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INFORMATION, MEASURABILITY, AND CONTINUOUS BEHAVIOR

by

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Information, Measurability, and Continuous Behavior

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Abstract

The stability of optimal plans with respect to information is studied given the representation of information as sub-σ-fields of a probability space. A decision maker is constrained to choose a plan measurable with respect to her information. Continuity is derived by characterizing the continuity of the measurability constraint correspondence and then applying a generalized maximum theorem. This approach can be simpler and require fewer assumptions than an approach based on continuity of conditional expectations.

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Contents

1 Introduction

In many economic models with uncertainty, the information structure is an endogenous variable or an exogenous parameter. When the set of information structures is infinite, one may need a topology on information with respect to which an agent's state-dependent choices and payoffs depend continuously on her information. This paper studies continuity with respect to information when information is represented by sub-σ-fields of a probability space. It builds on a literature initiated by Allen (1983), who was the first to apply topologies on sub-σ-fields to economics, using a metric introduced by Boylan (1971). Cotter (1986, 1987) showed that suitable continuity properties could be obtained with a weaker topology, called the pointwise convergence topology, and Stinchcombe (1990) derived further results about both topologies.

There are two ways we can view information and how it affects behavior. In the *ex-post* or *Bayesian* view, an agent chooses an action *after* observing her information and updating her belief. Information affects state-dependent actions through posteriors. In the *ex-ante* or *measurability* view, an agent formulates a plan of what action to take in each state *before* observing her information, subject to the constraint that the plan be measurable with respect to her information. Information affects actions through this measurability constraint. There are well-known conditions (reviewed in Section 8) under which these two viewpoints are equivalent, in the sense that an agent's state-dependent choices are the same whether they solve the ex-ante constrained maximization problem or solve state-by-state the ex-post maximization problem given updated beliefs.

However, the two viewpoints suggest different mathematical methods for characterizing continuity of behavior with respect to information. The methods used by Allen (1983), Cotter (1986, 1987), and Stinchcombe (1990) are based on the ex-post view. Consider first Stinchcombe (1990), who imposed assumptions on the probability space such that state-dependent posteriors (regular conditional probabilities—see Ash (1972, Sec. 6.6)) given a sub-σ-field are well-defined. He showed that the mapping from information to state-dependent posteriors is continuous under suitable topologies. The mapping from posteriors to actions is also continuous, and hence the composition of these two mappings—which takes information to state-dependent actions—is continuous. When state-dependent posteriors are not well-defined, one can use the method in Allen (1983) and Cotter (1986, 1987), who showed that the mapping from information to conditional expected utility is continuous and then used the fact that the mapping from utility to actions is continuous.[1]

The ex-ante viewpoint suggests instead that we derive continuity in the way one derives continuity of consumer demand with respect to prices—by application of the maximum theorem. This theorem provides conditions for the continuity of solutions to a constrained maximization problem with respect to exogenous parameters that affect the constraint. The purpose of this paper is to develop this method for the pointwise convergence topology introduced by Cotter (1986). Because the set of plans lies in an infinite-dimensional vector space, we use the generalized two-topology maximum theorem in Horsley et al. (1998a). The main step in this paper is to show that the measurability constraint correspondence is lower hemicontinuous and closed in suitable topologies.

This ex-ante method has several practical advantages over the ex-post method. First, it can be simpler and more direct to apply in abstract models, in which decision problems are often framed from an ex-ante viewpoint (particularly when agents must choose information

[1] See Van Zandt (1989, Chap. 1) for a more precise overview of this work and comparison with the current paper.

in a first stage). An example of this simplicity is in a proof by Cotter (1994) that the set of type-correlated equilibria (an extension of correlated equilibria to Bayesian games) depends upper hemicontinuously on information. Second, it does not require additive separability of preferences. Third, it can be applied in contracting problems in which the ex-post viewpoint is not valid because feasible choices in one state depend on choices made in other states. At a purely mathematical level, developing this method links the topological proximity of information and the proximity of the sets of measurable functions. Furthermore, it provides an example of maximum theorem arguments in a case where the parameter and choice spaces are complex. The one disadvantage of the method is that continuity is with respect to the weak topology on plans, whereas the results using the ex-post method are with respect to the L^1-norm. However, in practice such a weak topology is needed anyway for compactness.

A related exercise appears in Van Zandt (1993), but compared to the current paper it has fewer applications in economics. That paper considered the Boylan metric on information studied by Allen (1983), whose topology is stronger than the pointwise convergence topology and has been used to study rates of convergence of martingales. Van Zandt (1993) showed that, for the topology of convergence in measure on measurable functions and the associated Hausdorff metric on subsets of measurable functions, the distance between two σ-fields is directly related to the distance between their respective sets of measurable functions. From this result, continuity of the value of information in the metric follows under very general assumptions, but there are no results on the continuity of state-dependent plans with respect to information.

2 Uncertainty, information, and the decision problem

An overview of the exercise is followed by details on the notation and terminology.

Overview An agent faces a static decision problem in which uncertainty is represented by a probability space $\langle \Omega, \Sigma, \mu \rangle$. The agent observes information represented by a sub-σ-field $\mathcal{F}$ of Σ; we emphasize that this is an ex-ante representation of information and so it denotes not what an agent has observed but rather what an agent would observe in each state. After observing the realization of her information, the agent chooses an action in a non-empty, convex, and compact set $X \subset \mathbb{R}^m$. Before observing her information, she formulates a state-contingent plan $f\colon \Omega \to X$. In order to be informationally feasible, this plan must be measurable with respect to $\mathcal{F}$, meaning that $f^{-1}(B) \in \mathcal{F}$ for each Borel subset B of X. The agent may face additional economic constraints that depend on a parameter p in a set P; we denote by $\beta(p)$ the set of plans that are economically feasible when the measurability constraint is not taken into account. The agent's preferences over plans are given by the complete preorder $\succsim$. Her decision problem can thus be written

$$
\begin{aligned}
(1) \qquad & \max_{f\colon \Omega \to X} \quad \succsim \\
& \text{subject to:} \quad f \in \beta(p) \\
& \qquad\qquad\qquad f \text{ is } \mathcal{F}\text{-measurable.}
\end{aligned}
$$

We let ψ be the solution correspondence, which maps parameters p and information $\mathcal{F}$ to the set of solutions to this decision problem. The purpose of this paper is to characterize the continuity of ψ.

Equivalence classes of plans We referred to a plan as a measurable function from $\langle \Omega, \Sigma \rangle$ into X, where X is endowed with its Borel σ-field. Our formal definition of a *plan* is an element of the set $\mathfrak{X}$ of *equivalence classes* of such measurable functions, modulo being equal almost everywhere. Thus, we implicitly assume that the agent is indifferent between plans that are equal a.e., so we can define preferences directly on equivalence classes of plans.[2]

Equivalence classes of information We referred to information as a sub-σ-field of $\mathcal{F}$. In our formal definition, we also group information into equivalence classes such that the agent is indifferent between pieces of information in the same class. Specifically, we say that two sub-σ-fields $\mathcal{F}$ and $\mathcal{G}$ are *equivalent* if (a) for every $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that the symmetric difference $(F \setminus G) \cup (G \setminus F)$ is null, and (b) vice versa (for every $G \in \mathcal{G}$, there is $\ldots$). If $\mathcal{F}$ and $\mathcal{G}$ are equivalent, then for every $\mathcal{F}$-measurable plan f there is a $\mathcal{G}$-measurable plan g that is equal to f a.e., and vice versa (Van Zandt (1989, Chap. 1, Lemma A.14)); this implies that the set of $\mathcal{F}$-measurable plans and the set of $\mathcal{G}$-measurable plans correspond to the same equivalence classes in $\mathfrak{X}$. It follows also (or see Boylan (1971, Thm. 2)) that, if $\mathcal{F}$ and $\mathcal{G}$ are equivalent, then $E[f \,|\, \mathcal{F}] = E[f \,|\, \mathcal{G}]$ a.e. for any integrable function or plan f. We let $\mathfrak{F}$ denote the set of equivalence classes and define *information* to be an element of $\mathcal{F}$.

Comments on the representation of information Our representation of information is the standard one in statistical decision theory. It is a generalization to infinite state spaces of the representation of information as partitions, and it is a generalization of the representation of information as random variables. Specifically, the observation of a random variable $g\colon \Omega \to \mathbb{R}$ yields the information

$$\sigma(g) \;\equiv\; \left\{g^{-1}(B) \;\middle|\; B \text{ is a Borel subset of } \mathbb{R}\right\} ,$$

in the sense that a plan $f\colon \Omega \to X$ is $\sigma(g)$-measurable if and only if there is a Borel-measurable decision rule $h\colon \mathbb{R} \to X$ such that $f = h \circ g$ (Van Zandt (1989, Lemma A.7)). Furthermore, expectations conditional on a random object are equivalent to expectations conditional on the sub-σ-field generated by the random object (Ash (1972, Comm. 6.4.4)). Sub-σ-fields are only slightly more general than random variables. For example, if Σ is generated by a countable field—as is the set of Borel subsets of a separable metric space—then every sub-σ-field is equivalent to one generated by a random variable (Stinchcombe (1990, Lemma 3.2.2)). The real advantage to treating abstract information as sub-σ-fields is parsimony: two random variables that are not equal a.e. may generate the same sub-σ-field (and thus the same information), while for each two non-equivalent sub-σ-fields, there is an equivalence class of plans that is measurable with respect to one but not the other.

Formal definition of the decision problem We can now define the components of the decision problem in equation (1) more precisely:

- the economic constraint correspondence is $\beta\colon P \twoheadrightarrow \mathfrak{X}$;
- the measurability constraint correspondence $M\colon \mathfrak{F} \twoheadrightarrow \mathfrak{X}$ is defined by

$$M(\mathcal{F}) \;=\; \{f \in \mathfrak{X} \mid f \text{ is } \mathcal{F}\text{-measurable}\} \;;$$

- the overall constraint correspondence $\phi\colon P \times \mathfrak{F} \twoheadrightarrow \mathfrak{X}$ is defined by

$$\phi(p, \mathcal{F}) = \beta(p) \cap M(\mathcal{F}) \;;$$

[2]The existence of a probability measure μ for which this indifference holds is the only sense in which the agent is required to act probabilistically. If μ is replaced by any other probability measure with the same null sets, then the topology we define on information will be the same, as will the topologies on plans.

- the solution correspondence $\psi\colon P \times \mathfrak{F} \twoheadrightarrow \mathfrak{X}$ is defined by

$$\psi(p, \mathcal{F}) \;=\; \{f \in \phi(p, \mathcal{F}) \mid f \succsim g \;\forall g \in \phi(p, \mathcal{F})\} \;.$$

3 Continuity of correspondences and of preferences

Since terminology varies in the literature, we state here our definitions of hemicontinuity and closedness of correspondences.

Definition 1 Let Y and Z be topological spaces. A correspondence $\zeta\colon Y \twoheadrightarrow Z$ is

1. *lower hemicontinuous* if, for each open $U \subset Z$, $\{y \in Y \mid \zeta(y) \cap U \neq \varnothing\}$ is open;
2. *upper hemicontinuous* if, for each open $U \subset Z$, $\{y \in Y \mid \zeta(y) \subset U\}$ is open;
3. *continuous* if it is both lower and upper hemicontinuous;
4. *closed* if the graph of ζ, $\mathrm{Gr}(\zeta) \equiv \{\langle y, z\rangle \in Y \times Z \mid z \in \zeta(y)\}$, is a closed subset of $Y \times Z$.

We will refer to the following basic properties(see e.g. Klein and Thompson (1984)).

1. A function is continuous if and only if it is continuous as a correspondence.
2. The composition of two lower (resp., upper) hemicontinuous correspondences is lower (resp., upper) hemicontinuous.

We use also the following result.

Lemma 1 *Let Y_1, Y_2, and Z be topological spaces. Suppose that the function $h\colon Y_1 \times Y_2 \to Z$ is continuous. Then the correspondence $y_2 \mapsto h(Y_1, y_2)$ is lower hemicontinuous.*

Proof: Let $U \subset Z$ be open. Observe that $\{y_2 \in Y_2 \mid \zeta(y_2) \cap U \neq \varnothing\} = \mathrm{Proj}_{y_2}(h^{-1}(U))$. Since h is continuous, $h^{-1}(U)$ is open. The projection map is open, and hence $\mathrm{Proj}_{y_2}(h^{-1}(U))$ is open. □

The following definition of semicontinuity of a preference preorder is standard.

Definition 2 Let Z be a topological space and let $\succsim$ be a complete preorder on Z. We say that $\succsim$ is *lower* (resp., *upper*) *semicontinuous* if the weakly-worse-than set $\{z' \in Z \mid z \succsim z'\}$ (resp., weakly-better-than set $\{z' \in Z \mid z' \succsim z\}$) is closed for all $z \in Z$. We say that $\succsim$ is *continuous* if it is both lower and upper semicontinuous.

4 Applying a maximum theorem

In this section we outline how to characterize continuity of ψ by applying a maximum theorem and then we discuss associated topological issues.

We assume that P is endowed with a Hausdorff topology that remains fixed throughout this paper. In Section 5, we define a topology on $\mathfrak{F}$ that also remains fixed; for now, we take as given that $\mathfrak{F}$ is a topological space. The crucial topologies under discussion in this section are on $\mathfrak{X}$. We need two: the stronger one $\mathcal{S}$ is that of the L_1-norm $\|\cdot\|_1$; the weaker one $\mathcal{W}$

is the weak topology $\sigma(L_1(\mathbb{R}^m), L_\infty(\mathbb{R}^m))$.[3] Whenever we state a condition that depends on the topology of $\mathfrak{X}$, we will preface it by $\mathcal{W}$ or $\mathcal{S}$, meaning that the condition holds when $\mathfrak{X}$ has the indicated topology and also P and $\mathfrak{F}$ have their respective topologies.

The topology $\mathcal{W}$ has a basis consisting of all sets of the form

$$\left\{ f' \in \mathfrak{X} \;\middle|\; \left|\int g_i(f'-f)\right| < \epsilon_i \;\forall i = 1,\ldots,n \right\} ,$$

where $f \in \mathfrak{X}$, n is a positive integer, and (for $i = 1,\ldots,n$) $g_i \in L_\infty(\mathbb{R}^m)$ and $\epsilon_i > 0$. Because X is convex and compact, $\mathfrak{X}$ is $\mathcal{W}$-compact.

The conventional maximum theorem (Berge (1963, pp. 115–116)) states that if ϕ is lower hemicontinuous and has a closed graph and if $\succsim$ is continuous, then ψ has a closed graph. For this result to be useful (for example, in order to ensure that closedness of ψ implies upper hemicontinuity and in order to apply a fixed-point theorem in an equilibrium model), $\mathfrak{X}$ should be endowed with a compact topology or at least with a topology such that each $p \in P$ has a neighborhood on which the range of β has a compact closure. For this purpose, the suitable topology is $\mathcal{W}$ because $\mathfrak{X}$ is $\mathcal{W}$-compact if X is bounded.

However, to assume that $\succsim$ is $\mathcal{W}$-continuous is too restrictive. The weakest topology generally considered appropriate for continuity of $\succsim$ is $\mathcal{S}$. For example, additively separable preferences are $\mathcal{S}$-continuous but not $\mathcal{W}$-continuous if $\langle \Omega, \Sigma, \mu \rangle$ is non-atomic.

Lemma 2 *Let $v\colon X \times \Omega \to \mathbb{R}$ have the following properties:*

1. *for all $x \in X$, $v(x,\cdot)\colon \Omega \to \mathbb{R}$ is measurable;*
2. *for a.e. $\omega \in \Omega$, $v(\cdot,\omega)\colon X \to \mathbb{R}$ is continuous;*
3. *the function $\omega \mapsto \max_{x\in X} |v(x,\omega)|$ is integrable.*

Then the mapping $V\colon \mathfrak{X} \to \mathbb{R}$, defined by

$$V(f) \;=\; \int_\Omega v(f(\omega),\omega)\, d\mu(\omega) \;,$$

is $\mathcal{S}$-continuous. If $\langle \Omega, \Sigma, \mu \rangle$ is non-atomic, if X is not a singleton, and if $v(\cdot,\omega)\colon X \to \mathbb{R}$ is strictly concave a.e., then V is not $\mathcal{W}$-continuous.

Proof: The result on $\mathcal{S}$-continuity is in Van Zandt (1989, Lemma 3.3). The result on lack of $\mathcal{W}$-continuity is in Bewley (1972, p. 539). □

There is thus a tension between the need for a weak topology on $\mathfrak{X}$ to ensure compactness and a strong topology to ensure continuity of preferences. To resolve this tension, we resort to the maximum theorem in Horsley et al. (1998a), which makes use of two unrelated topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathfrak{X}$. The assumptions of lower semicontinuity of preferences and lower hemicontinuity of ϕ are stated with respect to $\mathcal{T}_1$, and then the assumptions of

[3] We defined these topologies treating $\mathfrak{X}$ as a subset of $L_1(\mathbb{R}^m)$ and using the dual system $\langle L_1(\mathbb{R}^m), L_\infty(\mathbb{R}^m)\rangle$. Because X is compact, they have equivalent definitions based on the dual system $\langle L_\infty(\mathbb{R}^m), L_1(\mathbb{R}^m)\rangle$. The topology $\mathcal{S}$ coincides on $\mathfrak{X}$ with the Mackey topology $\tau(L_\infty(\mathbb{R}^m), L_1(\mathbb{R}^m))$ and also with the topology of convergence in measure—see Hu and Papageorgiou (2000, VI.4.7) or Van Zandt (2000) for a complete proof. The topology $\mathcal{W}$ coincides on $\mathfrak{X}$ with the weak* topology $\sigma(L_\infty(\mathbb{R}^m), L_1(\mathbb{R}^m))$—the weak* topology is clearly stronger; they coincide because a compact Hausdorff topology and a weaker Hausdorff topology must be the same.

upper semicontinuity of preferences and closedness of ϕ are stated with respect to $\mathcal{T}_2$. The conclusion is that the solution correspondence ψ has a closed graph for the topology $\mathcal{T}_2$.

For the reasons just stated, we want $\mathcal{T}_2$ to be a compact topology and we let it be $\mathcal{W}$. It is acceptable to assume that $\succsim$ is $\mathcal{W}$-upper semicontinuous because a convex and $\mathcal{S}$-closed set is also $\mathcal{W}$-closed; hence, $\succsim$ is $\mathcal{W}$-upper semicontinuous if it is $\mathcal{S}$-continuous and convex. We then let $\mathcal{T}_1$ be the topology $\mathcal{S}$. With this assignment of topologies, we have the following analog of Horsley et al. (1998a, Cor. 8).

Lemma 3 *Assume that ϕ is $\mathcal{S}$-lower hemicontinuous and $\mathcal{W}$-closed and that it has non-empty and $\mathcal{W}$-compact values. Assume $\succsim$ is $\mathcal{S}$-lower semicontinuous and $\mathcal{W}$-upper semicontinuous. Then ψ is $\mathcal{W}$-closed and has non-empty and $\mathcal{W}$-compact values.*

5 Continuity of the measurability constraint

In order to apply this two-topology maximum theorem, we must define a topology on information and show that the measurability constraint M is $\mathcal{S}$-lower hemicontinuous and $\mathcal{W}$-closed.

We begin with some preliminaries and a basic result. Let $E[\cdot|\cdot]\colon \mathfrak{X} \times \mathfrak{F} \to \mathfrak{X}$ be the function defined by $\langle f, \mathcal{F}\rangle \mapsto E[f\,|\,\mathcal{F}]$. Note that this is well-defined because (a) $\mathfrak{X} \subset L_1(\mathbb{R}^m)$ and (b) since X is convex, $E[f\,|\,\mathcal{F}] \in \mathfrak{X}$ for $f \in \mathfrak{X}$ and $\mathcal{F} \in \mathfrak{F}$. We make frequent use of the following fact: for $f \in L_1(\mathbb{R}^m)$ and $\mathcal{F} \in \mathfrak{F}$, f is $\mathcal{F}$-measurable if and only if $f = E[f\,|\,\mathcal{F}]$ a.e.

Proposition 1 *Suppose $\mathfrak{F}$ and $\mathfrak{X}$ are endowed with topologies such that $E[\cdot|\cdot]$ is jointly continuous. Then M is lower hemicontinuous. If also the topology on $\mathfrak{X}$ is Hausdorff, then M is closed.*

Proof: Assume that $E[\cdot|\cdot]$ is jointly continuous.

Let $\mathcal{F} \in \mathfrak{F}$. Because a function $f \in \mathfrak{X}$ is $\mathcal{F}$-measurable if and only if $E[f\,|\,\mathcal{F}] = f$ a.e., $M(\mathcal{F})$ is the image of $\mathfrak{X}$ under $E[\cdot|\mathcal{F}]$. That is, $M(\mathcal{F}) = E[\mathfrak{X}|\mathcal{F}]$. According to Lemma 1, M is lower hemicontinuous if $E[\cdot|\cdot]$ is continuous.

Furthermore, $\mathrm{Gr}(M) = \{\langle \mathcal{F}, f\rangle \mid E[f\,|\,\mathcal{F}] = f \text{ a.e.}\}$. This is the inverse image of the diagonal in $\mathfrak{X} \times \mathfrak{X}$ under the map $\langle \mathcal{F}, f\rangle \mapsto \langle E[f\,|\,\mathcal{F}], f\rangle$, which is continuous as the product of the continuous maps $\langle \mathcal{F}, f\rangle \mapsto E[f\,|\,\mathcal{F}]$ and $\langle \mathcal{F}, f\rangle \mapsto f$. If the topology on $\mathfrak{X}$ is Hausdorff, then the diagonal of $\mathfrak{X} \times \mathfrak{X}$ is closed and hence $\mathrm{Gr}(M)$ is closed. □

To make use of this proposition, we should define a topology on $\mathfrak{F}$ such that, at the very least, $\mathcal{F} \mapsto E[f\,|\,\mathcal{F}]$ is $\mathcal{S}$-continuous for each $f \in \mathfrak{X}$. We use the weakest topology such that the mapping $\mathcal{F} \mapsto E[f\,|\,\mathcal{F}]$ from $\mathcal{F}$ to L_1 is continuous in the L_1-norm for each $f \in L_1(\mathbb{R})$. This topology, called the pointwise convergence topology and denoted by $\mathcal{P}$, was introduced and characterized by Cotter (1986, 1987); see Stinchcombe (1990) and Van Zandt (1989, Chap. 1) for further equivalent definitions. The topology $\mathcal{P}$ has a basis consisting of all sets of the form

$$\left\{\mathcal{G} \in \mathfrak{F} \;\middle|\; \|E[f_i|\mathcal{F}] - E[f_i|\mathcal{G}]\|_1 < \epsilon_i \;\forall i = 1, \ldots, n\right\},$$

where $\mathcal{F} \in \mathfrak{F}$, n is a positive integer, and (for $i = 1, \ldots, n$) $f_i \in L_1(\mathbb{R}^m)$ and $\epsilon_i > 0$. For the rest of this paper, $\mathfrak{F}$ is endowed with $\mathcal{P}$.[4]

[4]One drawback of the pointwise convergence topology is that the operation of combining information,

Then $\mathcal{F} \mapsto E[f|\mathcal{F}]$ is $\mathcal{S}$-continuous and hence $\mathcal{W}$-continuous (since $\mathcal{W}$ is weaker than $\mathcal{S}$) for each $f \in \mathfrak{X}$. Furthermore, for each $\mathcal{F} \in \mathfrak{F}$, the expectations operation $E[\cdot|\mathcal{F}]$, defined by $f \mapsto E[f|\mathcal{F}]$, is a norm-continuous linear operator from $L_1(\mathbb{R}^m)$ to $L_1(\mathbb{R}^m)$; and hence it is also continuous in the weak topology on $L_1(\mathbb{R}^m)$ (Dunford and Schwartz (1958, Thm. V.3.15)). Joint continuity of $E[\cdot|\cdot]$ for the topology $\mathcal{S}$ was established by Cotter (1986). We now show that $E[\cdot|\cdot]$ is jointly continuous in the topology $\mathcal{W}$.

The proof makes use of the following fact.

Lemma 4 *Let $p, q \in [1, \infty]$ satisfy $1/p + 1/q = 1$. For $f \in L_p(\mathbb{R}^m)$, $g \in L_q(\mathbb{R}^m)$, and $\mathcal{F} \in \mathfrak{F}$, $\int gE[f|\mathcal{F}] = \int E[g|\mathcal{F}]f$.*

Proof: For $p = q = 2$, this result says that the expectations operator is self-adjoint, which holds since it is the orthogonal projection on the subspace of $\mathcal{F}$-measurable functions. More generally, recall the following basic property of conditional expectations: if g is $\mathcal{F}$-measurable then $\int gE[f|\mathcal{F}] = \int gf$ (e.g. Neveu (1965, Prop. I-2-12)). Since $E[g|\mathcal{F}]$ is $\mathcal{F}$-measurable, it follows that $\int E[g|\mathcal{F}]E[f|\mathcal{F}] = \int E[g|\mathcal{F}]f$. Reversing the roles of f and g, we have $\int E[f|\mathcal{F}]E[g|\mathcal{F}] = \int E[f|\mathcal{F}]g$. Hence, $\int gE[f|\mathcal{F}] = \int E[g|\mathcal{F}]f$. □

Proposition 2 *The expectations operator $\langle f, \mathcal{F}\rangle \mapsto E[f|\mathcal{F}]$ is jointly $\mathcal{W}$-continuous from $\mathfrak{X} \times \mathfrak{F}$ into $\mathfrak{X}$.*

Proof: Let $f \in \mathfrak{X}$ and let $\mathcal{F} \in \mathfrak{F}$. Recall that a typical basis $\mathcal{W}$-neighborhood of $E[f|\mathcal{F}]$ in $\mathfrak{X}$ is

$$N \equiv \left\{ f' \in \mathfrak{X} \,\middle|\, \left|\int g_i E[f|\mathcal{F}] - \int g_i f'\right| < \epsilon_i \ \forall i = 1, \ldots, n \right\},$$

where n is a positive integer and (for $i = 1, \ldots, n$) $g_i \in L_\infty(\mathbb{R}^m)$ and $\epsilon_i > 0$. We show that we can construct a $\mathcal{W}$-neighborhood N_f of f in $\mathfrak{X}$ and a neighborhood $N_{\mathcal{F}}$ of $\mathcal{F}$ in $\mathfrak{F}$ such that $E[\cdot|\cdot]$ maps $N_f \times N_{\mathcal{F}}$ into N. Hence, $E[\cdot|\cdot]$ is $\mathcal{W}$-continuous.

Let k, g_i, and ϵ_i be as given in the definition of N. Then

$$N_f \equiv \left\{ f' \in \mathfrak{X} \,\middle|\, \left|\int E[g_i|\mathcal{F}]f - \int E[g_i|\mathcal{F}]f'\right| < \epsilon_i/2 \ \forall i = 1, \ldots, n \right\}$$

is a $\mathcal{W}$-neighborhood of f in $\mathfrak{X}$, and

$$N_{\mathcal{F}} \equiv \left\{ \mathcal{F}' \in \mathfrak{F} \mid \|E[g_i|\mathcal{F}] - E[g_i|\mathcal{F}']\|_1 < \epsilon_i/(2\,\|\mathfrak{X}\|_\infty) \ \forall i = 1, \ldots, n \right\}$$

is a neighborhood of $\mathcal{F}$ in $\mathfrak{F}$. Let $f' \in N_f$ and let $\mathcal{F}' \in N_{\mathcal{F}}$. Then, for all $i = 1, \ldots, n$,

$$\left|\int g_i E[f|\mathcal{F}] - \int g_i E[f'|\mathcal{F}']\right|$$

$$= \left|\int E[g_i|\mathcal{F}]f - \int E[g_i|\mathcal{F}']f'\right| \tag{2}$$

$$\leq \left|\int E[g_i|\mathcal{F}]f - \int E[g_i|\mathcal{F}]f'\right| + \left|\int E[g_i|\mathcal{F}]f' - \int E[g_i|\mathcal{F}']f'\right| \tag{3}$$

$$< \epsilon_i/2 + \|E[g_i|\mathcal{F}] - E[g_i|\mathcal{F}']\|_1 \cdot \|f'\|_\infty \tag{4}$$

$$< \epsilon_i \,. \tag{5}$$

(The first relation (2) is from Lemma 4; the second relation (3) is the triangle inequality; the third relation (4) is from the definition of N_f and the fact that $\left|\int gh\right| \leq \|g\|_1 \cdot \|h\|_\infty$ for $g \in L_1$ and $h \in L_\infty$; and the fourth relation (5) is from the definition of $N_{\mathcal{F}}$.) Therefore, using the definition of N, we have $E[f'|\mathcal{F}'] \in N$. □

$\langle \mathcal{F}, \mathcal{G}\rangle \mapsto \mathcal{F} \vee \mathcal{G}$, is not continuous (Cotter (1986)). However, this operation is uniformly continuous in the Boylan metric (Allen and Van Zandt (1992)). Because the Boylan metric topology is stronger than $\mathcal{P}$, the results of this paper hold if the Boylan metric topology is substituted for $\mathcal{P}$.

Corollary 1 *M is continuous and closed whether $\mathfrak{X}$ is endowed with $\mathcal{S}$ or with $\mathcal{W}$.*

Proof: Both $\mathcal{S}$ and $\mathcal{W}$ are Hausdorff topologies. Apply Propositions 1 and 2. □

Remark 1 It is the proof of Proposition 2 that uses the assumption that X is compact.

We now take up continuity of the correspondence ϕ, which is the intersection of $\langle p, \mathcal{F}\rangle \mapsto M(\mathcal{F})$ and $\langle p, \mathcal{F}\rangle \mapsto \beta(p)$. The intersection of two closed correspondences is closed, and so we have the following.

Proposition 3 *If β is $\mathcal{S}$-closed or $\mathcal{W}$-closed, then so is ϕ.*

However, the intersection of two lower hemicontinuous correspondences is not necessarily lower hemicontinuous. We can still establish lower hemicontinuity of ϕ as long as β satisfies a condition we call state independence.

Definition 3 We say that β is *state-independent* if, for all $p \in P$, all $f \in \beta(p)$, and all $\mathcal{F} \in \mathfrak{F}$, we have $E[f \,|\, \mathcal{F}] \in \beta(p)$.

In Section 7, we discuss this condition and the consequences of relaxing it.

Proposition 4 *Assume that β is state-independent. Then ϕ is lower hemicontinuous for any topology on $\mathfrak{X}$ such that β is lower hemicontinuous and $E[\cdot \,|\, \cdot]$ is continuous.*

Proof: This proof is related to the proof of Proposition 1. First we show that $\phi(p, \mathcal{F}) = E[\beta(p) \,|\, \mathcal{F}]$ for $\langle p, \mathcal{F}\rangle \in P \times \mathfrak{F}$, as follows. (a) Suppose $f \in \phi(p, \mathcal{F})$. Then $f \in \beta(p)$ and, since f is $\mathcal{F}$-measurable, $f = E[f \,|\, \mathcal{F}]$. Hence, $f \in E[\beta(p) \,|\, \mathcal{F}]$. (b) Suppose $f \in E[\beta(p) \,|\, \mathcal{F}]$. Then there is an $f' \in \beta(p)$ such that $f = E[f' \,|\, \mathcal{F}]$. Because β is state-independent, $f \in \beta(p)$. Because f is $\mathcal{F}$-measurable, $f \in \phi(p, \mathcal{F})$.

Assume that β is lower hemicontinuous and that $E[\cdot \,|\, \cdot]$ is continuous. Hence, ϕ is the composition of $\langle p, \mathcal{F}\rangle \mapsto \beta(p) \times \{\mathcal{F}\}$ and the function $E[\cdot \,|\, \cdot]$. If β is lower hemicontinuous then so is the first correspondence, because the product of two lower hemicontinuous correspondences is lower hemicontinuous (Klein and Thompson (1984, Thm. 7.3.12)).[5] The correspondence $\langle f, \mathcal{F}\rangle \mapsto \{E[f \,|\, \mathcal{F}]\}$ is continuous the function $E[\cdot \,|\, \cdot]$ is continuous. The composition of lower hemicontinuous correspondences is lower hemicontinuous. Hence, ϕ is lower hemicontinuous. □

Corollary 2 *Assume that β is state-independent. If β is $\mathcal{S}$- or $\mathcal{W}$-lower hemicontinuous, then so is ϕ.*

6 Continuity of behavior and applications

Combining Lemma 3 and Proposition 3 and 4 yields the following theorem.

Theorem 1 *Assume that*

[5]Note that the product of two *upper* hemicontinuous correspondences is not necessarily upper hemicontinuous. Hence, this type of proof cannot be use to show that ϕ is upper hemicontinuous without assuming that $\mathfrak{X}$ is compact.

1. $\succsim$ *is* $\mathcal{S}$*-lower semicontinuous and* $\mathcal{W}$*-upper semicontinuous,*

2. β *is* $\mathcal{S}$*-lower hemicontinuous and* $\mathcal{W}$*-closed and has non-empty values, and*

3. β *is state-independent.*

Then ψ *is* $\mathcal{W}$*-closed and has non-empty and* $\mathcal{W}$*-compact values.*

PROOF: According to Propositions 3 and 4, ϕ is $\mathcal{S}$-lower semicontinuous and $\mathcal{W}$-closed.

Furthermore, ϕ has non-empty values: Let $p \in P$ and $\mathcal{F} \in \mathfrak{F}$; then we have $f \in \beta(p)$ by assumption 2 and $E[f\,|\,\mathcal{F}] \in \beta(p)$ by assumption 3. Since $E[f\,|\,\mathcal{F}]$ is $\mathcal{F}$-measurable, it follows that $E[f\,|\,\mathcal{F}] \in \phi(p,\mathcal{F})$.

By Lemma 3, ψ is $\mathcal{W}$-closed and has non-empty and $\mathcal{W}$-compact values. □

Allen (1983) and Cotter (1986) studied consumer demand under uncertainty as a function of information and of state-independent prices and wealth, assuming that no state-contingent contracting is possible before observing information. The budget constraint must be satisfied state by state. Applying Theorem 1 to this problem, we can prove the following.

Proposition 5 *Assume that* $\mathbf{0} \in X$ *and that* $\succsim$ *is* $\mathcal{S}$*-lower semicontinuous and* $\mathcal{W}$*-upper semicontinuous. Assume that the economic constraint correspondence is* $\beta\colon \mathbb{R}^m_{++} \times \mathbb{R}_{++} \twoheadrightarrow \mathfrak{X}$*, defined by*

$$\beta(p,w) \;\equiv\; \{f \in \mathfrak{X} \mid pf(\omega) \leq w \text{ for a.e. } \omega \in \Omega\} \;.$$

Then the demand correspondence $\psi\colon \mathbb{R}^m_{++} \times \mathbb{R}_{++} \times \mathfrak{F} \twoheadrightarrow \mathfrak{X}$ *is* $\mathcal{W}$*-closed and has non-empty and* $\mathcal{W}$*-compact values.*

Rather than proving this proposition directly, we prove instead a more general result for the class of decision problems in which the economic constraint must be satisfied state by state and is state-independent. For $A \subset X$, define $L(A) \equiv \{f \in \mathfrak{X} \mid f(\omega) \in A \text{ for a.e. } \omega \in \Omega\}$.

Lemma 5 *Suppose that there is a correspondence* $\hat{\beta}\colon P \twoheadrightarrow X$ *such that, for* $p \in P$*,* $\beta(p) = L(\hat{\beta}(p))$*.*

1. *If* $\hat{\beta}$ *is lower hemicontinuous, then* β *is* $\mathcal{S}$*-lower hemicontinuous.*

2. *If* $\hat{\beta}$ *is closed and has convex values, then* β *is* $\mathcal{W}$*-closed.*

3. *If* $\hat{\beta}$ *has closed and convex values, then* β *is state-independent.*

PROOF: 1. During the proof of part 1, $\mathfrak{X}$ is endowed with the topology $\mathcal{S}$ without further qualification.

Let $U_{\mathfrak{X}} \subset \mathfrak{X}$ be open. Let $U_P \equiv \{p \in P \mid \beta(p) \cap U_{\mathfrak{X}} \neq \varnothing\}$. We have to show that U_P is open. This is trivial if U_P is empty. Otherwise, let $p \in U_P$. We find a neighborhood N_p of p such that $N_p \subset U_P$.

We show below that (a) $\beta(p) \cap U_{\mathfrak{X}}$ contains a simple function f. We can write $f = \sum_{i=1}^n 1_{F_i} x_i$, where $F_i \in \Sigma$ and 1_{F_i} is the indicator function of F_i. Let ϵ be the diameter of a ball (in the L_1-norm) in $U_{\mathfrak{X}}$ centered at f. We also show below that (b) there is a neighborhood N_p such that, for all $p' \in N_p$ and for all $i = 1, \ldots, n$, there is an $x'_i \in \hat{\beta}(p')$ for which $\|x'_i - x_i\| < \epsilon$. Given such p' and $\{x'_i \mid i = 1, \ldots, n\}$, the simple function $f' \equiv$

$\sum_{i=1}^{n} 1_{F_i} x_i'$ is an element of $\beta(p')$ and is within distance ϵ of f. Therefore, $f' \in \beta(p') \cap U_{\mathfrak{X}}$, from which it follows that $\beta(p') \cap U_{\mathfrak{X}} \neq \varnothing$ and that $p' \in U_p$. Hence, $N_p \subset U_P$.

Proof of claim (a): Recall that $\beta(p) = L(\hat{\beta}(p))$. Any subset of $\mathbb{R}^m$ is separable (since any subset of a second-countable space is second-countable). Furthermore, for the topology of convergence in measure, the set of simple functions is dense in the set of all measurable functions into a separable metric space. Therefore, the set of simple functions in $L(\hat{\beta}(p))$ is dense in $L(\hat{\beta}(p))$. Because also $U_{\mathfrak{X}}$ is open and $L(\hat{\beta}(p)) \cap U_{\mathfrak{X}}$ is non-empty (by the definition of U_P), there exists a simple function f in $L(\hat{\beta}(p)) \cap U_{\mathfrak{X}}$.

Proof of claim (b): Let $i \in \{1, \dots, n\}$. Since $x_i \in \hat{\beta}(p)$ and since $\hat{\beta}$ is lower hemicontinuous, there is a neighborhood N_i of p such that, if $p' \in N_i$, then there exists $x_i' \in \hat{\beta}(p')$ for which $\|x_i' - x_i\| < \epsilon$. Let $N_p \equiv \bigcap_{i=1}^{n} N_i$. Then N_p is a neighborhood of p with the required property.

2. During the proof of part 2, $\mathfrak{X}$ is endowed with the topology $\mathcal{W}$ without further qualification.

Let $\langle p, f \rangle \in \mathrm{Gr}(\beta)^{\mathrm{c}}$ (where Gr denotes the graph of a correspondence and Y^{c} denotes the complement of a set Y). We construct a neighborhood of $\langle p, f \rangle$ that does not intersect $\mathrm{Gr}(\beta)$, implying that $\mathrm{Gr}(\beta)$ is closed. We first claim—and later prove—that there is a hyperplane that strictly separates $f(\omega)$ and $\hat{\beta}(p)$ with positive probability; that is, there exist $z \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that $\mu\{\omega \in \Omega \mid z f(\omega) > \alpha\} > 0$ and $zx < \alpha$ for all $x \in \hat{\beta}(p)$. Because X is compact, closedness of $\hat{\beta}$ implies upper hemicontinuity and that $N_p \equiv \big\{p' \in P \bigm| \hat{\beta}(p') \subset \{x \in X \mid zx < \alpha\}\big\}$ is open. Thus, N_p is a neighborhood of p.

Let $F \equiv \{\omega \in \Omega \mid z f(\omega) > \alpha\}$, which has strictly positive measure. Define $g \in L_\infty(\mathbb{R}^m)$ by $g(\omega) \equiv z$ if $\omega \in F$ and $g(\omega) \equiv \mathbf{0}$ otherwise. Then $f' \to \int g f'$ is a continuous linear functional on $L_1(\mathbb{R}^m)$, and so $N_f \equiv \big\{f' \in \mathfrak{X} \bigm| \int g f' > \alpha/\mu(F)\big\}$ is a neighborhood of f. If $\langle p', f' \rangle \in N_p \times N_f$, then $\int g f' > \alpha/\mu(F)$, which implies that $\mu\{\omega \in F \mid z f'(\omega) > \alpha\} > 0$. Since $zx > \alpha$ implies that $x \notin \hat{\beta}(p')$, we have $\mu\big\{\omega \in \Omega \bigm| f'(\omega) \notin \hat{\beta}(p')\big\} > 0$ and hence $f' \notin L(\hat{\beta}(p'))$ and $\langle p', f' \rangle \notin \mathrm{Gr}(\beta)$.

Now we prove the claim. For each $z \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, let $H(z, \alpha)$ be the hyperplane $\{x \in \mathbb{R}^m \mid zx = \alpha\}$ and let $H^+(z, \alpha)$ (resp., $H^-(z, \alpha)$) be the open half-space above (resp., below) $H(z, \alpha)$. Let $\mathcal{H} = \big\{H^+(z, \alpha) \bigm| z, \alpha \text{ are rational and } \hat{\beta}(p) \subset H^-(z, \alpha)\big\}$. Because $\hat{\beta}(p)$ is convex and compact, any point in X outside $\hat{\beta}(p)$ can be separated by a hyperplane $H(z, \alpha)$ with rational coefficients, and so $\hat{\beta}(p)^{\mathrm{c}} \subset \bigcup \mathcal{H}$. By assumption, $f \notin L(\hat{\beta}(p))$, and so $\mu\big\{\omega \in \Omega \bigm| f(\omega) \in \hat{\beta}(p)^{\mathrm{c}} > 0\big\}$. Since $\mathcal{H}$ is countable, we have $H(z, \alpha) \in \mathcal{H}$ such that $\mu\{\omega \in \Omega \mid f(\omega) \in H^+(z, \alpha)\} > 0$. By the definition of $\mathcal{H}$, $\hat{\beta}(p) \subset H^-(z, \alpha)$.

3. Let $p \in P$ and $\mathcal{F} \in \mathfrak{F}$. Because $\hat{\beta}(p)$ is closed and convex, if $f \in L(\hat{\beta}(p))$ then $E[f \,|\, \mathcal{F}] \in L(\hat{\beta}(p))$. □

Combining Lemma 5 and Theorem 1 yields the following corollary.

Corollary 3 *Suppose that there is a correspondence $\hat{\beta}\colon P \twoheadrightarrow X$ such that, for $p \in P$, $\beta(p) = L(\hat{\beta}(p))$. Assume that*

1. *$\succsim$ is $\mathcal{S}$-lower semicontinuous and $\mathcal{W}$-upper semicontinuous, and*
2. *$\hat{\beta}$ is lower hemicontinuous and closed and has non-empty and convex values.*

Then ϕ is $\mathcal{W}$-closed and has non-empty and $\mathcal{W}$-compact values.

Since the budget constraint $\hat{\beta}(p,w) = \{x \in X \mid px \leq w\}$ satisfies the assumptions in Corollary 3 when $\mathbf{0} \in X$ and when the domain of $\hat{\beta}$ is $\mathbb{R}^m_{++} \times \mathbb{R}_{++}$, Corollary 3 provides the proof of Proposition 5.

The assumption on β in Lemma 5 and Corollary 3 has two components. One is that it is state-independent; the other is that what the agent can do in one state does not depend on what he can do in other states—that is, there is no ex-ante contracting. We now consider an alternative to this second component.

In contrast to Proposition 5, suppose that the agent can insure across states and hence needs to satisfy her budget constraint only on average, so that this constraint becomes $\int pg(\omega) \leq w$. We continue to restrict attention to state-independent prices. This exercise might be part of an equilibrium model in which agents face individual risks and are offered insurance by risk-neutral insurance companies, but the insured parties choose how much publicly verifiable information to acquire about medical conditions or about whatever risks affect their state-dependent preferences over the consumption goods.

Proposition 6 *Assume that $\mathbf{0} \in X$ and that $\succsim$ is $\mathcal{S}$-lower semicontinuous and $\mathcal{W}$-upper semicontinuous. Assume that the economic constraint correspondence is $\beta\colon \mathbb{R}^m_{++} \times \mathbb{R}_{++} \twoheadrightarrow \mathfrak{X}$, defined by*

$$\beta(p,w) \equiv \left\{ f \in \mathfrak{X} \;\middle|\; p \textstyle\int g \leq w \right\}$$

Then the demand correspondence $\psi\colon \mathbb{R}^m_{++} \times \mathbb{R}_{++} \times \mathfrak{F} \twoheadrightarrow \mathfrak{X}$ is $\mathcal{W}$-closed and has non-empty and $\mathcal{W}$-compact values.

Proof: It is trivial that β has non-empty and convex values. Horsley et al. (1998b) showed that β is $\mathcal{S}$-lower hemicontinuous and $\mathcal{W}$-closed. Since $\int f = \int E[f \mid \mathcal{F}]$ for any $f \in \mathfrak{X}$ and $\mathcal{F} \in \mathfrak{F}$, it follows that β is state-independent. □

As a final application, consider a variant of Theorem 1 in which the information is an endogenous choice. We redefine some of our notation for this purpose only. The preference ordering $\succsim$ is defined over $\mathfrak{F} \times \mathfrak{X}$, reflecting the disutility of acquiring information owing to the personal time or other resources needed to acquire and understand it. (A similar exercise would suppose that information is purchased and would put the cost of information in the budget constraint.) The only exogenous parameter is $p \in P$. The agent is restricted to a subset $\mathfrak{F}^* \subset \mathfrak{F}$ of information. The overall constraint correspondence is $\phi\colon P \twoheadrightarrow \mathfrak{F}^* \times \mathfrak{X}$, defined by $\phi(p) = \{\langle \mathcal{F}, f\rangle \in \mathfrak{F} \times \mathfrak{X} \mid f \in \beta(p) \cap M(\mathcal{F})\}$. The solution correspondence $\psi\colon P \twoheadrightarrow \mathfrak{F}^* \times \mathfrak{X}$ maps $p \in P$ to the $\succsim$-maximal elements of $\phi(p)$.

Proposition 7 *Assume that*

1. *$\succsim$ is $\mathcal{S}$-lower semicontinuous and $\mathcal{W}$-upper semicontinuous;*
2. *β is $\mathcal{S}$-lower hemicontinuous and $\mathcal{W}$-closed and has non-empty values;*
3. *β is state-independent;*
4. *$\mathfrak{F}^*$ is non-empty and compact.*

Then ψ is $\mathcal{W}$-closed and has non-empty and $\mathcal{W}$-compact values.

Proof: Applying again Horsley et al. (1998a, Cor. 8), we have to show that ϕ is $\mathcal{S}$-lower hemicontinuous and $\mathcal{W}$-closed and has non-empty values.

That ϕ has non-empty values follows directly from the last three assumptions of the proposition.

Observe that $\phi(p)$ is equal to the composition of the correspondence $p \underset{\zeta_1}{\longmapsto} \mathfrak{F} \times \beta(p)$ and the function $\langle \mathcal{F}, f \rangle \underset{\zeta_2}{\longmapsto} \langle \mathcal{F}, E[f \,|\, \mathcal{F}] \rangle$:

1. Let $\langle \mathcal{F}, f \rangle \in \phi(p)$. Then (a) $f \in \beta(p)$ and so $\langle \mathcal{F}, f \rangle \in \zeta_1(p)$, and (b) f is $\mathcal{F}$-measurable and so $\zeta_2(\mathcal{F}, f) = \langle \mathcal{F}, f \rangle$. Hence, $\langle \mathcal{F}, f \rangle \in \zeta_2 \circ \zeta_1(p)$.
2. Conversely, let $\langle \mathcal{F}, f \rangle \in \zeta_2 \circ \zeta_1(p)$. Then there is an $f' \in \mathfrak{X}$ such that (a) $f' \in \beta(p)$ and (b) $f = E[f' \,|\, \mathcal{F}]$. Property (a) and state independence of β imply that $f \in \beta(p)$; property (b) implies that $f \in M(\mathcal{F})$. Hence, $\langle \mathcal{F}, f \rangle \in \phi(p)$.

Since β is $\mathcal{S}$-lower hemicontinuous, so is ζ_1. The function ζ_2 is $\mathcal{S}$-continuous as shown in Cotter (1986). Hence, the composition of ζ_1 and ζ_2 is $\mathcal{S}$-lower hemicontinuous.

The condition $f \in M(\mathcal{F})$ in the definition of ϕ can be written as $\langle \mathcal{F}, f \rangle \in \mathrm{Gr}(M)$. Thus, $\mathrm{Gr}(\phi) = \{\langle p, \mathcal{F}, f \rangle \in P \times \mathcal{F}^* \times \mathfrak{X} \mid \langle p, f \rangle \in \mathrm{Gr}(\beta) \text{ and } \langle \mathcal{F}, f \rangle \in \mathrm{Gr}(M)\}$. Since $\mathrm{Gr}(\beta)$ and $\mathrm{Gr}(M)$ are $\mathcal{W}$-closed, so is $\mathrm{Gr}(\phi)$. □

We have not emphasized game theory applications, but note that Cotter (1994) used the continuity properties of the measurability constraint to show that the set of type-correlated equilibria (an extension of correlated equilibria to games of incomplete information) depends upper hemicontinuously on the players' information.

7 On the state independence of the economic constraint

One interpretation of state independence of the economic constraint β is that it does not reveal information; another is that information is not necessary in order to satisfy the constraint. When the economic constraint reveals information—such as in a microeconomic rational expectations general equilibrium model in which there is no forward contracting and prices are state-dependent and hence reveal information—the proper approach is to include such information in the agent's overall information. There is thus some mapping from $\sigma\colon P \to \mathcal{F}$ such that $\sigma(p)$ is the information revealed by economic parameter p, the measurability constraint is that f be $(\sigma(p) \vee \mathcal{F})$-measurable (where $\vee$ is the join operator), and state independence becomes

$$\forall p \in P,\ \forall f \in \beta(p),\ \forall \mathcal{F} \in \mathfrak{F}\colon\ E[f \,|\, \sigma(p) \vee \mathcal{F}] \in \beta(p)\ ,$$

preserving the spirit of this condition.

However, there remain obstacles to using the tools in this paper (and in related papers that use an ex-post approach) when the parameter p reveals information. One needs a topology on information such that both the mapping σ and the combining of information $\langle \mathcal{F}, \mathcal{G} \rangle \mapsto \mathcal{F} \vee \mathcal{G}$ are continuous. For concreteness, suppose we want to prove a general equilibrium existence theorem in a microeconomic rational expectations model with endogenous acquisition of information. The agent obtains information by observing prices and by acquiring information directly. The mapping from state-dependent prices to information should be continuous, but it is well known that this mapping can be discontinuous: two random variables can be very close to each other (in e.g. the L_1-norm), yet one is fully revealing because it takes on a different value in every state and the other reveals no information because it has the same value in every state. However, within a subclass of noisy random variables continuity can be shown with respect to the pointwise convergence topology (see Cotter (1986)), but there are no such results for the Boylan metric. On the other hand, the combining of information is continuous in the Boylan metric but not in the pointwise convergence topology.

8 Comparison with the ex-post method

This section sketches some results in Allen (1983), Cotter (1986), Stinchcombe (1990), and Hellwig (1996), and it relates their methods to the one in this paper.

Let $\mathcal{U}$ be the set of strictly concave continuous functionals on X, endowed with the norm topology of compact convergence. Let $u\colon \Omega \to \mathcal{U}$ be integrable. For $\mathcal{F} \in \mathfrak{F}$, $E[u\,|\,\mathcal{F}]$ denotes the strong conditional expectation for Banach-space valued random objects, as defined in Scalora (1961); the linear operator $u \mapsto E[u\,|\,\mathcal{F}]$ is norm-continuous. The function $v(x,\omega) \equiv u(\omega)(x)$ satisfies the assumptions of Lemma 2; hence, if preferences over plans are represented by the utility function $V(f) = \int u(\omega)(f(\omega))\,d\mu(\omega)$, then they are $\mathcal{S}$-continuous and we can apply the results of the previous section.

Allen (1983) and Cotter (1986) used an alternative strategy that takes advantage of the additive separability of preferences. Suppose that, as in Lemma 5, there is a correspondence $\hat{\beta}\colon P \twoheadrightarrow X$ with closed and convex values such that $\beta(p) = L(\hat{\beta}(p))$ for $p \in P$. Then $\psi(p,\mathcal{F})$ is a singleton $\{f\}$, where f is the unique plan such that $f(\omega)$ solves $\max_{x\in\beta(p)} E[u\,|\,\mathcal{F}](\omega)(x)$ for a.e. $\omega \in \Omega$ (see e.g. Van Zandt (1989, Prop. 7.1)). That is, the behavior is the same whether (a) the agent commits to a plan that yields the highest *ex-ante* expected utility subject to the measurability constraint or (b) the agent observes her information, updates her beliefs, and then in every state chooses an action that maximizes her *ex-post* conditional expected utility. Mathematically, this gives the following definition of ψ. Let $\hat{x}\colon P \times \mathcal{U} \to X$ be defined by $\hat{x}(p,v) = \arg\max_{x\in\beta(p)} v(x)$. Then $\psi(p,\mathcal{F}) = \hat{x}(p, E[u\,|\,\mathcal{F}])$ and hence is (loosely) the composition of $\mathcal{F} \mapsto E[u\,|\,\mathcal{F}]$ and $\hat{x}$. If $\hat{\beta}$ is lower hemicontinuous and closed and has non-empty and convex values, then $\hat{x}$ is continuous for the norm topology on $\mathcal{U}$. Allen (1983) and Cotter (1986) showed that $\mathcal{F} \mapsto E[u\,|\,\mathcal{F}]$ is continuous in the L_1-norm on $L_1(\mathcal{U})$ (Allen for a topology that is stronger than $\mathcal{P}$ and Cotter for the topology $\mathcal{P}$). Hence, this composition is continuous with respect to the L_1-norm on $\mathfrak{X}$—that is, in the topology $\mathcal{S}$. We may summarize as follows.

Proposition 8 *Suppose that*

1. *there is an integrable function $u\colon \Omega \to \mathcal{U}$ such that, for $f, g \in \mathfrak{X}$,*

$$f \succsim g \iff \int u(\omega)(f(\omega))\,d\mu(\omega) \geq \int u(\omega)(g(\omega))\,d\mu(\omega) \ ;$$

2. *there is a correspondence $\hat{\beta}\colon P \twoheadrightarrow X$ such that, for $p \in P$, $\beta(p) = L(\hat{\beta}(p))$;*

3. *$\hat{\beta}$ is lower hemicontinuous and closed and has non-empty and convex values.*

Then ψ is an $\mathcal{S}$-continuous function.

Stinchcombe (1990) obtained the same result with a slightly different method. The posteriors did not explicitly appear in the preceding discussion because the mapping from states to posteriors given a sub-σ-field is not always well-defined. However, it is if Ω is a compact separable metric space and Σ is its Borel field. Let $\mathcal{M}(\Omega)$ be the set of probability measures on Ω, endowed with the topology of weak convergence. Redefine $\hat{x}$ to be a mapping from $P \times \mathcal{M}(\Omega)$ to X given by $\hat{x}(p,\nu) = \max_{x \in \hat{\beta}(p)} \int u(\omega)(x)\, d\nu(\omega)$. Define $\hat{\nu}\colon \mathfrak{F} \to L(\mathcal{M}(\Omega))$ to be the mapping from information to state-dependent posteriors. Then $\psi(p,\mathcal{F})(\omega) = \hat{x}(p, \hat{\nu}(\mathcal{F})(\omega))$ a.e. The $\mathcal{S}$-continuity of ψ then follows from continuity of $\hat{x}$ and continuity of $\hat{\nu}$ when $L(\mathcal{M}(\Omega))$ is endowed with the topology of convergence in measure.

Compared to Corollary 3, Proposition 8 imposes stronger assumptions—mainly, the additive separability of preferences. On the other hand, it obtains a stronger conclusion—continuity of ϕ in the topology $\mathcal{S}$ rather than $\mathcal{W}$. However, we are not aware of applications in which it is important to have continuity with respect to the stronger topology.

A drawback of the ex-post approach is that it cannot be applied when there is contracting—that is, when what is feasible in one state depends on what one intends to do in the other states. Hence, there is no analog to Proposition 6.

Whereas this paper and those just cited concern a static decision problem under uncertainty, Hellwig (1996) (extending Jordan (1977)) studied a discrete-time stochastic control problem. He used a framework and method that is quite different from the ones discussed so far in this paper, yet it is based on an ex-ante measurability constraint. To compare it with the one in this paper, I use a static version of his model and I suppress (for simplicity) the economic parameter p. Some notation used previously in the paper is redefined to play an analogous but distinct role.

Rather than fixing a probability space and varying information sub-σ-fields, Hellwig fixed the sample space of random variables and varied the probability measure on them. Let X, Y_1, and Y_2 be compact separable metric spaces; then $x \in X$ is the action and the pair $\langle y_1, y_2 \rangle \in Y_1 \times Y_2$ defines the payoff-relevant state of the world. The agent observes y_1 before choosing and action, perhaps with randomization, from X. That is, a plan maps Y_1 to the set $\mathcal{M}(X)$ of probability measures on X. Given a joint distribution μ on $Y_1 \times Y_2$, a plan determines a joint distribution ν on $X \times Y_1 \times Y_2$.[6] The agent's preferences on the set $\mathcal{M}(X \times Y_1 \times Y_2)$ of such joint distributions are assumed to be continuous with respect to the topology of weak convergence on the set. For example, the agent's utility function V on $\mathcal{M}(X \times Y_1 \times Y_2)$ is defined by $\int_{X \times Y_1 \times Y_2} u(x, y_1, y_2)\, d\nu$, where $u\colon X \times Y_1 \times Y_2 \to \mathbb{R}$ is continuous. A joint distribution ν is induced by a plan if and only if, stated loosely,

- μ is the marginal of ν on $Y_1 \times Y_2$ and
- $\nu(\langle x, y_2 \rangle | y_1) = \nu(x|y_1)\nu(y_2|y_1)$ almost surely.

Let $M(\mu)$ be the elements of $\mathcal{M}(X \times Y_1 \times Y_2)$ that satisfy these constraints. Then the agent's problem, given μ, is to choose $\nu \in M(\mu)$ to maximize $V(\nu)$. Let $\psi\colon \mathcal{M}(Y_1 \times Y_2) \twoheadrightarrow \mathcal{M}(X \times Y_1 \times Y_2)$ be the solution correspondence.

Continuity of ψ can now be characterized by applying a one-topology maximum theorem. Endow $\mathcal{M}(X \times Y_1 \times Y_2)$ with the topology of weak convergence. Then this space is compact and V is continuous. Hellwig sought a topology on $\mathcal{M}(Y_1 \times Y_2)$ such that M is continuous. He called this the topology of convergence of information, and showed that it is strictly stronger than the topology of weak convergence. Note that this framework allows not just

[6] Hellwig used A instead of X, X instead of Y, μ instead of ν, and ν instead of μ.

the information about the state to vary but also the underlying probability measure on the state.

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