}$,

[26]Other mechanisms, such as market segmentation, institutional restrictions, transaction costs, illiquidity, imperfect divisibility of securities, may lead to the same observable behavior. That is, the modeling framework of Merton (1987) can be viewed as a reduced-form representation of all these microfoundations leading investors to optimally investing in only a subset of available securities.

[27]We have made the following changes to the notation used in Merton (1987) so that it is consistent with the notation in our paper. We denote an investor's risk aversion by γ instead of δ; we denote the total number of assets by N instead of n; we index individual assets by i instead of k; and we denote the asset-specific risk premium by a_i instead of λ_k.

where ϵ_i are asset-specific shocks. Here, I_i denotes the amount of physical investment in firm i and μ_i, η_i, and s_i represent parameters of firm i 's production technology.

Let V_i denote the equilibrium value of firm i at the beginning of the period. If R_i is the equilibrium return per dollar from investing in firm i over the period, then $R_i = C_i/V_i$, and

$$R_i = \mathbb{E}(R_i) + b_i Y + \sigma_i \epsilon_i, \tag{11}$$

where b_i and σ_i are functions of the parameters of the production technology of firm i.

There are two additional securities in the economy, both assumed to be in zero net supply: a security that is a risk-free security with return R_f and the $(N + 1)$st risky security, which combines the risk-free security and a forward contract with cash settlements on the factor Y. Without loss of generality, the forward price of the contract is assumed to be such that the standard deviation of the equilibrium returns on the security is unity. As a result, the return on this security is

$$R_{N+1} = \mathbb{E}(R_{N+1}) + Y. \tag{12}$$

There is a sufficiently large number of investors with a sufficiently disperse distribution of wealth so that each investor acts as a price taker. Each investor is risk averse and exhibits mean-variance preferences over the end-of-period wealth:

$$U^j = E(R^j W^j) - \frac{\gamma^j}{2W^j}\,\mathrm{var}(R^j W^j),$$

where W^j denotes the value of the initial endowment of investor j evaluated at equilibrium prices, R^j denotes the return per dollar on the investor j's optimal portfolio, and $\gamma^j > 0$ is the risk-aversion of investor j.

Investors differ in their information sets. The common part of investors' information sets includes: (i) the return on the risk-free security, (ii) the structure of securities' return given in expression (11), and (iii) the expected return and variance of the forward contract security given in (12). However, different investors have knowledge about the parameters b_i and σ_i for different subsets of securities. The investors who know about security i agree on its characteristics. To simplify the analysis, investors are assumed to have identical risk aversion $\gamma^j = \gamma$ and identical initial wealth $W^j = W$.

The optimal solution of the each investor's portfolio problem allows us to obtain the aggregate demand for each security. Equating this to the aggregate supply for each security leads to the equilibrium expected return for asset i (Merton, 1987, eq. (16)):

$$\mathbb{E}(R_i) = R_f + \gamma b_i b + \gamma x_i \sigma_i^2 / q_i, \quad \text{for} \quad i = \{1, \ldots, N\}, \tag{13}$$

where x_i is the fraction of the market portfolio invested in asset i, $b = \sum_{i=1}^{N} x_i b_i$, and q_i is the fraction of investors who know about security i.

Denoting the return on the market as $R_m = \sum_{i=1}^{N} x_i R_i$, Merton (1987, eq. (24)) obtains the equilibrium expected excess return on the market:

$$\mathbb{E}(R_m) - R_f = \gamma \operatorname{var}(R_m) + a_m, \tag{14}$$

where $a_m = \sum_{i=1}^{N} x_i a_i$,

$$\begin{aligned} a_i &= (1 - q_i)\Delta_i, \\ \Delta_i &= \mathbb{E}(R_i) - R_f - b_i(\mathbb{E}(R_{N+1}) - R_f). \end{aligned}$$

Equations (11) and (14) then imply

$$R_i - R_f = \beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m + b_i Y + \sigma_i \epsilon_i, \tag{15}$$

where β_i denotes the covariance of the return on security i with the return on the market portfolio divided by the variance of the market return. Equation (15) contains Y on the right-hand side. We substitute out Y by using the definition of the market portfolio return along with equations (11) and (13), to obtain

$$R_i - R_f = a_i - \beta_i a_m + \beta_i(\mathbb{E}(R_m) - R_f) + \frac{b_i}{b}(R_m - \mathbb{E}(R_m)) + \sigma_i \epsilon_i.$$

We now derive the SDF in this economy. In particular, we consider the case where the number of available assets is large, that is, $N \to \infty$. In this case, as we show in Appendix B in the proof for Proposition 7: (i) $\beta_i \to b_i/b$, (ii) $a_m \to 0$, and (iii) the market return is asymptotically orthogonal to all asset-specific shocks, ϵ_i. The proposition below then shows that this leads to equilibrium asset returns and an SDF that have the same functional form as those for the APT model, as specified in equations (1) and (6), respectively.

Proposition 7. *When the number of assets is large, $N \to \infty$, equilibrium asset returns are*

$$\begin{aligned} R_i - R_f &= a_i + \beta_i(\mathbb{E}(R_m) - R_f) + \beta_i(R_m - \mathbb{E}(R_m)) + \sigma_i \epsilon_i, \qquad (16) \\ &= a_i + \beta_i(R_m - R_f) + \sigma_i \epsilon_i, \end{aligned}$$

and the equilibrium SDF is

$$M = \underbrace{-\frac{1}{R_f} \sum_{i=1}^{N} \left(\frac{a_i}{\sigma_i} \epsilon_i\right)}_{M^a} + \underbrace{\frac{1}{R_f} - \frac{\mathbb{E}(R_m) - R_f}{R_f \cdot \operatorname{var}(R_m)}(R_m - \mathbb{E}(R_m))}_{M^\beta}. \tag{17}$$

The SDF in (17) consists of two components representing adjustments for risk: the first one represents an aggregate measure of asset-specific risk M^a, and the second one systematic risk M^β, exactly as prescribed by the SDF under the APT. Note that a_i in (16) represents the compensation for asset-specific risk, because

$$a_i = -\operatorname{cov}\left(R_i - R_f, -\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{a_i}{\sigma_i}\epsilon_i\right)\right) \times R_f,$$

which coincides with the elements of the vector a in the APT. Naturally, the other part of the risk premium in (16), $\beta_i(\bar{R}_m - R_f)$, is compensation for exposure to systematic risk, represented by market risk because of the assumption of a single common factor:

$$\beta_i(\mathbb{E}(R_m) - R_f) = -\operatorname{cov}\left(R_i - R_f, -\frac{\mathbb{E}(R_m) - R_f}{R_f \cdot \operatorname{var}(R_m)}(R_m - \mathbb{E}(R_m))\right) \times R_f.$$

If all investors were fully informed about all N assets, that is, $q_i = 1$, then $a_i = 0$, and the results in (16) and (17) simplify to the expressions for security returns and the SDF under the CAPM, respectively.

Thus, the above discussion shows that there are equilibrium models that support the notion that asset-specific risk is priced. Moreover, Proposition 7 shows that the result that asset-specific risk is priced is not limited to an economy with a finite number of assets.

7 Conclusion

A fundamental challenge in finance is to price assets. The main difficulty when pricing assets is to determine how exactly to adjust their returns for risk. The literature has proposed a large number of alternative factor models to accomplish this task. Despite the proliferation of systematic risk factors, referred to as the factor zoo (Cochrane, 2011), there is still a sizable pricing error, called alpha. This leads one to the question posed in the title of this paper: "What is missing in asset-pricing factor models?"

We challenge the conventional wisdom that only systematic sources of risk receive compensation in financial markets by showing that asset-specific risk is also compensated. That is, the pricing error alpha implied by factor models includes compensation not only for missing common risk factors but also for asset-specific risk. Theoretically, we demonstrate this key insight through the lens of the SDF under the assumptions of the APT and support it by demonstrating that an equilibrium model such as Merton (1987) is consistent with our insight. Empirically, we show that an aggregate measure of asset-specific risk, a component of the admissible SDF represented by a linear combination of asset-specific shocks, accounts for 56% of the variation in the admissible SDF.

What is missing in virtually all factor models is compensation for this asset-specific risk. We show that even though some conventional factors, for example, Value, have sizeable correlations with the aggregate measure of asset-specific risk, adding an arbitrary number of factors to a candidate factor model will not lead to an admissible SDF.

The methodology we develop in this paper applies widely—to reduced-form factor models, but also to partial- and general-equilibrium asset pricing models—without needing to identify which factors (strong or weak) are missing. In terms of estimation, the methodology is designed and feasible for a large number of assets; in fact, its performance improves with the number of assets considered. Our novel insight, which establishes the importance of asset-specific risk, is crucial both for empiricists wanting to resolve the factor zoo and for theorists wishing to develop microfounded models of asset pricing.

A Appendix: Can one recover M^a_{t+1} using observable variables?

In this appendix, we show that when the asset-specific correction term M^a_{t+1} is correlated with some observed factors, then these factors are necessarily weak (Lettau and Pelger, 2020). Thus even if one could find the complete set of observed factors spanning M^a_{t+1}, one could not estimate accurately the corresponding risk premia.[28] As a result, the component M^a_{t+1} cannot be estimated the same way as $M^{\beta,can}_{t+1}$ and $M^{\beta,mis}_{t+1}$.

Proposition A1. *Under Assumptions 1 and 2, assume that there are no missing systematic risk factors, that is $K^{mis} = 0$, implying*

$$R_{t+1} - R_{ft} = a + \beta^{\text{can}} f^{\text{can}}_{t+1} + e_{t+1},$$

where

$$T^{-1}\sum_{t=1}^{T}(e_t - \bar{e})(e_t - \bar{e})' \to_p V_e,$$

$$T^{-1}\sum_{t=1}^{T}(f^{can}_t - \bar{f}^{can})(f^{can}_t - \bar{f}^{can})' \to_p V^{can}_f > 0,$$

as $T \to \infty$.

If, for an observed factor f^{idio}_{t+1}, $a'V_e^{-1}e_{t+1} = f^{\text{idio}}_{t+1} - \mathbb{E}(f^{\text{idio}}_{t+1})$, then f^{idio}_{t+1} must be a weak factor.

Proof: Without loss of generality, given that the e_t is uncorrelated with f^{can}_t by Assumption 1, assume that f^{can}_t and f^{idio}_t are orthogonal in sample, that is $\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})f^{\text{can}}_t = 0_{K^{\text{can}}}$. Considering the time-series regression

$$R_{it} - R_{ft} = \beta_0 + \beta^{\text{can}} f^{\text{can}}_t + \beta^{\text{idio}} f^{\text{idio}}_t + u_t,$$

the OLS estimator of β^{idio} satisfies

$$\hat{\beta}^{\text{idio}}_i = \frac{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})(R_{it} - R_{ft})}{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})^2} = \frac{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})(a_i + \beta^{can\prime}_i f^{can}_t + e_{it})}{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})^2}$$

$$= \frac{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})a_i}{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})^2} + \frac{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})e_{it}}{\sum_{t=1}^{T}(f^{\text{idio}}_t - \bar{f}^{\text{idio}})^2}$$

[28] See Anatolyev and Mikusheva (2021) for the full asymptotic analysis of estimated risk premia associated with observable weak factors.

$$= \frac{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))a_i}{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))^2} + \frac{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))e_{it}}{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))^2}$$

$$= 0 + \frac{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))e_{it}}{\sum_{t=1}^{T}(a'V_e^{-1}(e_{t+1}-\bar{e}))^2} \rightarrow_p \frac{a'V_e^{-1}V_e\iota_i}{a'V_e^{-1}V_eV_e^{-1}a} = \frac{a_i}{a'V_e^{-1}a} = \frac{a_i}{\delta_{apt}},$$

where we set ι_i equal to the ith row/column of I_N, and where we assumed, without loss of generality, that the APT constraint binds, i.e., $a'V_e^{-1}a = \delta_{apt}$. □

Proposition A1 implies that f_t^{idio} is a weak factor because its loading satisfies $\beta^{\text{idio}\prime}\beta^{\text{idio}} < \delta < \infty$ for any N. This makes estimation of the corresponding risk premium problematic. In fact, its second-pass estimator satisfies

$$\hat{\lambda}^{idio} = 1 + \frac{O_p(N^{-1}T^{-\frac{1}{2}})}{O(N^{-1})}. \tag{A1}$$

Therefore, $\hat{\lambda}^{idio}$ is meaningless especially when N is large. For a formal analysis see Anatolyev and Mikusheva (2021). To see how (A1) arises, it is enough to study the behaviour of the simple case of the two-pass estimator when $K^{can} = K^{idio} = 1$, satisfying

$$\hat{\lambda} = \begin{bmatrix} \hat{\lambda}^{can} \\ \hat{\lambda}^{\text{idio}} \end{bmatrix} = \begin{bmatrix} \beta^{can\prime}\beta^{can} & \beta^{can\prime}\beta^{\text{idio}} \\ \beta^{\text{idio}\prime}\beta^{can} & \beta^{\text{idio}\prime}\beta^{\text{idio}} \end{bmatrix}^{-1} \begin{bmatrix} \beta^{can\prime}(\bar{R}-\bar{R}_f) \\ \beta^{\text{idio}\prime}(\bar{R}-\bar{R}_f) \end{bmatrix}$$

$$= \begin{bmatrix} \beta^{can\prime}\beta^{can} & \beta^{can\prime}a \\ a'\beta^{can} & a'a \end{bmatrix}^{-1} \begin{bmatrix} \beta^{can\prime}(\bar{R}-\bar{R}_f) \\ a'(\bar{R}-\bar{R}_f) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{can} + \bar{f}^{can} - \mathbb{E}f_t^{can} \\ 1 \end{bmatrix} + \begin{bmatrix} \beta^{can\prime}\beta^{can} & \beta^{can\prime}a \\ a'\beta^{can} & a'a \end{bmatrix}^{-1} \begin{bmatrix} \beta^{can\prime}\bar{e} \\ a'\bar{e} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{can} + \bar{f}^{can} - \mathbb{E}f_t^{can} \\ 1 \end{bmatrix} +$$

$$\frac{1}{\frac{a'a}{N}\frac{\beta^{can\prime}\beta^{can}}{N} - (\frac{\beta^{can\prime}a}{N})^2} \begin{bmatrix} \frac{a'a}{N} & -\frac{\beta^{can\prime}a}{N} \\ -\frac{\beta^{can\prime}a}{N} & \frac{\beta^{can\prime}\beta^{can}}{N} \end{bmatrix} \begin{bmatrix} \frac{\beta^{can\prime}\bar{e}}{N} \\ \frac{a'\bar{e}}{N} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{can} + \bar{f}^{can} - \mathbb{E}f_t^{can} \\ 1 \end{bmatrix} + \frac{1}{O(\frac{1}{N})} \begin{bmatrix} O(\frac{1}{N}) & O(\frac{1}{N^{\frac{1}{2}}}) \\ O(\frac{1}{N^{\frac{1}{2}}}) & O(1) \end{bmatrix} \begin{bmatrix} O_p(\frac{1}{(NT)^{\frac{1}{2}}}) \\ O_p(\frac{1}{NT^{\frac{1}{2}}}) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^{can} + \bar{f}^{can} - \mathbb{E}f_t^{can} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{O_p(N^{-\frac{3}{2}}T^{-\frac{1}{2}})}{O(N^{-1})} \\ \frac{O_p(N^{-1}T^{-\frac{1}{2}})}{O(N^{-1})} \end{bmatrix}.$$

where we set $\delta_{apt} = 1$ for simplicity.

Whereas the two-pass estimator for the candidate risk factor is always consistent for $\lambda^{can} + \bar{f}^{can} - \mathbb{E}f_t^{can}$ when either N or T or both diverge, coinciding with λ^{can} when $T \to \infty$, the estimated risk premium $\hat{\lambda}^{\text{idio}}$ of f_{t+1}^{idio} has an undefined limit, especially when N becomes large. Therefore, the traditional two-pass regression approach does not permit to accurately estimate M_{t+1}^a. In contrast, the methodology described in this paper explains how to construct an accurate estimate of M_{t+1}^a, which does not even rely on the existence and identification of f_{t+1}^{idio}.

Theorem A1 extends to the multivariate case, that is when f_{t+1}^{idio} is a vector. It also extends to the case when f_t^{idio} spans the asset-specific risk *imperfectly*, that is for

$$(f_{t+1}^{\text{idio}} - E(f_{t+1}^{\text{idio}}) = \gamma a' V_e^{-1} e_{t+1} + \eta_{t+1}$$

where $E\eta_{t+1} = 0, corr(a'V_e^{-1}e_{t+1}, \eta_{t+1}) = 0, \text{var}(\eta_{t+1}) = \sigma_\eta^2 = \text{var}(f_{t+1}^{\text{idio}})(1 - \rho^2_{f^{\text{idio}}, M^a})$. Then

$$\hat{\beta}_i^{\text{idio}} = \beta_i^{\text{idio}} + O_p(T^{-1/2}) = \frac{\gamma a_i}{\gamma^2 a' V_e^{-1} a + \sigma_\eta^2} + O_p(T^{-1/2}),$$

implying that f_{t+1}^{idio} continues to be a weak factor, given $\beta^{\text{idio}\prime}\beta^{\text{idio}} < \infty$, and estimation of its risk premia is still problematic, as discussed above.

B Appendix: Proofs

This appendix contains the proofs for all the propositions in the manuscript. We use the following notation: I_a denotes the identity matrix of dimension $a \times a$; $0_{a\times b}$ denotes a matrix of zeros with a rows and b columns and 0_a denotes a vector of zeros with a entries ; $A > 0$ means that the matrix A is positive definite; $\|\cdot\|$ denotes the Euclidean norm; $\mathbb{E}(\cdot)$ denotes the expectation operator; $a_N = O(b_N)$ and $a_N = o(b_N)$ with $b_N > 0$ it means that $|a_N|/b_N$ is bounded and $|a_N|/b_N \to 0$, respectively, as $N \to \infty$; $\xrightarrow{p}$ denotes convergence in probability; $a_N = O_p(b_N)$ and $a_N = o(_p b_N)$ with $b_N > 0$ it means that $|a_N|/b_N$ is bounded in probability and $|a_N|/b_N \xrightarrow{p} 0$, respectively, as $N \to \infty$.

B.1 Lemmas

We start by providing a set of lemmas needed to prove our results.

Lemma B.1. *For a random vector $z \sim N(\mu_z, \Sigma_z)$, and any constant vector d, one gets:*

(i)

$$\mathbb{E}(e^{d'z}) = e^{d'\mu_z + \frac{1}{2}d'\Sigma_z d}.$$

(ii)

$$\mathbb{E}(ze^{d'z}) = \mu^* e^{\frac{1}{2}(\mu^{*\prime}\Sigma_z^{-1}\mu^* - \mu_z'\Sigma_z^{-1}\mu_z)},$$

setting

$$\mu^* = (\mu_z + \Sigma_z d).$$

An alternative expression is

$$\mathbb{E}(ze^{d'z}) = (\mu_z + \Sigma_z d)e^{(\frac{1}{2}d'\Sigma_z d + \mu_z' d)}.$$

Proof: (i) is well-known. For (ii), denoting by n_z the dimensionality of the vector z,

$$\mathbb{E}(ze^{d'z}) = \frac{1}{(\sqrt{2\pi})^{n_z}|\Sigma_z|^{\frac{1}{2}}} \int ze^{d'z}e^{-\frac{1}{2}(z-\mu_z)'\Sigma_z^{-1}(z-\mu_z)}dz.$$

Then

$$\begin{aligned}
e^{d'z}e^{-\frac{1}{2}(z-\mu_z)'\Sigma_z^{-1}(z-\mu_z)} &= e^{d'z - \frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \mu_z'\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + (\Sigma_z d + \mu_z)'\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}z'\Sigma_z^{-1}z - \frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \mu^{*\prime}\Sigma_z^{-1}z} \\
&= e^{-\frac{1}{2}\mu_z'\Sigma_z^{-1}\mu_z + \frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*}e^{-\frac{1}{2}z'\Sigma_z^{-1}z + \mu^{*\prime}\Sigma_z^{-1}z - \frac{1}{2}\mu^{*\prime}\Sigma_z^{-1}\mu^*}
\end{aligned}$$

$$= e^{-\frac{1}{2}\mu_z' \Sigma_z^{-1} \mu_z + \frac{1}{2}\mu^{*\prime} \Sigma_z^{-1} \mu^*} e^{-\frac{1}{2}(z-\mu^*)' \Sigma_z^{-1}(z-\mu^*)},$$

implying

$$\mathbb{E}(ze^{d'z}) = e^{-\frac{1}{2}\mu_z' \Sigma_z^{-1} \mu_z + \frac{1}{2}\mu^{*\prime} \Sigma_z^{-1} \mu^*} \left(\frac{1}{(\sqrt{2\pi})^{n_z} |\Sigma_z|^{\frac{1}{2}}} \int z e^{-\frac{1}{2}(z-\mu_z^*)' \Sigma_z^{-1}(z-\mu_z^*)} dz \right).$$

□

Lemma B.2. *Let $V_R = \beta V_f \beta' + V_\varepsilon$ with $N \times N$ and $K \times K$ matrices $V_\varepsilon > 0$ and $V_f > 0$, and a full-column rank $N \times K$ matrix β satisfying $\beta' V_\varepsilon{}^{-1} \beta / N \to D > 0$ for some matrix D. Then:*

$$\beta' V_R^{-1} \beta \to V_f^{-1}.$$

Proof: By the Sherman Morrison formula,

$$V_R^{-1} = V_\varepsilon{}^{-1} - V_\varepsilon{}^{-1} \beta (V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1},$$

pre-multiplying by β', and re-arranging terms, yields

$$\begin{aligned}
\beta' V_R^{-1} &= \beta' V_\varepsilon{}^{-1} - \beta' V_\varepsilon{}^{-1} \beta (V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1} \\
&= (I_K - \beta' V_\varepsilon{}^{-1} \beta (V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1}) \beta' V_\varepsilon{}^{-1} \\
&= ((V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta) - \beta' V_\varepsilon{}^{-1} \beta)(V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1} \\
&= V_f^{-1} (V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1}.
\end{aligned}$$

Post-multiplying by β and taking the limit as $N \to \infty$ gives

$$\beta' V_R^{-1} \beta \to V_f^{-1},$$

because $(V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1} \beta \to I_K$. □

Lemma B.3. *Under the assumptions of Lemma B.2 and for a random vector e with mean zero and covariance V_ε:*

$$\beta' V_R^{-1} e = O_p(N^{-\frac{1}{2}}).$$

Proof: Pre-multiplying by β' and post-multiplying by e one obtains:

$$\beta' V_R^{-1} e = V_f^{-1} (V_f^{-1} + \beta' V_\varepsilon{}^{-1} \beta)^{-1} \beta' V_\varepsilon{}^{-1} e.$$

The result follows noticing that $\beta' V_\varepsilon{}^{-1} e = O_p((\beta' V_\varepsilon{}^{-1} \beta)^{\frac{1}{2}})$ using the result $X = O_p((\mathbb{E}(X))^{\frac{1}{2}})$ for any random variable X with finite second moment. □

Lemma B.4. *Under the assumptions of Lemma B.2 and letting $V_\varepsilon = \beta^{\text{mis}}\beta^{\text{mis}\prime} + V_e$ for a column full-rank $N \times K^{\text{mis}}$ matrix β^{mis} and a $N \times N$ matrix $V_e > 0$ such that $\beta^{\text{mis}\prime} V_e^{-1}\beta^{\text{mis}}/N \to E > 0$ for some matrix E, then:*

$$\beta^{\text{mis}\prime} V_\varepsilon^{-1}\beta = O(1).$$

Proof: Along the same lines of the proof to Lemma B.2

$$\beta^{\text{mis}\prime} V_\varepsilon^{-1}\beta = (I_{K^{\text{mis}}} + \beta^{\text{mis}\prime} V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime} V_e^{-1}\beta = (\frac{I_{K^{\text{mis}}}}{N} + \frac{\beta^{\text{mis}\prime} V_e^{-1}\beta^{\text{mis}}}{N})^{-1}\frac{\beta^{\text{mis}\prime} V_e^{-1}\beta}{N} = O(1),$$

where, by the Schwartz inequality, $\| \beta^{\text{mis}\prime} V_e^{-1}\beta \| \leq \| \beta^{\text{mis}\prime} V_e^{-1}\beta^{\text{mis}} \|^{\frac{1}{2}} \| \beta' V_e^{-1}\beta \|^{\frac{1}{2}}$. □

Lemma B.5. *Under the assumptions of Lemma B.4:*

$$\beta^{\text{mis}\prime} V_\varepsilon^{-1}\beta^{\text{mis}} \to I_{K^{\text{mis}}}.$$

Proof: This is a special case of Lemma B.2. □

Lemma B.6. *Under the assumptions of Lemma B.4 and for a random vector e with mean zero and covariance V_e: Then:*

$$\beta^{\text{mis}\prime} V_\varepsilon^{-1} e = O_p(N^{-\frac{1}{2}}).$$

Proof: This is a special case of Lemma B.3. □

Lemma B.7. *Under the assumptions of Lemma B.4, setting $V_R = \beta V_f \beta' + V_\varepsilon$, $\alpha = a + \beta^{\text{mis}}\lambda_m$ for an $K^{\text{can}} \times 1$ vector of constants λ_m and $a' V_e^{-1} a = O(1)$, then:*

$$\alpha' V_R^{-1}\beta = O(N^{-\frac{1}{2}}).$$

Proof: Given

$$\begin{aligned}
\alpha' V_R^{-1}\beta &= a' V_R^{-1}\beta + \lambda_m' \beta^{\text{mis}\prime} V_R^{-1}\beta \\
&= a'(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta(V_f^{-1} + \beta' V_\varepsilon^{-1}\beta)^{-1}\beta' V_\varepsilon^{-1})\beta \\
&\quad + \lambda_m' \beta^{\text{mis}\prime}(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta(V_f^{-1} + \beta' V_\varepsilon^{-1}\beta)^{-1}\beta' V_\varepsilon^{-1})\beta \\
&= a' V_\varepsilon^{-1}\beta(V_f^{-1} + \beta' V_\varepsilon^{-1}\beta)^{-1}V_f^{-1} + \lambda_m' \beta^{\text{mis}\prime} V_\varepsilon^{-1}\beta(V_f^{-1} + \beta' V_\varepsilon^{-1}\beta)^{-1}V_f^{-1} \\
&= O(N^{-\frac{1}{2}}) + O(N^{-1}),
\end{aligned}$$

by Lemma B.4, the bound $\| a' V_\varepsilon^{-1}\beta \| \leq \| a' V_\varepsilon^{-1} a \|^{\frac{1}{2}} \| \beta' V_\varepsilon^{-1}\beta \|^{\frac{1}{2}}$ and

$$|a' V_\varepsilon^{-1} a| = |a'(V_e^{-1} - V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime} V_e^{-1})a|$$

$$\leq |a'V_e{}^{-1}a| + |aV_e{}^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e{}^{-1}a|$$

$$\leq |a'V_e{}^{-1}a| + |aV_e{}^{-1}a|^{\frac{1}{2}} \parallel \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}} \parallel^{\frac{1}{2}} \parallel (I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}})^{-1} \parallel |aV_e{}^{-1}a|^{\frac{1}{2}} \parallel \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}} \parallel^{\frac{1}{2}}$$

$$= |a'V_e{}^{-1}a| + |aV_e{}^{-1}a| \parallel \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}} \parallel\parallel (I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e{}^{-1}\beta^{\text{mis}})^{-1} \parallel = O(1),$$

together with $\parallel \beta' V_\varepsilon{}^{-1}\beta \parallel^{\frac{1}{2}} = O(N^{\frac{1}{2}})$. □

Proof of Proposition 1

The idea of the proof is to apply the APT mathematics to $R_{t+1} - \beta\big(f_{t+1} + \lambda - \mathbb{E}(f_{t+1})\big) = \alpha + \varepsilon_{t+1}$. By Chamberlain and Rothschild (1983, Theorem 4) the error covariance matrix has an *approximate* factor structure, and satisfies

$$V_\varepsilon = \beta^{\text{mis}}\beta^{\text{mis}\prime} + V_e,$$

where $V_e > 0$ with uniformly bounded eigenvalues, and by Chamberlain and Rothschild (1983, Corollary 2) there exists a vector λ^{mis} such that $(\alpha - \beta^{\text{mis}}\lambda^{\text{mis}})'V_e^{-1}(\alpha - \beta^{\text{mis}}\lambda^{\text{mis}})$ is bounded for any N, where β^{mis} is the $N \times K^{mis}$ matrix made by the K^{mis} dominant eigenvectors of V_ε (the eigenvectors associated with the largest K^{can} eigenvalues), each multiplied by the square-root of the corresponding eigenvalues. We set $a = \alpha - \beta^{\text{mis}}\lambda^{\text{mis}}$.

By the Sherman-Morrison-Woodbury decomposition,

$$V_\varepsilon^{-1} = V_e^{-1} - V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}.$$

Therefore, by substitution,

$$\begin{aligned}
\alpha' V_\varepsilon^{-1}\alpha &= \alpha' V_e^{-1}\alpha - \alpha' V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\alpha \\
&= (\beta^{\text{mis}}\lambda^{\text{mis}} + a)'V_e^{-1}(\beta^{\text{mis}}\lambda^{\text{mis}} + a) \\
&\quad - (\beta^{\text{mis}}\lambda^{\text{mis}} + a)'V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}(\beta^{\text{mis}}\lambda^{\text{mis}} + a) \\
&= \lambda^{\text{mis}\prime}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} \\
&\quad - \lambda^{\text{mis}\prime}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{mis}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} \\
&\quad + a'V_e^{-1}a - a'V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}a \\
&\quad + 2a'V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} - 2a'V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}.
\end{aligned}$$

We now show that $\alpha' V_\varepsilon^{-1}\alpha$ is bounded for any N. We study each of the terms on the right-hand side of the last equality sign, one by one. The first and second term satisfy

$$\lambda^{\text{mis}\prime}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}} - \lambda^{\text{mis}\prime}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}})^{-1}\beta^{\text{mis}\prime}V_e^{-1}\beta^{\text{mis}}\lambda^{\text{mis}}$$

$$
\begin{aligned}
&= \lambda^{\text{mis}\prime}(I_N - \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1}) \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}} \lambda^{\text{mis}} \\
&= \lambda^{\text{mis}\prime}(I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}} \lambda^{\text{mis}} \leq \lambda^{\text{mis}\prime} \lambda^{\text{mis}},
\end{aligned}
$$

because $I_{K^{\text{can}}} - (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}}$ is positive semidefinite. Next, for the third term, denoting by $g_{NN}(V_e)$ the smallest eigenvalue of V_e,

$$
a' V_e^{-1} a \leq a'a / g_{NN}(V_e) = O(1).
$$

Now, the jth element of $a' V_e^{-1} \beta^{\text{mis}}$, obtained by considering the jth column of β^{mis}, for every $1 \leq j \leq K^{\text{mis}}$, satisfies

$$
|a' V_e^{-1} \beta_j^{\text{mis}}| \leq (a' V_e^{-1} a)^{\frac{1}{2}} (\beta_j^{\text{mis}\prime} V_e^{-1} \beta_j^{\text{mis}})^{\frac{1}{2}} = O(N^{\frac{1}{2}}),
$$

Moreover, the (i,j)th element, for every $1 \leq i,j \leq K^{\text{can}}$, of $(\beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})$ is equal to $\beta_j^{\text{mis}\prime} V_e^{-1} \beta_i^{\text{mis}}$. Therefore, $(I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1}$ decreases at rate $O(N^{-1})$. On the other hand, along the same lines, the elements of the vector $\beta^{\text{mis}\prime} V_e^{-1} a$ diverge at most at rate $O(N^{\frac{1}{2}})$. Collecting terms, the fourth term satisfies:

$$
|a' V_e^{-1} \beta^{\text{mis}} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} a| = O(1).
$$

Concerning the last two terms, it turns out that their difference converges to zero. In fact,

$$
\begin{aligned}
&\left| 2a' V_e^{-1} \beta^{\text{mis}} \lambda^{\text{mis}} - 2a' V_e^{-1} \beta^{\text{mis}} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}} \lambda^{\text{mis}} \right| \\
&= 2 \left| a' V_e^{-1} \beta^{\text{mis}} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \lambda^{\text{mis}} \right| \\
&\leq (a' V_e^{-1} \beta^{\text{mis}} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_e^{-1} a)^{\frac{1}{2}} (\lambda^{\text{mis}\prime} (I_{K^{\text{can}}} + \beta^{\text{mis}\prime} V_e^{-1} \beta^{\text{mis}})^{-1} \lambda^{\text{mis}})^{\frac{1}{2}} \\
&= O(N^{-\frac{1}{2}}).
\end{aligned}
$$

□

Observe that Proposition 1 assumes the presence of at least one omitted systematic risk factor. The case when there are no missing systematic factors, that is, $K^{\text{mis}} = 0$, in turn implied when V_ε has all bounded eigenvalues, coincides with the classical APT.

Proof of Proposition 2

In general, the SDF can always be re-written as *linear* in payoffs or excess returns. Without loss of generality we assume that the candidate factors are traded and expressed as excess returns, or as difference of excess returns (such as highest decile portfolio minus lowest decile portfolio) implying $\mathbb{E}f_{t+1}^{\text{can}} = \lambda^{\text{can}}$. In fact, if the factors are non traded, by standard arguments one replaces them with the corresponding (traded) mimicking portfolios.

Therefore, given that in the APT payoffs (excess returns) are linear in f_t^{can}, $\varepsilon_{N,t}$, and 1, then the SDF under the APT satisfies:

$$M_{t+1} = \mathbb{E}(M_{t+1}) + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}, \tag{B1}$$

for some given coefficient vector b, which is $K^{\text{can}} \times 1$, and coefficient vector c, which is $N \times 1$. We determine b and c below whereas $\mathbb{E}(M_{t+1}) = R_{ft}^{-1}$, and given that we assumed the existence of the risk-free asset, $R_{ft} = 1 + r_{ft}$, it must be that:

$$\begin{aligned} 0_{K^{\text{can}}} &= \mathbb{E}(M_{t+1} f_{t+1}^{\text{can}}), \\ 0_N &= \mathbb{E}(M_{t+1}(R_{t+1} - R_{ft} 1_N)), \end{aligned}$$

leading to a total of $K^{\text{can}} + N$ constraints. Substituting M_{t+1} from (B1) one gets:

$$\begin{aligned} 0_{K^{\text{can}}} &= \mathbb{E}\left[\left(\mathbb{E}(M_{t+1}) + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right) f_{t+1}^{\text{can}}\right] \\ &= \mathbb{E}\left[\left(R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right) f_{t+1}^{\text{can}}\right] \\ &= R_{ft}^{-1}\mathbb{E}(f_{t+1}^{\text{can}}) + \mathbb{E}\left(f_{t+1}^{\text{can}}(f_{t+1}^{\text{can}} - \lambda^{\text{can}})'b\right) \\ &\quad + \mathbb{E}\left(f_{t+1}^{\text{can}}\varepsilon_{N,t+1}'\right)c \\ &= R_{ft}^{-1}\lambda^{\text{can}} + V_{f^{\text{can}}} b, \end{aligned}$$

implying that

$$b = -R_{ft}^{-1} V_{f^{\text{can}}}^{-1} \lambda^{\text{can}}.$$

Next,

$$\begin{aligned} 0_N &= \mathbb{E}\left[\left(R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right)\left(R_{t+1} - R_{ft}1_N\right)\right] \\ &= \mathbb{E}\Big[\left(R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right) \times \\ &\qquad \left(\alpha + \beta^{\text{can}}\lambda^{\text{can}} + \beta^{\text{can}}(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + \varepsilon_{t+1}\right)\Big] \\ &= R_{ft}^{-1}(\alpha + \beta^{\text{can}}\lambda^{\text{can}}) + \beta^{\text{can}} V_{f^{\text{can}}} b + V_\varepsilon c \\ &= R_{ft}^{-1}(\alpha + \beta^{\text{can}}\lambda^{\text{can}}) - R_{ft}^{-1}\beta^{\text{can}} V_{f^{\text{can}}} V_{f^{\text{can}}}^{-1}\lambda^{\text{can}} + V_\varepsilon c \\ &= R_{ft}^{-1}(\alpha + \beta^{\text{can}}\lambda^{\text{can}}) - R_{ft}^{-1}\beta^{\text{can}}\lambda^{\text{can}} + V_\varepsilon c \\ &= R_{ft}^{-1}\alpha + V_\varepsilon c, \end{aligned}$$

implying that

$$c = -R_{ft}^{-1} V_\varepsilon^{-1} \alpha.$$

□

Proof of Proposition 3

Assuming, without loss of generality, that the candidate model has only tradable factors represented by either factor returns in excess of the risk-free rate (for example, market factor) or long-minus-short strategies, we define the non-negative SDF to be:

$$M_{\exp,t+1} = \exp(\mu_m^+ + (b^+)'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + (c^+)'\varepsilon_{t+1}),$$

which implies that to identify $M_{\exp,t+1}$, we need to find: μ_m^+, b^+, and c^+.

Imposing the following $1 + K^{\text{can}} + N$ constraints,

$$\begin{aligned} R_{ft}^{-1} &= \mathbb{E}(M_{\exp,t+1}), \\ 0_{K^{\text{can}}} &= \mathbb{E}(M_{\exp,t+1} f_{t+1}^{\text{can}}), \\ 0_N &= \mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_{ft}1_N)), \end{aligned}$$

allows one to identify $M_{\exp,t+1}$, as we show below. Starting with the first restriction, using Lemma B.1 below, we get:

$$\begin{aligned} R_{ft}^{-1} &= \mathbb{E}(M_{\exp,t+1}) = \mathbb{E}(\exp[\mu_m^+ + (b^+)'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c^{+\prime}\varepsilon_{t+1}]) \\ &= \exp[\mu_m^+] \exp[(\frac{1}{2} b^{+\prime} V_{f^{\text{can}}} b^+ + \frac{1}{2} c^{+\prime} \Sigma_{N,t} c^+] \end{aligned}$$

implying

$$\exp[\mu_m^+] = R_{ft}^{-1} \exp[-(\frac{1}{2} b^{+\prime} V_f b^+ + \frac{1}{2} c^{+\prime} V_\varepsilon c^+)].$$

Next, considering the K^{can} restrictions and using Lemma B.1 again, we obtain:

$$\begin{aligned} 0_{K^{\text{can}}} &= \mathbb{E}(M_{\exp,t+1} f_{t+1}^{\text{can}}) \\ &= \mathbb{E}(M_{\exp,t+1}(f_{t+1}^{\text{can}} - \lambda^{\text{can}})) + \lambda^{\text{can}} \mathbb{E}(M_{\exp,t+1}) \\ &= \lambda^{\text{can}} \mathbb{E}(M_{\exp,t+1}) + e^{\mu_m^+} \mathbb{E}(e^{c^{+\prime}\varepsilon_{t+1}}) \mathbb{E}(e^{b^{+\prime}(f_{t+1}^{\text{can}} - \lambda^{\text{can}})})(f_{t+1}^{\text{can}} - \lambda^{\text{can}})) \\ &= \lambda^{\text{can}} R_{ft}^{-1} + e^{(\mu_m^+ + \frac{1}{2} c^{+\prime} V_\varepsilon c^+ + \frac{1}{2} b^{+\prime} V_{f^{\text{can}}} b^+)} V_{f^{\text{can}}} b^+ \\ &= \lambda^{\text{can}} R_{ft}^{-1} + R_{ft}^{-1} V_{f^{\text{can}}} b^+ \end{aligned}$$

yielding

$$b^+ = -V_{f^{\text{can}}}^{-1} \lambda^{\text{can}}.$$

Finally, imposing the N restrictions and using Lemma B.1 again, we get:

$$\begin{aligned} 0_N &= \mathbb{E}(M_{\exp,t+1}(R_{t+1} - R_{ft}1_N)) \\ &= \mathbb{E}\big(M_{\exp,t+1}(\alpha + \beta^{\text{can}} \lambda^{\text{can}} + \beta^{\text{can}}(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + \varepsilon_{t+1})\big) \end{aligned}$$

$$= (\alpha + \beta^{\text{can}}\lambda^{\text{can}})\mathbb{E}(M_{\exp,t+1}) + \mathbb{E}(M_{\exp,t+1}\beta^{\text{can}}(f^{\text{can}}_{t+1} - \lambda^{\text{can}})) + \mathbb{E}(M_{\exp,t+1}\varepsilon_{t+1})$$
$$= (\alpha + \beta^{\text{can}}\lambda^{\text{can}})\mathbb{E}(M_{\exp,t+1}) - \beta^{\text{can}}\lambda^{\text{can}}\mathbb{E}(M_{\exp,t+1}) + R_{ft}^{-1}V_\varepsilon c^+,$$

implying that

$$c^+ = -V_\varepsilon{}^{-1}\alpha,$$

where we used $\mathbb{E}(M_{\exp,t+1}\varepsilon_{t+1}) = e^{(\mu_m^+ + \frac{1}{2}b^{+\prime}V_{f\text{can}}b^+)}\mathbb{E}(e^{c^{+\prime}\varepsilon_{t+1}}\varepsilon_{t+1}) = R_{ft}^{-1}V_\varepsilon c^+$. Putting terms together

$$M_{\exp,t+1} = R_{ft}^{-1}e^{-[\lambda^{\text{can}\prime}V^{-1}_{f\text{can}}(f^{\text{can}}_{t+1}-\lambda^{\text{can}})+\frac{1}{2}\lambda^{\text{can}\prime}V^{-1}_{f\text{can}}\lambda^{\text{can}}]}e^{-[\alpha' V_\varepsilon{}^{-1}\varepsilon_{t+1}+\frac{1}{2}\alpha' V_\varepsilon{}^{-1}\alpha]}.$$

□

Proof of Proposition 4

The result follows from Lemmas B.2 and B.3. □

Proof of Proposition 5

Consider

$$\hat{M}^{\alpha}_{\exp,t+1} = \exp\Big[-\alpha' V_R^{-1}(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}\alpha' V_R^{-1}\alpha\Big]$$
$$= \exp\Big[-\alpha'(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})(R_{t+1} - \mathbb{E}[R_{t+1}]) - \frac{1}{2}\alpha' V_R^{-1}\alpha\Big],$$

where the first term of the exponent can be written as the sum of three components:

$$\alpha'(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})(R_{t+1} - \mathbb{E}[R_{t+1}])$$
$$= \alpha' V_\varepsilon^{-1}\varepsilon_{t+1} - \alpha' V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})\varepsilon_{t+1}$$
$$+ \alpha'(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})\beta^{\text{can}}(f^{\text{can}}_{t+1} - \lambda^{\text{can}}).$$

As $N \to \infty$, the second and third components vanish and only the first component remains. In fact, given $\varepsilon_{t+1} = \beta^{\text{mis}}(f^{\text{mis}}_{t+1} - \lambda^{\text{mis}}) + e_{t+1}$ and $\alpha = \beta^{\text{mis}}\lambda^{\text{mis}} + a$, one obtains

$$\alpha' V_\varepsilon^{-1}\varepsilon_{t+1} - a' V_e^{-1}\varepsilon_{t+1} \xrightarrow{p} 0, \text{and}$$
$$\alpha' V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})\varepsilon_{t+1} =$$
$$\alpha' V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})\beta^{\text{mis}}(f^{\text{mis}}_{t+1} - \lambda^{\text{mis}})$$
$$+ \alpha' V_\varepsilon^{-1}\beta^{\text{can}}(V^{-1}_{f\text{can}} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})e_{t+1}$$
$$o(N^{1/2}N^{-1}N^{1/2}) + o_p(N^{1/2}N^{-1}N^{1/2}) = o_p(1), \text{and}$$

$$\alpha'(V_\varepsilon^{-1} - V_\varepsilon^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}\beta^{\text{can}\prime}V_\varepsilon^{-1})\beta^{\text{can}}(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) =$$
$$\alpha' V_\varepsilon^{-1}\beta^{\text{can}}(V_{f^{\text{can}}}^{-1} + \beta^{\text{can}\prime}V_\varepsilon^{-1}\beta^{\text{can}})^{-1}V_{f^{\text{can}}}^{-1}(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) = o_p(N^{-\frac{1}{2}}),$$

making use of $\beta^{\text{can}\prime}V_\varepsilon{}^{-1}e_{t+1} = O_p((\beta^{\text{can}\prime}V_\varepsilon^{-1}V_eV_\varepsilon^{-1}\beta^{\text{can}})^{\frac{1}{2}}) = O_p(N^{\frac{1}{2}})$ and $\alpha' V_\varepsilon^{-1}\beta^{\text{can}} = a'V_\varepsilon^{-1}\beta^{\text{can}} + \lambda^{\text{mis}\prime}\beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{can}} = o(N^{\frac{1}{2}}) + o(N^{\frac{1}{2}})$, recalling $V_\varepsilon = \beta^{\text{can}}\beta^{\text{can}\prime} + V_e$. □

Proof of Proposition 6

By Proposition 5, $\hat{M}_{t+1}^{\beta,mis} \xrightarrow{p} -R_{ft}^{-1}\lambda^{\text{mis}\prime}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}}))$, setting for simplicity $M_{1_T} = I_T - 1_T1_T'/T$, one obtains, given $M_{1_T}1_T = O_{T\times T}$,

$$\hat{\gamma}_1 \xrightarrow{p} -R_{ft}^{-1}(G'M_{1_T}G)^{-1}G'M_{1_T}(F^{\text{mis}} - 1_T\mathbb{E}(f_{t+1}^{\text{mis}\prime}))\lambda^{\text{mis}}$$
$$= -R_{ft}^{-1}(QF^{\text{mis}\prime}M_{1_T}F^{\text{mis}}Q')^{-1}QF^{\text{mis}\prime}M_{1_T}F^{\text{mis}}\lambda^{\text{mis}} = -R_{ft}^{-1}(Q')^{-1}\lambda^{\text{mis}} = \gamma_1.$$

Thus, regarding the limit of R_g^2, its numerator simplifies to

$$\gamma_1'(G'M_{1_T}G)\gamma_1 = (R_{ft})^{-2}\lambda^{\text{mis}\prime}Q^{-1}Q(F^{\text{mis}\prime}M_{1_T}F^{\text{mis}})Q'(Q')^{-1}\lambda^{\text{mis}}$$
$$= (R_{ft})^{-2}\lambda^{\text{mis}\prime}(F^{\text{mis}\prime}M_{1_T}F^{\text{mis}})\lambda^{\text{mis}},$$

and its denominator becomes

$$= (R_{ft})^{-2}\lambda^{\text{mis}\prime}(F^{\text{mis}} - 1_T\mathbb{E}(f_{t+1}^{\text{mis}\prime}))'M_{1_T}(F^{\text{mis}} - 1_T\mathbb{E}(f_{t+1}^{\text{mis}\prime}))\lambda^{\text{mis}}$$
$$= (R_{ft})^{-2}\lambda^{\text{mis}\prime}(F^{\text{mis}\prime}M_{1_T}F^{\text{mis}})\lambda^{\text{mis}},$$

and thus identical to the numerator of the limit of R_g^2.

The case of orthogonal G and F^{mis} is straightforward and so we omit details. □

Proof of Proposition 7

The equilibrium process for asset returns, given by expressions (2) and (24) in the manuscript, is

$$R_i - R_f = \beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m + b_i Y + \sigma_i\epsilon_i.$$

We posit that the SDF is

$$M = \xi + \chi Y + \sum_{i=1}^{N}\zeta_i\epsilon_i,$$

where ξ, χ, and ζ_i need to be determined. From the Law of One Price, we have $N+2$ equations to determine ξ, χ, and ζ_i, $i = \{1,\ldots,N\}$:

$$E[M] = \frac{1}{R_f}, \tag{B2}$$

$$E[M(R_{N+1} - R_f)] = 0 \tag{B3}$$

$$E[M(R_i - R_f)] = 0, \quad \text{for} \quad i = \{1, \ldots, N\}, \tag{B4}$$

where, from (3) and (11) in the manuscript,

$$R_{N+1} = R_f + \gamma b + Y.$$

From expression (B2), we get

$$\xi = \frac{1}{R_f}.$$

From expression (B3), we get

$$\chi = -\frac{\gamma b}{R_f}.$$

From expression (B4), for each $i = \{1, \ldots, N\}$ we have

$$\xi\beta_i(\mathbb{E}(R_m) - R_f) + \xi(a_i - \beta_i a_m) + \chi\beta_i + \zeta\sigma_i = 0.$$

As a result,

$$\zeta = -\frac{1}{R_f}\frac{\beta_i(\mathbb{E}(R_m) - R_f) + a_i - \beta_i a_m - b_i\gamma b}{\sigma_i}.$$

Recall that

$$R_m = \sum_{i=1}^{N} x_i R_i$$

and use (2) and (16) from the manuscript to obtain

$$\begin{aligned} R_m - R_f &= \sum_{i=1}^{N} x_i(\gamma b_i b + \gamma x_i\sigma_i^2/q_i) + \sum_{i=1}^{N} x_i b_i Y + \sum_{i=1}^{N} x_i\sigma_i\epsilon_i \\ &= \gamma b^2 + \gamma\sum_{i=1}^{N} x_i^2\sigma_i^2/q_i + bY + \sum_{i=1}^{N} x_i\sigma_i\epsilon_i. \end{aligned}$$

From the last expression, we obtain

$$bY = (R_m - R_f) - \gamma b^2 - \gamma\sum_{i=1}^{N} x_i^2\sigma_i^2/q_i - \sum_{i=1}^{N} x_i\sigma_i\epsilon_i.$$

As a result, the SDF is

$$M = \frac{1}{R_f} - \frac{\gamma}{R_f}\left((R_m - R_f) - b^2\gamma - \gamma\sum_{i=1}^{N} x_i^2\sigma_i^2/q_i - \sum_{i=1}^{N} x_i\sigma_i\epsilon_i\right)$$

$$-\frac{1}{R_f}\sum_{t=1}^{N}\frac{\beta_i(\mathbb{E}(R_m)-R_f)+a_i-\beta_i a_m-b_i\gamma b}{\sigma_i}\epsilon_i.$$

Grouping together similar terms, we obtain

$$\begin{aligned}M=&\frac{1}{R_f}+\frac{\gamma^2 b^2}{R_f}+\frac{\gamma^2\sum_{i=1}^{N}x_i^2\sigma_i^2/q_i}{R_f}-\frac{\gamma}{R_f}(R_m-R_f)\\&-\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{\beta_i(\mathbb{E}(R_m)-R_f)+a_i-\beta_i a_m-b_i\gamma b-\gamma x_i\sigma_i^2}{\sigma_i}\epsilon_i\right).\end{aligned}$$

Finally, we use expressions (22) and (24) in Merton (1987) to simplify the loading of M on ϵ_i and obtain

$$-\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{\beta_i(\mathbb{E}(R_m)-R_f)+a_i-\beta_i a_m-b_i\gamma b-\gamma x_i\sigma_i^2}{\sigma_i}\right)=-\frac{1}{R_f}\sum_{i=1}^{N}\frac{a_i}{\sigma_i}.$$

Using de-meaned returns on the market portfolio as a factor in the SDF, along with expressions (15), (19), and (24), we obtain

$$M=\underbrace{-\frac{1}{R_f}\sum_{i=1}^{N}\left(\frac{a_i}{\sigma_i}\epsilon_i\right)}_{M^a}+\underbrace{\frac{1}{R_f}-\frac{(\mathbb{E}(R_m)-R_f)}{R_f\,\mathrm{var}(R_m)}(R_m-\mathbb{E}(R_m))}_{M^\beta}.$$

If the number of available assets is large, that is, $N\to\infty$, then

$$\begin{aligned}\beta_i&=\frac{b_i b+x_i\sigma_i^2}{b^2+\sum_{i=1}^{N}x_i^2\sigma_i^2}\to\frac{b_i b}{b^2}=\frac{b_i}{b}\\a_m&=\sum_{i=1}^{N}x_i a_i=\sum_{i=1}^{N}x_i(1-q_i)\Delta_i=\sum_{i=1}^{N}\gamma x_i^2\sigma_i^2\frac{(1-q_i)}{q_i}\to 0,\\cov(\sum_{i=1}^{N}x_i\sigma_i\epsilon_i,\epsilon_i)&=\sum_{i=1}^{N}x_i\sigma_i\to 0.\end{aligned}$$

Thus given $N\to\infty$, (i) $\beta_i\to b_i/b$, (ii) $a_m\to 0$, and (iii) the market return is asymptotically orthogonal to all asset-specific shocks, ϵ_i.

Making these substitutions gives the results in (16) and (17).

B.2 Proposition B1

Proposition B1 (Parameter estimates of APT). *Suppose that the vector of asset returns R_{t+1} satisfies the data-generating process given in equations (1) and (2). Assume that the* *number of missing factors in the candidate model, K^{mis}, is known, that the sample covariance matrix of candidate factors $\hat{V}_{f^{ecan}}$ is nonsingular, with $\hat{V}_{f^{\text{can}}} = \hat{M}_{f^{\text{can}}} - \bar{f}^{\text{can}}\bar{f}^{\text{can}\prime}$, setting $\hat{M}_{f^{\text{can}}} = T^{-1}\sum_{t=1}^{T} f_t^{\text{can}} f_t^{\text{can}\prime}$, $\bar{f}^{\text{can}} = T^{-1}\sum_{t=1}^{T} f_t^{\text{can}}$, and setting $\hat{M}_{Rf^{\text{can}}} = \frac{1}{T}\sum_{t=1}^{T}(R_t - R_{ft}1_N) f_t^{\text{can}\prime}$, $\bar{R} = \frac{1}{T}\sum_{t=1}^{T} R_t$, and $\bar{R}_f = \frac{1}{T}\sum_{t=1}^{T} R_{ft}$.*

(i) *If the optimal value of the Karush-Kuhn-Tucker multiplier $\hat{\kappa}$ is greater than zero, then*

$$\operatorname{vec}(\hat{\beta}^{\text{can}}) = \left((\hat{M}_{f^{\text{can}}} \otimes I_N) - (\bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} \otimes G)\right)^{-1} \operatorname{vec}\left(\hat{M}_{Rf^{\text{can}}} - G(\bar{R} - R_{ft}1_N)\bar{f}^{\text{can}\prime}\right), \tag{B5}$$

$$\hat{\lambda}^{\text{mis}} = (\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\hat{V}_{\varepsilon}^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\bar{f}^{\text{can}}\Big), \tag{B6}$$

$$\hat{a} = \frac{1}{\hat{\kappa}+1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\bar{f}^{\text{can}} - \hat{\beta}^{\text{mis}}\hat{\lambda}^{\text{mis}}\Big). \tag{B7}$$

in which

$$G = \frac{1}{(\hat{\kappa}+1)} I_N + \frac{\hat{\kappa}}{(\hat{\kappa}+1)} \hat{\beta}^{\text{mis}}(\hat{\beta}^{\text{mis}\prime}\, \hat{V}_{\varepsilon}^{-1}\, \hat{\beta}^{\text{mis}})^{-1}\hat{\beta}^{\text{mis}\prime}\, \hat{V}_{\varepsilon}^{-1},$$

$$\hat{V}_{\varepsilon} = \hat{\beta}^{\text{mis}}\hat{\beta}^{\text{mis}\prime} + \hat{V}_e,$$

where $\hat{\beta}^{\text{mis}}$ and $\hat{V}_e$ do not admit a closed-form solution and $\hat{\lambda}^{\text{can}}$ and $\hat{V_{f^{\text{can}}}}$ coincide with the sample mean and sample covariance of the factors f_t^{can}.

(ii) *If the optimal value of the Karush-Kuhn-Tucker multiplier satisfies $\hat{\kappa} = 0$ one can estimate only $\alpha = \beta^{\text{mis}}\lambda^{\text{mis}} + a$ but not the two components separately, and one obtains*

$$\hat{\alpha} = \bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\, \bar{f}^{\text{can}\, e}, \tag{B8}$$

and the expression for $\operatorname{vec}(\hat{\beta}^{\text{can}})$ can be obtained by setting $\hat{\kappa} = 0$ in (B5). The expressions for $\hat{\lambda}^{\text{can}}$ and $\hat{V_{f^{\text{can}}}}$ are unchanged, and, as before, the expressions for the estimators of $\hat{\beta}^{\text{mis}}$ and $\hat{V}_e$ do not admit a closed-form solution.

Proof of Proposition B1

Defining by $\tilde{\theta}$ the constrained maximum likelihood estimator corresponding to $\tilde{\kappa} = 0$, this is infeasible whenever $\tilde{a}'\tilde{V_{\varepsilon}}^{-1}\tilde{a} > \delta_{\text{apt}}$. Similarly, the case $\hat{\kappa} > 0$ is infeasible whenever, for every $\hat{\kappa} > 0$,

$$\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\, \bar{f}^{\text{can}} - \hat{\beta}^{\text{mis}}\, \hat{\lambda}^{\text{mis}}\Big)'\hat{V}_{\varepsilon}^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\, \bar{f}^{\text{can}} - \hat{\beta}^{\text{mis}}\, \hat{\lambda}^{\text{mis}}\Big) < \delta_{\text{apt}},$$

because

$$(1+\hat{\kappa})^2 = \frac{\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\, \bar{f}^{\text{can}} - \hat{\beta}^{\text{mis}}\, \hat{\lambda}^{\text{mis}}\Big)'\hat{V}_{\varepsilon}^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}^{\text{can}}\, \bar{f}^{\text{can}} - \hat{\beta}^{\text{mis}}\, \hat{\lambda}^{\text{mis}}\Big)}{\delta_{\text{apt}}}.$$

When both cases are feasible, the optimal value for the Karush-Kuhn-Tucker multiplier $\tilde{\kappa}$ will be either greater that zero or equal to zero, depending on which case maximizes the log-likelihood, namely depending on whether $L(\hat{\boldsymbol{\theta}})$ or $L(\tilde{\boldsymbol{\theta}})$ is largest, respectively. Note that when $\hat{\kappa} > 0$ then $\hat{a}'\hat{V}_\varepsilon^{-1}\hat{a} = \delta_{\text{apt}}$ by construction.

We now derive the formulae for the estimators. Assume for now that case $\hat{\kappa} > 0$ holds. Differentiating the penalized log-likelihood with respect to λ^{mis}, a, and the Lagrange multiplier κ, the first $K^{\text{mis}} + N$ equations (after some algebra) are:

$$\begin{pmatrix} \beta^{\text{mis}\prime}V_\varepsilon^{-1} \\ I_N \end{pmatrix}\left(\bar{R} - \bar{R}_f 1_N - \beta^{\text{can}}\bar{f}^{\text{can}})\right) = \begin{pmatrix} \beta^{\text{mis}\prime}V_\varepsilon^{-1}\beta^{\text{mis}} & \beta^{\text{mis}\prime}V_\varepsilon^{-1} \\ \beta^{\text{mis}} & (1+\hat{\kappa})I_N \end{pmatrix}\begin{pmatrix} \hat{\lambda}^{\text{mis}} \\ \hat{a} \end{pmatrix},$$

where recall that $V_\varepsilon = \beta^{\text{mis}}\beta^{\text{mis}\prime} + V_e$. It is straightforward to see that, because of the APT restriction, λ^{mis} and a can now be identified separately, as long as $\hat{\kappa} > 0$. In fact, the above system of linear equations can be solved because the matrix premultiplying $\hat{\lambda}^{\text{mis}}$ and $\hat{a}$ is non-singular for every $\hat{\kappa} > 0$, leading to the closed-form solution (B6) and (B7).

Turning now to the first-order condition with respect to the generic (a,b)th element of β^{can}, denoted by B_{2ab} with $1 \leq a \leq N$, $1 \leq b \leq K^{\text{can}}$, one obtains

$$-\frac{1}{T}\sum_{t=1}^{T}\left(R_t - R_{ft}1_N - \beta^{\text{mis}}\lambda^{\text{mis}} - a - \hat{\beta}^{\text{can}}f_t^{\text{can}}\right)' V_\varepsilon^{-1}\left(-\frac{\partial\beta^{\text{can}}}{\partial B_{2ab}} f_t\right) = 0,$$

which can be re-arranged as

$$\hat{M}_{Rf^{\text{can}}} - (a + \beta^{\text{mis}}\lambda^{\text{mis}})\bar{f}^{\text{can}\prime} - \hat{\beta}^{\text{can}}\hat{M}_{f^{\text{can}}} = 0_{N\times K^{\text{can}}}.$$

Inserting (B6) and (B7), and G, and re-arranging terms yields

$$\hat{\beta}^{\text{can}}\hat{M}_{f^{\text{can}}} - G\hat{\beta}^{\text{can}}\bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} = \hat{M}_{Rf^{\text{can}}} - G(\bar{R} - \bar{R}_f 1_N)\bar{f}^{\text{can}\prime},$$

which can be rewritten more succinctly as

$$\frac{1}{T}\sum_{t=1}^{T} f_t^{\text{can}} g_t' = 0_{K^{\text{can}}\times N},$$

with $g_t = \left(R_t - R_{ft}1_N - G(\bar{R} - \bar{R}_f 1_N) - \hat{\beta}^{\text{can}}f_t^{\text{can}} + G\hat{\beta}^{\text{can}}\bar{f}^{\text{can}}\right)$. Taking the vec operator and solving for $\hat{\beta}^{\text{can}}$ gives the desired expression in (B5).

We need to show that a solution for $\hat{\beta}^{\text{can}}$ exists. This requires one to establish that the matrix $\left((\hat{M}_{f^{can}} \otimes I_N) - (\bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} \otimes G)\right)$ is invertible. This matrix can be written as

$$\left((\hat{M}_{f^{\text{can}}} \otimes I_N) - (\bar{f}^{\text{can}}\bar{f}^{\text{can}\prime} \otimes G)\right)$$

$$= \Big(((\hat{M}_{f^{\text{can}}} - \bar{f}^{\text{can}}\,\bar{f}^{\text{can}\,\prime}) \otimes I_N) + (\bar{f}^{\text{can}}\,\bar{f}^{\text{can}\,\prime} \otimes (I_N - G)) \Big).$$

The first matrix on the right hand side is non-singular by assumption. One then just needs to show that the second matrix is positive semi-definitive. This follows because $(I_N - G)$ is positive semi-definite.

$$\begin{aligned}
I_N - G &= I_N - \frac{1}{(\hat{\kappa}+1)} I_N - (\frac{\hat{\kappa}}{1+\hat{\kappa}}) \beta^{\text{mis}} (\beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_\varepsilon^{-1} \\
&= (\frac{\hat{\kappa}}{1+\hat{\kappa}})(I_N - \beta^{\text{mis}} (\beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_\varepsilon^{-1}) \\
&= (\frac{\hat{\kappa}}{1+\hat{\kappa}}) V_\varepsilon (V_\varepsilon^{-1} - V_\varepsilon^{-1} \beta^{\text{mis}} (\beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} V_\varepsilon^{-1}) \\
&= (\frac{\hat{\kappa}}{1+\hat{\kappa}}) V_\varepsilon (V_\varepsilon^{-1})^{\frac{1}{2}} (I_N - (V_\varepsilon^{-1})^{\frac{1}{2}} \beta^{\text{mis}} (\beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} (V_\varepsilon^{-1})^{\frac{1}{2}}) (V_\varepsilon^{-1})^{\frac{1}{2}}.
\end{aligned}$$

The right-hand side is the product of positive-definite matrices, namely V_ε and $(V_\varepsilon^{-1})^{\frac{1}{2}}$, and of the matrix $I_N - (V_\varepsilon^{-1})^{\frac{1}{2}} \beta^{\text{mis}} (\beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}})^{-1} \beta^{\text{mis}\prime} (V_\varepsilon^{-1})^{\frac{1}{2}}$, which is the projection matrix orthogonal to $(V_\varepsilon^{-1})^{\frac{1}{2}} \beta^{\text{mis}}$, and therefore, positive semidefinite itself.

Therefore, plugging $\hat{\beta}^{\text{can}}$ into $\hat{\lambda}^{\text{mis}}$ and $\hat{a}$, one obtains that

$$\hat{\lambda}^{\text{mis}} = \hat{\lambda}^{\text{mis}}(\beta^{\text{mis}}, V_e), \quad \hat{a} = \hat{a}(\beta^{\text{mis}}, V_e) \quad \text{and} \quad \hat{\kappa} = \hat{\kappa}(\beta^{\text{mis}}, V_e).$$

Substituting them, together with $\hat{\beta}^{\text{can}} = \hat{\beta}^{\text{can}}(\beta^{\text{mis}}, V_e)$,, into $L(\theta) - \kappa(a' V_\varepsilon{}^{-1} a - \delta_{\text{apt}})$, gives the concentrated likelihood function, which is a function of only β^{mis} and V_e which will be maximized numerically, providing $\hat{\beta}^{\text{mis}}$ and $\hat{V}_e$.

Suppose now that $\hat{\kappa} = 0$ holds, and recall that in this case the MLC is indicated by $\hat{\theta}$. One can clearly obtain a unique solution to the linear system of equations $(\beta^{\text{mis}}, I_N) \begin{pmatrix} \hat{\lambda}^{\text{mis}} \\ \hat{a} \end{pmatrix} = \beta^{\text{mis}} \hat{\lambda}^{\text{mis}} + \hat{a}$. However, to solve for $\hat{\lambda}^{\text{mis}}$ and $\hat{a}$ separately, one needs to invert the matrix

$$\begin{pmatrix} \beta^{\text{mis}\prime} V_\varepsilon^{-1} \\ I_N \end{pmatrix} (\beta^{\text{mis}}, I_N) = \begin{pmatrix} \beta^{\text{mis}\prime} V_\varepsilon^{-1} \beta^{\text{mis}} & \beta^{\text{mis}\prime} V_\varepsilon^{-1} \\ \beta^{\text{mis}} & I_N \end{pmatrix},$$

which is not possible because it is of dimension $(N + K^{\text{mis}}) \times (N + K^{\text{mis}})$ but of rank N, as the left-hand side shows that it is obtained from the product of two matrices of dimension $(N + K^{\text{mis}}) \times N$. All the other parameters are identified separately, and their expressions follow from differentiating $L(\theta)$ and solving the resulting first-order conditions. For instance, the formula for $\hat{\beta}^{\text{can}}$ follows by setting $G = I_N$ into (B5). $\square$

C Online Appendix with Additional Results

This appendix contains additional results and generalizations of some of the results reported in the main text.

C.1 The SDF with Non-Orthogonal Components

All the previous results were derived under the assumption that the candidate risk factors f_t^{can} and the unobserved idiosyncratic shock ε_{t+1} are (conditionally) orthogonal, as formalized in Assumption 1. However, one can envisage situations where orthogonality does not necessarily hold, the best example being when there are missing systematic risk factors that are *hidden* in the idiosyncratic shock and are correlated with the observed risk factors.[29]

In this case, note that an *observationally equivalent* representation of the SDF M_{t+1} exists such that the observed risk factors f_{t+1}^{can} and the unobserved idiosyncratic shock ε_{t+1} are orthogonal. In particular, recalling that $\varepsilon_{t+1} = e_{t+1} + \beta^{\text{mis}}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}}))$, as before, with e_{t+1} and f_{t+1}^{mis} being mutually uncorrelated,

$$M_{t+1} = R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + c'\epsilon_{t+1}, \tag{C1}$$

$$= R_{ft}^{-1} + \tilde{b}'(f_{t+1}^{\text{can}} - E_t(f_{t+1}^{\text{can}})) + c'\tilde{\epsilon}_{t+1}, \tag{C2}$$

setting the $K^{\text{can}} \times K^{\text{mis}}$ matrix of covariances $Q = \text{cov}(f_{t+1}^{\text{can}}, f_{t+1}^{\text{mis}\prime})$ with

$$\tilde{b} = b + V_{f^{\text{can}}}^{-1} Q \beta^{\text{mis}\prime} c,$$

$$\tilde{\epsilon}_{t+1} = e_{t+1} + \beta^{\text{mis}}\big(\tilde{f}_{t+1}^{\text{mis}} - \mathbb{E}(\tilde{f}_{t+1}^{\text{mis}})\big), \quad \text{where}$$

$$\tilde{f}_{t+1}^{\text{mis}} = (I_{K^{\text{mis}}}, -Q'V_{f^{\text{can}}}^{-1}) \begin{pmatrix} f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}}) \\ f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}}) \end{pmatrix}.$$

Notice that by contruction $\text{cov}(f_{t+1}^{\text{can}}, \tilde{f}_{t+1}^{\text{mis}\prime}) = 0_{K^{\text{can}} \times K^{\text{mis}}}$, because $\tilde{f}_{t+1}^{\text{mis}}$ represent the linear-projection residual from projecting $f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})$ on $f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})$.

Although the two representations (C1) and (C2) are observationally equivalent, the one based on correlated components, that is (C1), has the advantage of ensuring a cleaner interpretation of the parameters, such as the ones for loadings and risk premia. For instance, the loadings associated with f_{t+1}^{can} in representation (C2) differ from the (true) loadings of f_{t+1}^{can} in representation (C1), a consequence of the omitted-variable bias. This can be seen by comparing the extended APT in the orthogonal and non-orthogonal representations:

$$R_{t+1} - R_{ft}1_N = a + (\beta^{\text{mis}}, \beta^{\text{can}}) \begin{pmatrix} \lambda^{\text{mis}} \\ \lambda^{\text{can}} \end{pmatrix} + (\beta^{\text{mis}}, \beta^{\text{can}}) \begin{pmatrix} f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}}) \\ f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}}) \end{pmatrix} + e_{t+1},$$

[29] We report only the results valid for finite N. The large N analysis follow along the steps outlined in Section 2.3.2.

$$= a + (\tilde{\beta}^{\text{mis}}, \tilde{\beta}^{\text{can}}) \begin{pmatrix} \tilde{\lambda}^{\text{mis}} \\ \lambda^{\text{can}} \end{pmatrix} + (\tilde{\beta}^{\text{mis}}, \tilde{\beta}^{\text{can}}) \begin{pmatrix} \tilde{f}^{\text{mis}}_{t+1} - \mathbb{E}(\tilde{f}^{\text{mis}}_{t+1}) \\ f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1}) \end{pmatrix} + e_{t+1},$$

where

$$\tilde{\beta}^{\text{mis}} = \beta^{\text{mis}}(I_{K^{\text{mis}}} - Q'V^{-1}_{f^{\text{can}}}Q)^{\frac{1}{2}},$$

$$\tilde{\beta}^{\text{can}} = \beta^{\text{can}} + \beta^{\text{mis}}Q'V^{-1}_{f^{\text{can}}},$$

$$\tilde{\lambda}^{\text{mis}} = (I_{K^{\text{mis}}} - Q'V^{-1}_{f^{\text{can}}}Q)^{-\frac{1}{2}}(\lambda^{\text{mis}} - Q'V^{-1}_{f^{\text{can}}}\lambda^{\text{can}}),$$

and $\tilde{f}^{\text{mis}}_{t+1}$ has (conditionally) unit covariance matrix and is uncorrelated with f^{can}_{t+1}.[30] Notice that $(\beta^{\text{mis}}, \beta^{\text{can}}) \begin{pmatrix} \lambda^{\text{mis}} \\ \lambda^{\text{can}} \end{pmatrix} = (\tilde{\beta}^{\text{mis}}, \tilde{\beta}^{\text{can}}) \begin{pmatrix} \tilde{\lambda}^{\text{mis}} \\ \lambda^{\text{can}} \end{pmatrix}$ as the possibility of a non-zero Q does not affect expected excess returns $\mathbb{E}(R_{t+1} - R_{ft}1_N)$.

We now show how all our results can be generalized to allow for the case of correlated observed and missing factors. In particular, we need to generalize Assumption 1 to:

Assumption C1 (Linear factor model: correlated case)**.** *Assumption 1 holds with*

$$\mathbb{E}(f^{\text{can}}_{t+1}\varepsilon'_{t+1}) = P,$$

for some non-zero $K^{\text{can}} \times N$ matrix P such that perfect (conditional) correlation between f^{can}_{t+1} and ε_{t+1} is ruled out:

$$I_N - (V_\varepsilon^{-1})^{\frac{1}{2}} P' V^{-1}_{f^{\text{can}}} P (V_\varepsilon^{-1})^{\frac{1}{2}} > 0.$$

When $V_\varepsilon = \beta^{\text{mis}}\beta^{\text{mis}\prime} + V_e$, then

$$P = Q\beta^{\text{mis}}, \tag{C3}$$

but we keep the more general notation in terms of P in order to provide the SDF formulae. However, when constructing estimators for the model's parameters, we will impose (C3).

Although the expression for expected excess returns is unchanged, the variance for excess returns becomes:

$$\text{cov}(R_{t+1} - R_{ft}1_N) = V_R = \beta^{\text{can}} V_f \beta^{\text{can}\prime} + V_\varepsilon + P'\beta^{\text{can}\prime} + \beta^{\text{can}} P.$$

We show how the expressions for the linear and exponential SDF change in the absence of orthogonality.

[30] In particular, $\tilde{f}^{\text{mis}}_{t+1}$ is given by:

$$\tilde{f}^{\text{mis}}_{t+1} = (I_{K^{\text{mis}}} - Q'V^{-1}_{f^{\text{can}}}Q)^{-\frac{1}{2}} (I_{K^{\text{mis}}}, -Q'V^{-1}_{f^{\text{can}}}) \begin{pmatrix} f^{\text{mis}}_{t+1} - \mathbb{E}(f^{\text{mis}}_{t+1}) \\ f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1}) \end{pmatrix}.$$

Proposition C1 (SDF: Correlated case)**.** *Under Assumptions C1 and 2 of the APT, there exists an admissible SDF of the form*

$$
\begin{aligned}
M_{t+1} &= R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) + c'\varepsilon_{t+1}, \quad \textit{with} \\
b &= -R_{ft}^{-1}\left(V_{f^{\text{can}}}^{-1}\lambda^{\text{can}} - V_{f^{\text{can}}}^{-1}PH^{-1}(\alpha - P'V_{f^{\text{can}}}^{-1}\lambda^{\text{can}})\right), \\
c &= -R_{ft}^{-1}\left(H^{-1}(\alpha - P'V_{f^{\text{can}}}^{-1}\lambda^{\text{can}})\right), \quad \textit{where} \\
H &= V_\varepsilon - P'V_{f^{\text{can}}}^{-1}P.
\end{aligned}
$$

When expressed in terms of a linear projection on the set of payoffs $(1, R_{t+1} - R_{ft}1_N)$, *the SDF is*

$$
\begin{aligned}
\hat{M}_{t+1} &= R_{ft}^{-1} + (b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}])V_R^{-1}\big(R_{t+1} - \mathbb{E}(R_{t+1})\big) \\
&= R_{ft}^{-1} - (\alpha + \beta^{\text{can}}\lambda^{\text{can}})'V_R^{-1}\big(R_{t+1} - \mathbb{E}(R_{t+1}).
\end{aligned}
$$

Proof of Proposition C1

We start with the conjecture that the SDF is still linear in the observed factors f_{t+1}^{can} and idiosyncratic risk ε_{t+1}, although now these can be cross-correlated. Stating the $K^{\text{can}} + N$ pricing equations:

$$
\begin{aligned}
0_{K^{\text{can}}} &= \mathbb{E}(M_{t+1}f_{t+1}^{\text{can}}) \\
0_N &= \mathbb{E}(M_{t+1}(R_{t+1} - R_{ft}1_N)),
\end{aligned}
$$

yields,

$$
\begin{aligned}
0_{N+K^{\text{can}}} &= \begin{pmatrix} \mathbb{E}\left[\left(R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right) f_{t+1}^{\text{can}}\right] \\ \mathbb{E}\left[\left(R_{ft}^{-1} + b'(f_{t+1}^{\text{can}} - \lambda^{\text{can}}) + c'\varepsilon_{t+1}\right)\left(R_{t+1} - R_{ft}1_N\right)\right] \end{pmatrix} \\
&= R_{ft}^{-1}\begin{pmatrix} \lambda^{\text{can}} \\ \beta^{\text{can}}\lambda^{\text{can}} + \alpha \end{pmatrix} + \begin{pmatrix} V_{f^{\text{can}}}b + Pc \\ (\beta^{\text{can}}V_{f^{\text{can}}} + P')b + (V_\varepsilon + \beta^{\text{can}}P)c \end{pmatrix} \\
&= R_{ft}^{-1}\begin{pmatrix} \lambda^{\text{can}} \\ \beta^{\text{can}}\lambda^{\text{can}} + \alpha^{\text{can}} \end{pmatrix} + \begin{pmatrix} V_{f^{\text{can}}} & P \\ (\beta^{\text{can}}V_{f^{\text{can}}} + P') & (V_\varepsilon + \beta^{\text{can}}P) \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix}.
\end{aligned}
$$

Using the blockwise formula for the inverse of a matrix, in view of the lack of perfect correlation between the f_{t+1}^{can} and the ε_{t+1}, one obtains the solution:

$$
\begin{aligned}
\begin{pmatrix} b \\ c \end{pmatrix} &= -R_{ft}^{-1}\begin{pmatrix} V_{f^{\text{can}}} & P \\ (\beta^{\text{can}}V_{f^{\text{can}}} + P') & (V_\varepsilon + \beta^{\text{can}}V_{f^{\text{can}}}) \end{pmatrix}^{-1}\begin{pmatrix} \lambda^{\text{can}} \\ \beta^{\text{can}}\lambda^{\text{can}} + \alpha \end{pmatrix} \\
&= -R_{ft}^{-1}\begin{pmatrix} V_{f^{\text{can}}}^{-1} + V_{f^{\text{can}}}^{-1}PH^{-1}(\beta^{\text{can}} + P'V_{f^{\text{can}}}^{-1}) & -V_{f^{\text{can}}}^{-1}PH^{-1} \\ -H^{-1}(\beta^{\text{can}} + P'V_{f^{\text{can}}}^{-1}) & H^{-1} \end{pmatrix}\begin{pmatrix} \lambda^{\text{can}} \\ \beta^{\text{can}}\lambda^{\text{can}} + \alpha \end{pmatrix}
\end{aligned}
$$

$$= -R_{ft}^{-1}\begin{pmatrix} V_{f^{\text{can}}}^{-1}\lambda^{\text{can}} - V_{f^{\text{can}}}^{-1}PH^{-1}(\alpha - P'V_{f^{\text{can}}}^{-1}\lambda^{\text{can}}) \\ H^{-1}(\alpha - P'V_{f^{\text{can}}}^{-1}\lambda^{\text{can}}) \end{pmatrix}.$$

We now establish the result for the projection SDF $\hat{M}_{t+1}$. By construction, setting $X_{t+1} = (1, R'_{t+1} - R_{ft}1'_N)'$ and for simplicity using $\mu = \mathbb{E}(R_{t+1} - R_{ft}1_N)$,

$$\begin{aligned}
\hat{M}_{t+1} &= \mathbb{E}(M_{t+1}X'_{t+1})(\mathbb{E}(X_{t+1}X'_{t+1})^{-1}X_{t+1} \\
&= (R_{ft}^{-1}, R_{ft}^{-1}\mu' + b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}])\begin{pmatrix} 1 + \mu'V_R^{-1}\mu & -\mu'V_R^{-1} \\ -V_R^{-1}\mu & V_R^{-1} \end{pmatrix} X_{t+1} \\
&= (R_{ft}^{-1} - (b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}])V_R^{-1}\mu, (b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'V_\varepsilon)V_R^{-1})X_{t+1} \\
&= R_{ft}^{-1} + (b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}])V_R^{-1}(R_{t+1} - R_{ft}1_N - \mu),
\end{aligned}$$

where we apply the block formula for the inverse of a square matrix to $\mathbb{E}(X_{t+1}X'_{t+1})$, which exists because of our assumption of less-than-perfect correlation between observed factors and idiosyncratic shocks. Finally, by means of algebraic manipulations,

$$b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P] + c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}] = \lambda^{\text{can}\,\prime}\beta^{\text{can}\,\prime} + \alpha',$$

which gives the desired result. $\square$

Note that although the expressions for the coefficients in the SDF, namely b and c, differ from the case $P = 0_{K^{\text{can}}\times N}$, one still obtains the decomposition into the alpha- and beta-SDFs:

$$M_{t+1} = M_{t+1}^{\alpha} + M_{t+1}^{\beta,\text{can}},$$

where

$$M_{t+1}^{\beta,\text{can}} = R_{ft}^{-1} - R_{ft}^{-1}b'(f_{t+1}^{\text{can}} - \mathbb{E}(f_{t+1}^{\text{can}})) \quad \text{and} \quad M_{t+1}^{\alpha} = -R_{ft}^{-1}\varepsilon'_{t+1}c,$$

where b and c are defined in Proposition C1. In terms of the pricing of asset returns:

$$\begin{aligned}
\mathbb{E}\left(M_{t+1}^{\beta,\text{can}}\begin{bmatrix} 1 \\ R_{t+1} - R_{ft}1_N \end{bmatrix}\right) &= R_{ft}^{-1}\begin{bmatrix} 1 \\ (V_\varepsilon + \beta^{\text{can}}P)c \end{bmatrix} \\
\mathbb{E}\left(M_{t+1}^{\alpha}\begin{bmatrix} 1 \\ R_{t+1} - R_{ft}1_N \end{bmatrix}\right) &= R_{ft}^{-1}\begin{bmatrix} 0 \\ -(V_\varepsilon + \beta^{\text{can}}P)c \end{bmatrix}.
\end{aligned}$$

Notice that now $\text{cov}(M_{t+1}^{\alpha}, M_{t+1}^{\beta,can}) = R_{ft}^2 b'Pc \neq 0$. Despite this, as for the earlier orthogonal case, the misspecified M_{t+1}^{can} prices the observed factors correctly, that is $\mathbb{E}(M_{t+1}^{\beta,can} f_{t+1}^{\text{can}}) = 0_{K^{\text{can}}}$.

Likewise, one obtains the decomposition in terms of linear projections as:

$$\hat{M}_{t+1} = \hat{M}_{t+1}^{\alpha} + \hat{M}_{t+1}^{\beta,can}, \quad \text{with}$$

$$\hat{M}^{\beta,can}_{t+1} = R_{ft}^{-1} + b'[V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P]V_R^{-1}\big(R_{t+1} - \mathbb{E}(R_{t+1})\big),$$
$$\hat{M}^{\alpha}_{t+1} = c'[V_\varepsilon + P'\beta^{\text{can}\,\prime}])V_R^{-1}\big(R_{t+1} - \mathbb{E}(R_{t+1})\big),$$

where it can be shown that the previous large-N results extend also to the non-orthogonal case.[31]

C.2 The Nonnegative SDF with Non-Orthogonal Components

Given the strong analogies between the specifications of the linear and nonnegative SDF cases, we introduce the nonnegative SDF for the case of correlated components, and its corresponding decomposition in terms of (nonlinear) projections, without a formal proof.

Proposition C2 (Nonnegative SDF: Correlated case)**.** *Under Assumptions C1 and 2 of the APT and that returns are conditionally Gaussian, there exists an admissible SDF $M_{\exp,t+1}$ of the form*

$$M_{\exp,t+1} = \exp\left[\mu_m^+ + b^{+\prime}(f^{\text{can}}_{t+1} - \mathbb{E}(f^{\text{can}}_{t+1})) + c^{+\prime}\varepsilon_{t+1}\right], \quad \textit{with}$$
$$\mu_m^+ = \ln(R_{ft}^{-1}) - \frac{1}{2}(b^{+\prime}, c^{+\prime})\begin{pmatrix} V_{f^{\text{can}}} & P \\ P' & V_\varepsilon \end{pmatrix}\begin{pmatrix} b^+ \\ c^+ \end{pmatrix},$$
$$b^+ = -\left(V^{-1}_{f^{\text{can}}}\lambda^{\text{can}} - V^{-1}_{f^{\text{can}}}PH^{-1}(\alpha - P'V^{-1}_{f^{\text{can}}}\lambda^{\text{can}})\right),$$
$$c^+ = -\left(H^{-1}(\alpha - P'V^{-1}_{f^{\text{can}}}\lambda^{\text{can}})\right),$$

recalling $H = V_\varepsilon - P'V^{-1}_{f^{\text{can}}}P$.

Proof of Proposition C2

We omit the proof because it is similar to the proof for Proposition 3, but using Proposition C1. □

Note that the relevant decomposition of the nonnegative SDF in terms of (nonlinear) projections is given by:

$$\hat{M}_{\exp,t+1} = \hat{M}^{\alpha}_{\exp,t+1}\,\hat{M}^{\beta,\text{can}}_{\exp,t+1}, \quad \text{where}$$

$$\hat{M}^{\beta,\text{can}}_{\exp,t+1} = R_{ft}^{-1}\exp\left[b^{+\prime}(V_{f^{\text{can}}}\beta^{\text{can}\,\prime} + P)V_R^{-1}(R_{t+1} - \mathbb{E}(R_{t+1}) - \frac{1}{2}b^{+\prime}V_{f^{\text{can}}}b^+\right], \quad \text{and}$$
$$\hat{M}^{\alpha*+}_{\exp,t+1} = \exp\left[c^{+\prime}(V_\varepsilon + P'\beta^{\text{can}\,\prime})V_R^{-1}\big(R_{t+1} - \mathbb{E}(R_{t+1})\big) - \frac{1}{2}(b^{+\prime}, c^{+\prime})\begin{pmatrix} 0_{K^{\text{can}}\times K^{\text{can}}} & P \\ P' & V_\varepsilon \end{pmatrix}\begin{pmatrix} b^+ \\ c^+ \end{pmatrix}\right],$$

[31] Details are available upon request.

where the previous large-N results extend also to this non-negative non-orthogonal case.[32]

C.3 Estimation of the APT: the General Case

We now explain how to estimate the APT allowing for both tradable and nontradable factors, and for both asset-specific pricing errors and pricing errors arising from omitted systematic risk factors. Assume that

$$R_{t+1} - R_{ft}1_N = a + \beta^{\text{mis}}\lambda^{\text{mis}} + \beta_1^{\text{can}}(\lambda_1^{\text{can}} + f_{1t+1}^{\text{can}} - \mathbb{E}(f_{1t+1}^{\text{can}})) + \beta_2^{\text{can}} f_{2t+1}^{\text{can}} + \varepsilon_{t+1}, \quad \text{with}$$
$$\alpha = a + \beta^{\text{mis}}\lambda^{\text{mis}}, \;\; \varepsilon_{t+1} = \beta^{\text{mis}}(f_{t+1}^{\text{mis}} - \mathbb{E}(f_{t+1}^{\text{mis}})) + e_{t+1},$$

where we set $\beta^{\text{can}} = (\beta_1^{\text{can}}, \beta_2^{\text{can}})$, $V_f = \text{var}(f_{t+1})$, $f_{t+1} = (f_{1t+1}', f_{2t+1}^{e\prime})'$, with f_{1t+1} denoting the set of K_1^{can} nontradable observed factors and f_{2t+1}^{e} the set of K_2^{can} tradable observed factors, expressed as excess returns, where $K^{\text{can}} = K_1^{\text{can}} + K_2^{\text{can}}$. We assume that the missing factors are uncorrelated with the observed factors.[33] Given that f_{2t}^{can} are excess returns, or difference of, on tradable assets, their risk premia satisfy $\lambda_2^{\text{can}} = E(\mathbf{f}_{2t}^{\text{can}})$ and, to avoid confusion with the risk premia of the nontradable assets λ_1^{can}, we will use the expectation formulation for λ_2^{can}.

The joint log-likelihood function takes the following form:

$$\begin{aligned} L(\tilde{\theta}) = & -\frac{1}{2}\log(\det(\tilde{\beta^{\text{mis}}}\tilde{\beta^{\text{mis}}}' + \tilde{V}_e)) \qquad \text{(C4)} \\ & -\frac{1}{2T}\sum_{t=1}^{T}\Big(R_t - R_{ft}1_N - \tilde{\beta^{\text{mis}}}\tilde{\lambda}^{\text{mis}} - \tilde{a} - \tilde{\beta}_1^{\text{can}}(\tilde{\lambda}_1^{\text{can}} + f_{1t}^{\text{can}} - \tilde{\mathbb{E}}(f_{1t}^{\text{can}})) - \tilde{\beta}_2^{\text{can}} f_{2t}^{\text{can}}\Big)' \\ & \times (\tilde{\beta^{\text{mis}}}\tilde{\beta^{\text{mis}}}' + \tilde{V}_e)^{-1}\Big(R_t - R_{ft}1_N - \tilde{\beta^{\text{mis}}}\tilde{\lambda}^{\text{mis}} - \tilde{a} - \tilde{\beta}_1^{\text{can}}(\tilde{\lambda}_1^{\text{can}} + f_{1t}^{\text{can}} - \tilde{\mathbb{E}}(f_{1t}^{\text{can}})) - \tilde{\beta}_2^{\text{can}} f_{2t}^{\text{can}}\Big) \\ & -\frac{1}{2}\log(\det(\tilde{V}_f)) - \frac{1}{2T}\sum_{t=1}^{T}(f_t - \tilde{\mathbb{E}}(f_t))'\tilde{V}_f^{-1}(f_t - \tilde{\mathbb{E}}(f_t)). \end{aligned}$$

Without loss of generality, one can assume that the missing factors have unit variance, that is, $\text{var}(f_t^{\text{mis}}) = I_{K^{\text{mis}}}$, achieving identification of β^{mis}.

Proposition C3 (Parameter estimation of APT: General Case)**.** *Suppose that the vector of asset returns, R_t, satisfies Assumption 1 and that $\hat{M}_{f_2^{\text{can}}} - \bar{f}_2^{\text{can}}\bar{f}_2^{\text{can}\prime}$ is nonsingular, where $\hat{M}_{f_2^{\text{can}}} = T^{-1}\sum_{t=1}^{T} f_{2t}^{\text{can}} f_{2t}^{\text{can}\prime}$ and $\bar{f}_2^{\text{can}} = T^{-1}\sum_{t=1}^{T} f_{2t}^{\text{can}}$. Then*

$$\hat{\theta} = \underset{\tilde{\theta}}{\text{argmax}} \;\; L(\tilde{\theta}) \quad \textit{subject to} \quad \tilde{a}'\tilde{V}_\varepsilon^{-1}\tilde{a} \le \delta_{\text{apt}},$$

[32] Details are available upon request.

[33] The estimator can be extended to the case of correlated observed and omitted risk factors; details are available upon request.

where $L(\tilde{\theta})$ *is defined in* (C4), *and* $\hat{\theta} = (\hat{a}', \widehat{\lambda^{\text{mis}}}', \hat{\lambda}_1^{\text{can}\,\prime}, \hat{\mathbb{E}}(f_{1t}^{\text{can}})', \hat{\mathbb{E}}(f_{2t}^{\text{can}})', \text{vec}(\widehat{\beta^{\text{mis}}})', \text{vec}(\hat{\beta}^{\text{can}})', \text{vech}(\hat{V}_e)', \text{vech}(\hat{V}_f)')'$.

(i) If the optimal value of the Karush-Kuhn-Tucker multiplier satisfies $\hat{\kappa} > 0$, *setting*

$$D = (\beta^{\text{mis}}, \beta_1^{\text{can}}), \quad \lambda = (\lambda^{\text{mis}\,\prime}, \lambda_1^{\text{can}\,\prime})',$$

then, using $\otimes$ *to denote the Kronecker product,*

$$\begin{aligned}
\text{vec}(\hat{B}_2^{\text{can}}) &= \Big((\hat{M}_{f_2^{\text{can}}} \otimes I_N) - (\bar{f}_2^{\text{can}} \bar{f}_2^{\text{can}\,\prime} \otimes \hat{G})\Big)^{-1} \text{vec}\left(\hat{M}_{hf_2^{\text{can}}} - \hat{G}\bar{h}\bar{f}_2^{\text{can}\,\prime}\right), \qquad \text{(C5)}\\
\hat{\lambda} &= (\hat{D}'\,\hat{V}_\varepsilon^{-1}\,\hat{D})^{-1}\hat{D}'\,\hat{V}_\varepsilon^{-1}\Big(\bar{h} - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}}\Big),\\
\hat{a} &= \frac{1}{\hat{\kappa}+1}\Big(\bar{h} - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}} - \hat{D}\hat{\lambda}\Big),
\end{aligned}$$

where $\hat{V}_\varepsilon = \widehat{\beta^{\text{mis}}\beta^{\text{mis}}}' + \hat{V}_e$, $\hat{M}_{hf_2^{\text{can}}} = T^{-1}\sum_{t=1}^{T} h_t f_{2t}^{\text{can}\,\prime}$, $\bar{h} = T^{-1}\sum_{t=1}^{T} h_t$ *with* $h_t = R_t - R_f 1_N - \hat{\beta}_1^{\text{can}}(f_{1t}^{\text{can}} - \bar{f}_1^{\text{can}})$ *and* $\bar{f}_1^{\text{can}} = T^{-1}\sum_{t=1}^{T} f_{1t}^{\text{can}}$, *and*

$$\hat{G} = \frac{1}{(\hat{\kappa}+1)} I_N + \frac{\hat{\kappa}}{(\hat{\kappa}+1)}\hat{D}(\hat{D}'\,\hat{V}_\varepsilon^{-1}\,\hat{D})^{-1}\hat{D}'\,\hat{V}_\varepsilon^{-1}.$$

Note that $\hat{D} = (\widehat{\beta^{\text{mis}}}, \hat{B}_1^{\text{can}})$ *and* $\hat{V}_e$ *do not admit a closed-form solution and, as before,* $\hat{\mathbb{E}}(f_t)$ *and* $\hat{V}_f$ *coincide with the sample mean and sample covariance of the observed factors* f_t.

(ii) If the optimal value of the Karush-Kuhn-Tucker multiplier satisfies $\hat{\kappa} = 0$ *one can estimate only* $\alpha_N = a + D\lambda$ *but not the two components separately, and one obtains*

$$\hat{\alpha}_{N,MLC} = \bar{R} - \bar{R}_f 1_N - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}},$$

and the expression for $\text{vec}(\hat{\beta}_2^{\text{can}})$ *can be obtained by setting* $\hat{\kappa} = 0$ *in the terms that appear in* (C5). *The expressions for* $\hat{\mathbb{E}}(f_t)$ *and* $\hat{V}_f$ *are unchanged, and, as before, the expressions for the estimators of* $\hat{D}$ *and* $\hat{V}_e$ *do not admit a closed-form solution.*

Proof. Within this proof, for simplicity, we do not use the $\tilde{\cdot}$ notation to denote feasible parameter values.

Defining by $\tilde{\theta}$ the MLC corresponding to $\hat{\kappa} = 0$, this is unfeasible whenever we have that $\tilde{a}'\tilde{V}_\varepsilon^{-1}\tilde{a} > \delta_{\text{apt}}$. Similarly, case $\hat{\kappa} > 0$ is unfeasible whenever,

$$\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}} - \hat{D}\,\hat{\lambda}\Big)'\hat{V}_\varepsilon^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}} - \hat{D}\,\hat{\lambda}\Big) < \delta_{\text{apt}},$$

because $(1+\hat{\kappa})^2 = \frac{\left[\bar{R} - \bar{R}_f 1_N - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}} - \hat{D}\,\hat{\lambda}\right]'\hat{V}_\varepsilon^{-1}\left[\bar{R} - \bar{R}_f 1_N - \hat{\beta}_2^{\text{can}}\,\bar{f}_2^{\text{can}} - \hat{D}\,\hat{\lambda}\right]}{\delta_{\text{apt}}}$. When both cases are feasible, the optimal value for the Karush-Kuhn-Tucker multiplier will be greater than zero or equal to zero, depending on which case maximizes the log-likelihood, namely depending on whether $L(\hat{\theta})$ or $L(\tilde{\theta})$ is largest, respectively. Note that when $\kappa > 0$ then $\hat{a}'\hat{V_\varepsilon}^{-1}\hat{a} = \delta_{\text{apt}}$ by construction.

We now derive the formulae for the estimators. Assume for now that case $\hat{\kappa} > 0$ holds. Differentiating the penalized log-likelihood with respect to λ, a, and the Lagrange multiplier κ, the first $K^* + N$ equations, setting $K^* = K^{\text{mis}} + K_1^{\text{can}}$, (after some algebra) are:

$$\begin{pmatrix} D'V_\varepsilon{}^{-1} \\ I_N \end{pmatrix}\Big(\bar{R} - \bar{R}_f 1_N - \beta_2^{\text{can}}\bar{f}_2^{\text{can}}\Big) = \begin{pmatrix} D'V_\varepsilon{}^{-1}D & D'V_\varepsilon{}^{-1} \\ D & (1+\hat{\kappa})I_N \end{pmatrix}\begin{pmatrix} \hat{\lambda} \\ \hat{a} \end{pmatrix},$$

where recall that $V_\varepsilon = \beta^{\text{mis}}\beta^{\text{mis}\prime} + V_e$, and noting that all the expressions above and below are left as function of the feasible values for V_e and D (as opposed to their MLC values). It is straightforward to see that, because of the APT restriction, λ and a can now be identified separately, as long as $\hat{\kappa} > 0$. In fact, the above system of linear equations can be solved because the matrix pre-multiplying $\hat{\lambda}$ and $\hat{a}$ is non-singular for every $\hat{\kappa} > 0$, leading to the closed-form solution:

$$\hat{\lambda} = (D'V_\varepsilon{}^{-1}D)^{-1}D'V_\varepsilon{}^{-1}\Big(\bar{R} - \bar{R}_f 1_N - \beta_2^{\text{can}}\bar{f}_2^{\text{can}}\Big), \tag{C6}$$

$$\hat{a} = \frac{1}{\hat{\kappa}+1}\Big(\bar{R} - \bar{R}_f 1_N - \beta_2^{\text{can}}\bar{f}_2^{\text{can}} - D\hat{\lambda}\Big). \tag{C7}$$

Turning now to the first-order condition with respect to the generic (a,b)th element of $\beta_{2N,\text{MLC}}$, denoted by B_{2ab} with $1 \le a \le N$, $1 \le b \le K_2^{\text{can}}$, one obtains

$$\frac{1}{T}\sum_{t=1}^{T}\Big(R_t - R_{ft}1_N - \beta^{\text{mis}}\lambda^{\text{mis}} - a - \beta_1^{\text{can}}(\lambda_1^{\text{can}} + f_{1t}^{\text{can}} - \bar{f}_1^{\text{can}}) - \hat{\beta}_2^{\text{can}}f_{2t}^{\text{can}}\Big)' V_\varepsilon{}^{-1}(-\frac{\partial\beta_2^{\text{can}}}{\partial B_{2ab}}f_{2t}^{\text{can}}) = 0.$$

Now, inserting (C6) and (C7) into the above expression, setting

$$G = \frac{1}{(\hat{\kappa}+1)}I_N + \frac{\hat{\kappa}}{(\hat{\kappa}+1)}D(D'V_\varepsilon{}^{-1}D)^{-1}D'V_\varepsilon{}^{-1},$$

and re-arranging terms yields, setting $\hat{M}_{f_1^{\text{can}}f_2^{\text{can}}} = 1/T\sum_{t=1}^{T} f_{1t}^{\text{can}}f_{2t}^{\text{can}\prime}$,

$$\hat{\beta}_2^{\text{can}}\hat{M}_{f_2^{\text{can}}} - G\hat{\beta}_2^{\text{can}}\bar{f}_2^{\text{can}}\bar{f}_2^{\text{can}\prime} = \hat{M}_{Rf_2^{\text{can}}} - G(\bar{R} - \bar{R}_f 1_N)\bar{f}_2^{\text{can}\prime} - \beta_1^{\text{can}}(\hat{M}_{f_1^{\text{can}}f_2^{\text{can}}} - \bar{f}_1^{\text{can}}\bar{f}_2^{\text{can}\prime}),$$

which can be rewritten more succinctly as

$$\frac{1}{T}\sum_{t=1}^{T} f_{2t}^{\text{can}}g_t' = 0_{K_2^{\text{can}}\times N},$$

with $g_t = \left(h_t - G\bar{h} - \hat{\beta}_2^{\text{can}} f_{2t}^{\text{can}} + G\hat{\beta}_2^{\text{can}} \bar{f}_2^{\text{can}}\right)$. Taking the vec operator and solving for $\hat{\beta}_2^{\text{can}}$ gives the desired expression in (C5).

We need to show that a solution for $\hat{\beta}_2^{\text{can}}$ exists. This requires one to establish that the matrix $\left((\hat{M}_{f_2^{\text{can}}} \otimes I_N) - (\bar{f}_2^{\text{can}} \bar{f}_2^{\text{can}\prime} \otimes G)\right)$ is invertible. This matrix can be written as

$$\left((\hat{M}_{f_2^{\text{can}}} \otimes I_N) - (\bar{f}_2^{\text{can}} \bar{f}_2^{\text{can}\prime} \otimes G)\right) = \left(((\hat{M}_{f_2^{\text{can}}} \bar{f}_2^{\text{can}} \bar{f}_2^{\text{can}\prime}) \otimes I_N) + (\bar{f}_2^{\text{can}} \bar{f}_2^{\text{can}\prime} \otimes (I_N - G))\right).$$

The first matrix on the right hand side is non-singular, given the assumptions made. One then just needs to show that the second matrix is positive semi-definitive, which follows from the proof of Theorem B1.

Therefore, plugging $\hat{\beta}_2^{\text{can}} = \hat{\beta}_2^{\text{can}}(D, V_e)$ into $\hat{\lambda}$ and $\hat{a}$, and then $\hat{\lambda}$ into $\hat{a}$, one obtains that

$$\hat{\beta}_2^{\text{can}} = \hat{\beta}_2^{\text{can}}(D, V_e), \hat{\lambda} = \hat{\lambda}(D, V_e), \quad \hat{a} = \hat{a}(D, V_e) \quad \text{and} \quad \hat{\kappa} = \hat{\kappa}(D, V_e).$$

Substituting them into $L(\theta) - \kappa(a' V_\varepsilon{}^{-1} a - \delta_{\text{apt}})$, gives the concentrated log-likelihood function, which is a function of only D and V_e and it will be maximized numerically to obtain $\hat{D}$ and $\hat{V}_e$. Observe that the penalization term vanishes for the concentrated log likelihood function for either $\hat{\kappa} = 0$ and $\hat{\kappa} > 0$.

(ii) Suppose now that $\hat{\kappa} = 0$. One can clearly obtain a unique solution for $(D, I_N)\begin{pmatrix} \tilde{\lambda} \\ \tilde{a} \end{pmatrix} = D\tilde{\lambda} + \tilde{a}$. However, to solve for $\tilde{\lambda}$ and $\tilde{a}$ separately, one needs to invert the matrix

$$\begin{pmatrix} D' V_\varepsilon{}^{-1} \\ I_N \end{pmatrix} (D, I_N) = \begin{pmatrix} D' V_\varepsilon{}^{-1} D & D' V_\varepsilon{}^{-1} \\ D & I_N \end{pmatrix},$$

which is not possible because it is of dimension $(N+K^*) \times (N+K^*)$ but of rank N, because the left-hand side shows that it is obtained from the product of two matrices of dimension $(N + K^*) \times N$. Thus, only the sum $D\tilde{\lambda} + \tilde{a}$ can be estimated. All the other parameters are identified separately and their expressions follow from differentiating $L(\theta)$ and solving the resulting first-order conditions. For instance, the formula for $\tilde{\beta}_2^{\text{can}}$ follows from setting $G = I_N$ into (C5). $\square$

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