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Dynamic Pricing with Loss Averse Consumers and Peak-End Anchoring

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We analyze a dynamic pricing problem where consumer's purchase decisions are affected by representative past prices, summarized in a reference price. We propose a new, behaviorally motivated reference price mechanism, based on the peak-end memory model proposed by Kahneman et al. (1993). Specifically, we assume that consumers' reference price is a weighted average of the lowest and last price. Gain or loss perceptions with respect to this reference price affect consumer purchase decisions in the spirit of prospect theory, resulting in a non-smooth demand function. We investigate how these behavioral patterns in consumer anchoring and decision processes affect the optimal dynamic pricing policy of the firm. In contrast with previous literature, we show that peak-end anchoring leads to a range of optimal constant pricing policies even with loss neutral buyers. This range becomes wider if consumers are loss averse. In general, we show that skimming or penetration strategies are optimal, i.e. the transient pricing policy is monotone, and converges to a steady state, which depends on the initial price perception. The value of the steady state price decreases, the more consumers are sensitive to price changes, and the more they anchor on the lowest price.

Key words: Dynamic Pricing, Deterministic Dynamic Programming, Behavioral Consumer Theory, Prospect Theory, Peak-end Rule.

1. Introduction

In repeat-purchase markets, consumers form price expectations, also known as reference prices. Actual prices are perceived as discounts or surcharges relative to these reference prices, according to prospect theory (Tversky and Kahneman 1991); this perception affects the demand for a firm's product, and hence its profitability. As a result, for example, the often practised discounts by firms, while usually profitable in the short term, may erode consumers' price expectations and willingness to pay, and thereby negatively affect long run profitability. It is therefore important for a firm to

understand (1) how its pricing policy affects consumers' price expectations and purchase decisions, and (2) how to set prices over time to maximize profitability in this context.

There is a growing body of marketing, economics and operations literature studying how firms should optimally set prices when consumers anchor on past prices (Kopalle et al. 1996, Fibich et al. 2003, Heidhues and Kőszegi 2005, Popescu and Wu 2007). This literature builds on behavioral theories (prospect theory, mental accounting, etc.) to model demand dependence on reference prices, but pays little attention to the anchoring and memory processes, i.e. how reference prices are formed. A common assumption in the literature is that consumers' reference price is a weighted average of past prices (see e.g. Mazumdar et al. 2005). Specifically, the anchoring mechanism follows an exponentially smoothed adaptive expectations process.

This paper proposes a different, behaviorally motivated anchoring mechanism, whereby consumers anchor on the lowest and most recent prices observed, and the reference price is a weighted average of the two. This assumption is motivated by the *peak-end rule*, a "snapshot model" of remembered utility, proposed by Kahneman et al. (1993). The goal of this paper is to understand how this consumer anchoring process influences the optimal pricing strategies of the firm. We investigate the robustness of state of the art results on pricing with reference effects (in particular those obtained by Popescu and Wu, 2007), with respect to this reference price formation mechanism. We seek to understand how behavioral regularities, such as loss aversion and peak-end anchoring, interact in determining the structure of the optimal pricing policy. Finally, we aim to provide insights as to which aspects of consumer behavior the firm should assess in order to optimize pricing decisions.

Behavioral motivation and evidence. The marketing literature finds a strong empirical validation of reference dependence and prospect theory in consumer purchase decisions, reviewed in Kalyanaram and Winer (1995) and Mazumdar et al. (2005). The role of historic prices in forming price expectations is widely supported in empirical studies, for a wide range of products and services. Winer (1986), Greenleaf (1995), and Erdem et al. (2001) find evidence for an adaptive expectations framework, due to Nerlove (1958), where the reference price is an exponentially smoothed average of past prices. Due to its simplicity, this has been the memory model used throughout the

dynamic pricing literature. Nevertheless, as pointed out in Rabin (1996, p.8), this model is "not very well grounded in behavioral evidence".

Our work is motivated by a different memory model: the *peak-end rule*, proposed by Kahneman et al. (1993). This model hypothesizes a "moment based approach" to memory, whereby the overall evaluation of past experiences is based on representative moments of experience. Specifically, these authors propose an averaging scheme, whereby most experiences are assigned zero weight, and a few most relevant snapshots, specifically the peak and the end (i.e. the strongest and the most recent experience) receive positive weights. The psychological literature supporting the peak-end rule is reviewed in Kahneman (2000) and Fredrickson (2000). Peak-end anchoring models have been empirically validated in various psychological contexts, both under negative experiences (e.g. pain recall in Kahneman et al. 1993, annoyance in Schreiber and Kahneman 2000) and positive ones (e.g. enjoyment of video clips in Fredrickson and Kahneman 1993, music in Rozin et al. 2004, material goods in Do et al. 2008).

The moment based approach to remembered utility has also been evidenced in other economic contexts. Duesenberry (1952, Chapter 5) explains how households' saving patterns are affected not only by current income, but also by past income levels among which the highest one is the most relevant. Oest and Paap (2004) propose a model where each household's reference price is an average of past prices recalled by that household. Therefore, each household is hypothesized to anchor on selective moments to form its reference price. They apply the model to estimate the average recall probabilities of specific past prices based on scanner data.

In the pricing context, we posit that the representative peak-end moments in reference price formation are associated with the lowest and the last price, i.e. the highest and the most recent transaction utility derived from the purchase experience. While the empirical validity of the peak-end rule in the pricing context is yet to be investigated, several studies find support for anchoring on most recent and extreme prices. Krishnamurti et al. (1992) and Chang et al. (1999) find support for the last price as a reference point. Nwokoye (1975) reports that some consumers anchor on extreme prices, i.e. the lowest and the highest, in their price judgements. The reference price is also

an indicator of what consumers consider as a "fair price" (Mazumdar et al. 2005), and perception of fairness is typically anchored on the lowest prices (see e.g. Xia et al. 2004). Our focus here is on a model where consumers anchor on the minimum and last prices, consistent with the peak-end literature. Section 7 briefly discusses the implications of anchoring on the maximum, rather than the minimum price.

Contribution and relation with the literature. This paper fits in a growing body of behavioral operations literature, reviewed by Loch and Wu (2007) and Gino and Pisano (forthcoming). Our work contributes to the pricing and revenue management literature, for which a comprehensive reference is the book by Talluri and van Ryzin (2004); see also Bitran and Caldentey (2003) for a review. A recent review on modeling customer behavior in revenue management and auctions is due to Shen and Su (2007). Within the pricing and revenue management literature, our work contributes to the behavioral pricing stream, and comes closest to Kopalle et al. (1996), Fibich et al. (2003), and especially Popescu and Wu (2007), reviewed below.

Other work in behavioral operations that incorporates consumer learning models includes Gaur and Park (2007; consumers learn fill rates), Liu and van Ryzin (2007; consumers learn about rationing risk), and Ovchinnikov and Milner (2005; consumers learn about the likelihood of last-minute sales).

There are a few other papers that study the dependence of demand on past prices. Ahn et al. (2007) propose a model where demand in each period contains the residual latent demand which has not materialized in previous periods because of higher prices. Fleischmann, Hall and Pyke (2005) provide history dependent, endogenous pricing models that capture stock piling effects.

From the behavioral economics literature, reviewed by Camerer et al. (2004) and Ho et al. (2006), a related paper is Heidhues and Kőszegi (2005). They propose a new model where the reference price is defined as an internal, rational expectations equilibrium.

This paper builds directly on the existing literature on dynamic pricing with reference effects. We briefly review here the papers most closely related to ours. Greenleaf (1995) provides numerical insights into the firm's optimal pricing policy. The first analytic results are due to Kopalle

et al. (1996), and extended by Fibich et al. (2003). These authors prove the monotonicity and convergence of the optimal price paths under a piecewise linear demand model. Popescu and Wu (2007), henceforth PW, extend these findings to general demand functions and reference effects, and provide structural results. Like previous studies, they show that loss aversion leads to a range of steady states, which collapses to a single point for loss neutral buyers. Complementing previous research, they find that the transient pricing policy is monotone under convex (or linear) reference effects, but not in general. In their model, it is optimal for the firm to induce a consistent perception of gain or loss, by systematically pricing above, respectively below the reference price. In parallel work, Nasiry and Popescu (2008) extend these results to capture asymmetries in the exponential smoothing memory process, in the spirit of the model proposed by Gaur and Park (2007). Our work explores similar issues, under a different model of consumer memory and behavior.

This paper proposes and investigates a new price anchoring process, based on the peak-end rule. This makes our work different from the current literature on pricing with reference effects, which has focused exclusively on exponentially smoothed memory models. In what follows, we summarize our main findings, and relate them to existing literature.

Consistent with previous results for loss averse consumers, we find that a constant pricing policy is optimal for a range of relatively low initial price expectations, supporting an EDLP policy (every day low price; Lal and Rao 1997, Fibich et al. 2003). In our case, the range of constant optimal prices (called steady states), is due to the kinked anchoring process, and therefore persists when consumers are loss neutral (i.e equally sensitive to discounts and surcharges). This is in contrast with previous literature. The range of steady state prices is wider, the more loss averse consumers are, and the more sensitive to the lowest price anchor.

In general, the more consumers anchor on the minimum price, the lower the steady state prices and the firm's profits. For relatively high initial price expectations, the firm should follow a skimming strategy, converging on the long run to a unique optimal price, which is independent of consumers' initial price perception and their sensitivity to the minimum price. In general, we find that firms should follow either skimming or penetration strategies, i.e. the optimal pricing policy

of the firm is monotonic. In contrast with the literature, in particular PW, the result is true in our case under general, non-linear reference effects. This suggests that consumer memory processes, in addition to decision processes, are important in determining the optimal pricing policy of the firm. We characterize when it is relevant for the firm to measure these behavioral processes, and assess the pricing implications of ignoring or mis-judging these phenomena. Our insights extend when consumer behavior is heterogeneous, as well as under alternative, asymmetric memory models. From a methodological standpoint, we develop a non-standard approach to solve a dynamic program with kinked reward and transition functions.

Structure. The rest of the paper is structured as follows. In Section 2, we develop the general demand and reference price formation models, based on behavioral assumptions, and set up the dynamic pricing problem of the firm. Section 3 studies the optimal transient and long term pricing policies for loss averse buyers with linear reference effects. Section 4 studies heterogeneous markets and the sensitivity of optimal prices and profits with respect to behavioral parameters, in particular the minimum price anchor. Section 5 benchmarks the optimal pricing policy against that of a mis-calibrated firm, in particular a firm that ignores future profits or behavioral regularities, such as reference dependence, loss aversion or peak-end anchoring. Section 6 extends our findings to general non-linear reference effects, under behavioral assumptions motivated by prospect theory. Some extensions and future research directions are presented in Section 7. Finally, Section 8 summarizes our main findings in relation to the literature and concludes the paper.

2. The Model

This section provides the main setup for the paper. We describe how consumers make purchase decisions based on prices and reference prices, and how this decision affects the demand for a firm's product. We model demand dependence on reference prices based on prospect theory (Tversky and Kahneman 1991), following the general setup in PW.

Mental accounting theory (Thaler 1985) describes the utility from purchase experiences as consisting of two components: acquisition utility and transaction utility. Acquisition utility is the

monetary value of the product, captured by the difference between price and consumer's reservation price. Transaction utility corresponds to the psychological value of the deal, determined by the gap between the reference price and the price, $x = r - p$.

At the aggregate level, this motivates a demand model of the general form proposed in PW:

$$d(p, r) = d_0(p) + h(r - p, r). \tag{1}$$

Here $d_0(p) = d(p, p)$ is the base demand, i.e. demand in absence of reference effects, and $h(x, r)$ is the reference effect, capturing demand dependence on the reference point. In particular, $h(0, r) = 0$.

In a deterministic context, prospect theory (Tversky and Kahneman 1991) posits three main properties of the transaction utility, which are inherited by the aggregate reference dependent demand, and validated empirically in the pricing context (Kalyanaram and Winer 1995). First, *reference dependence* means that demand increases with the magnitude of the perceived discount, and decreases with that of the perceived surcharge, i.e. $h(x, r)$ is increasing in x. Second, *loss aversion* means that surcharges (perceived losses) loom larger than discounts (perceived gains) of the same magnitude, so demand is more sensitive to surcharges than discounts. Finally, there is *diminishing sensitivity* to both discounts and surcharges, i.e. $h(x, r)$ is concave over gains ($x \geq 0$) and convex over losses ($x \leq 0$).

The next section focuses on a loss averse demand model with linear reference effects, specifically: $h(x, r) = \lambda x$ for $x \leq 0$, and $h(x, r) = \gamma x$ for $x \geq 0$, where $\lambda, \gamma > 0$ by reference dependence and $\lambda \geq \gamma$ captures loss aversion. Empirical evidence suggests that the loss aversion index $\gamma / \lambda \simeq 0.5$ (Ho and Zhang 2008, Kahneman et al. 1990). Section 6 extends the results to general, non-linear reference effects, satisfying prospect theory assumptions.

The firm's short term profit is denoted $\pi(p, r) = p \, d(p, r)$, and the base profit is $\pi_0(p) = p \, d_0(p)$. These expressions implicitly assume zero cost; all our results extend for a non-zero marginal cost c. The following assumptions are made throughout the paper, and consistent with PW.

Assumption 1. *(a) Demand $d(p, r)$ is decreasing in p and increasing in r, and the reference effect, $h(x, r)$, is increasing in the perception gap, $x = r - p$. (b) The base demand, $d_0(p)$, is non-negative,*

bounded, continuous and decreasing in price, p. (c) The base profit $\pi_0(p)$ is non-monotone and strictly concave.

We assume that consumers' reference price, r_t, at each stage t is formed based on the peak-end rule (Kahneman et al. 1993), as a weighted average of the minimum price, m_{t-1}, and the last price of the product, p_{t-1}. That is,

$$r_t = \theta m_{t-1} + (1-\theta)p_{t-1}, \tag{2}$$

where $m_{t-1} = \min(m_0, p_1, \ldots, p_{t-1}) = \min(m_{t-2}, p_{t-1}), t > 1$, and $\theta \in (0,1]$ captures how much consumers anchor on the lowest price. This adaptation model is different from previous models used in the pricing literature (including PW), which mainly rely on exponential smoothing. In particular, for $\theta = 0$, we recover a special case of the model in PW, where consumers anchor solely on the previous period price; this limiting case is briefly discussed in Section 4.1.

Given initial conditions m_0 and p_0, the firm maximizes infinite horizon β-discounted revenues:

$$J(m_0, p_0) = \max_{p_t \in \mathbf{P}} \sum_{t=1}^{\infty} \beta^{t-1}\pi(p_t, r_t), \text{ where } r_t = \theta m_{t-1} + (1-\theta)p_{t-1}, m_t = \min(m_{t-1}, p_t), \beta \in (0,1).$$

We assume that prices are confined to a bounded interval $\mathbf{P} = [0, \bar{p}]$, where, for simplicity, $\bar{p}$ is such that $d_0(\bar{p}) = 0$ (this avoids trivial boundary solutions). The Bellman Equation for this problem is:

$$J(m_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ \pi(p_t, r_t) + \beta J\Big(\min(p_t, m_{t-1}), p_t\Big) \right\}, \tag{3}$$

$$\text{where} \quad r_t = \theta m_{t-1} + (1-\theta)p_{t-1}.$$

Intuitively, we expect that higher reference prices (i.e. memory of higher prices) should enable the firm to extract higher profits from the market. All proofs are in the Appendix.

Lemma 1. *The value function, $J(m, p)$, is increasing in both arguments.*

The infinite horizon model implicitly assumes that lowest prices can be remembered indefinitely. This is a reasonable approximation in a context where the frequency of transactions is high relative to the horizon length. Our model does not imply, however, that consumers remember all past prices. Lowest prices are recalled because of their salience: their extremeness makes them stand out

in the memory process. As explained in the introduction, the moment based approach posits that consumers forget most intermediate prices, but recall those which are most salient and/or recent. This amounts to the minimum and last prices under the peak-end rule. Our main insights remain valid when modeling the possibility of forgetting or updating the minimum price, as discussed in Section 7.1.3.

3. Linear Reference Effects

This section investigates the solution of Problem (3) when consumers' marginal sensitivity to discounts, respectively surcharges, is constant, i.e. reference effects are piecewise linear. We first identify the steady state prices (i.e. the optimal constant price policies), then describe the transient behavior of the price dynamics, and contrast these results with the literature. Our results extend under non-linear reference effects, as shown in Section 6.

According to prospect theory (Tversky and Kahneman 1991), prices are perceived as discounts or surcharges relative to a reference price, and surcharges have a stronger absolute impact on demand than discounts of the same magnitude. This effect, known as loss aversion, is validated in the pricing context (see Kalyanaram and Winer 1995) and captured by a kinked demand function. The reference dependent demand model used in this section is:

$$d(p,r) = d_0(p) - \lambda(p-r)^+ + \gamma(r-p)^+ = \begin{cases} d_0(p) - \lambda(p-r), & \text{if } p \geq r \\ d_0(p) - \gamma(p-r), & \text{if } p \leq r \end{cases}, \tag{4}$$

with $\lambda \geq \gamma > 0$, accounting for loss aversion and reference dependence. The profit function is:

$$\pi(p,r) = \Big[d_0(p) - \lambda(p-r)^+ + \gamma(r-p)^+ \Big] p = \begin{cases} \pi_\lambda(p,r), & \text{if } p \geq r \\ \pi_\gamma(p,r), & \text{if } p \leq r \end{cases}. \tag{5}$$

It is easy to see that, for $k \in \{\lambda, \gamma\}$, the smooth profit functions

$$\pi_k(p,r) = \Big[d_0(p) + k(r-p) \Big] p = \pi_0(p) + k(r-p)p, \tag{6}$$

are both supermodular in (p,r). A function $f(x,y)$ is said to be supermodular if $f(x^h, y^h) - f(x^l, y^h) \geq f(x^h, y^l) - f(x^l, y^l)$ for all $x^l < x^h$ and $y^l < y^h$ (see Topkis 1998). Together with loss aversion ($\lambda \geq \gamma$), this implies the following result, essential for our future developments.

Lemma 2. *The short-term profit,* $\pi(p,r) = \min\left(\pi_\lambda(p,r), \pi_\gamma(p,r)\right)$*, is supermodular in* (p,r)*.*

By Topkis' Theorem (Topkis 1998, Theorem 2.8.2), Lemma 2 confirms the intuition that myopic firms, i.e. those focused on short term profits, should charge higher prices when consumers have higher price expectations. Myopic policies are further compared to optimal ones in Section 5.1.

Lemma 2 allows to write the Bellman Equation for this section as follows:

$$J(m_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ \min(\pi_\lambda, \pi_\gamma)(p_t, r_t) + \beta J\left(\min(p_t, m_{t-1}), p_t\right) \right\}; \; r_t = \theta m_{t-1} + (1 - \theta) p_{t-1}. \quad (7)$$

3.1. Steady States

This section characterizes the long term pricing strategy of the firm facing loss averse consumers with demand given by (4). Identifying the steady states of Problem (7) requires a non-standard approach, because, in this problem, both the short-term profit and the transition in the value function (memory structure) are non-smooth. Our analysis is based on a bounding technique, which identifies the steady states of Problem (7) based on those of a series of smooth problems, for which standard methods can be applied.

For $\nu \in [0, 1]$, and $m \in \mathbf{P}$, consider the following smooth problem with one-dimensional state:

$$J_m^\nu(p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ (1 - \nu)\pi_\lambda(p_t, \theta m + (1 - \theta)p_{t-1}) + \nu\pi_\gamma(p_t, p_{t-1}) + \beta J_m^\nu(p_t) \right\}. \quad (8)$$

We first show that the family $J_m^\nu, \nu \in [0, 1]$, provides upper bounds for the value function J.

Lemma 3. *For any* $m \le p$*, we have* $J(m, p) \le J_m^\nu(p)$*.*

We next argue that by approximating the value function J by a smooth upper bound J_m^ν, for an appropriate subset of values ν, the firm will charge optimal prices on the long run. Technically, this amounts to matching supergradients of the original problem with gradients of an appropriate smooth upper bound Problem (8). We first identify steady states of Problem (8) which will help characterize those of Problem (7).

The structure of the problem leads us to consider three price-memory scenarios (low, medium and high): $\mathbf{R}_1 = [0, s]$, $\mathbf{R}_2 = [s, S]$, and $\mathbf{R}_3 = [S, \bar{p}]$, where the thresholds s, S solve respectively:

$$\pi_0'(p) - \lambda(1 - \beta(1 - \theta))p = 0, \quad (9)$$

$$\pi_0'(p) - \gamma(1-\beta)p = 0. \tag{10}$$

Uniqueness of s and S follows because the above left hand sides (LHS) are strictly decreasing in p, by concavity of π_0 (Assumption 1c). Moreover, $s \leq S$ because $1 - \beta(1-\theta) \geq 1 - \beta$ and $\lambda \geq \gamma > 0$.

Lemma 4. *(a) For $\nu \in [0,1]$ and $m \in \mathbf{P}$, Problem (8) admits a unique steady state, which solves:*

$$\pi_0'(p) - \Big[\lambda(1-\nu)(2-(1-\theta)(1+\beta)) + \nu\gamma(1-\beta)\Big]p + \lambda(1-\nu)\theta m = 0. \tag{11}$$

(b) For any $m \in [s, S]$, there exists $\nu \in [0,1]$ such that m is a steady state of the corresponding Problem (8).

Denote $p_\lambda^{**}(m)$ the unique steady state of Problem (8) for $\nu = 0$. By Lemma 4(a), $p_\lambda^{**}(m)$ solves:

$$\pi_0'(p) - \lambda(2-(1-\theta)(1+\beta))p + \lambda\theta m = 0, \tag{12}$$

in particular $p_\lambda^{**}(s) = s$. The thresholds s and S defined above, correspond to those values m for which the steady state of J_m^ν equals m, for $\nu = 0$, respectively $\nu = 1$. It turns out that (s,s) and (S,S) are steady states for our Problem (7). The next result identifies steady states of Problem (7) based on the steady states of Problem (8), identified in Lemma 4.

Lemma 5. *(a) For $m \in \mathbf{R}_1$, $(m, p_\lambda^{**}(m))$ is a steady state of Problem (7), where $p_\lambda^{**}(m)$ solves (12). (b) For $m \in \mathbf{R}_2$, (m,m) is a steady state of Problem (7).*

Lemma 5 suggests to partition the initial state space into the following regions: $\overline{\mathbf{R}}_{1a} = \{(m,p) \mid p \geq p_\lambda^{**}(m), m \leq s\}$, $\overline{\mathbf{R}}_{1b} = \{(m,p) \mid p \leq p_\lambda^{**}(m), m \leq s, p\}$, $\overline{\mathbf{R}}_2 = \{(m,p) \mid p \geq m, s \leq m \leq S\}$, and $\overline{\mathbf{R}}_3 = \{(m,p) \mid p \geq m, m \geq S\}$, as shown in Figure 1.

The main result in this section confirms that the steady states identified in Lemma 5 are indeed the only steady states of Problem (7).

Proposition 1. *The set of steady states of Problem (7) is $\{(m, p_\lambda^{**}(m)) \mid m \in \mathbf{R}_1\} \cup \{(m,m) \mid m \in \mathbf{R}_2\}$. In particular, the value of the steady state prices is decreasing in λ, and increasing in β.*

The result says that the value of the steady state is lower, the more sensitive consumers are to deviations from the reference price. Furthermore, a more patient firm (higher β) charges higher steady state prices. These sensitivity results are consistent with literature on exponentially smoothed memory models (PW, Fibich et al. 2003, Kopalle et al. 1996). Also consistent with the literature is the range of steady states observed under loss aversion. In our model, this range is due both to loss aversion and the kinked memory structure, as explained in Section 3.3. This section provides a detailed comparison with the literature. Sensitivity with respect to θ is discussed in Section 4.1.

3.2. Optimal Policy and Price Paths

This section investigates the transient pricing policy of the firm. Specifically, we study convergence and monotonicity properties of the price paths of Problem (7), starting at an arbitrary initial state (m_0, p_0), $m_0 \leq p_0$. Denote the optimal pricing policy of the firm by:

$$p^*(m_{t-1}, p_{t-1}) = \arg\max_{p_t} \left\{ \pi(p_t, \theta m_{t-1} + (1-\theta)p_{1-1}) + \beta J(\min(m_{t-1}, p_t), p_t) \right\}.$$

The optimal price path $\{p_t\}_t$ is given by $p_t = p^*(m_{t-1}, p_{t-1})$, with $m_t = \min(m_{t-1}, p_t), t \geq 1$, and the state path is $\{(m_t, p_t)\}_t$.
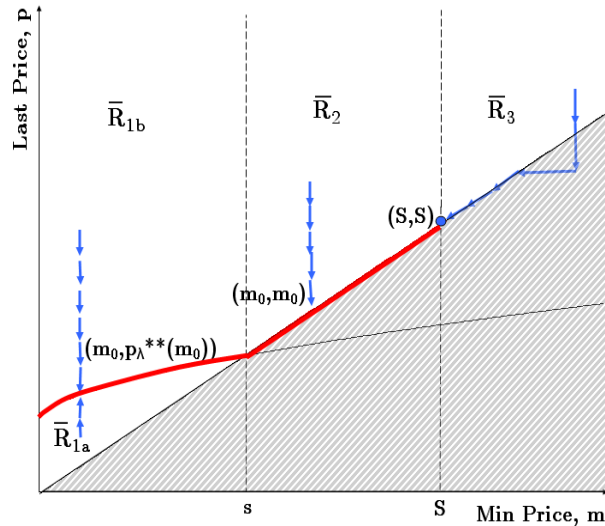

Figure 1   Steady states and optimal price path of Problem (7). The red line depicts the range of steady states, and the blue arrows show the optimal price paths starting at generic points in each region.

Our first result in this section shows that, if (m_0, p_0) is in any of the three regions $\mathbf{R}_i$, $i = 1, 2, 3$, defined in Section 3.1, the state path remains in that region.

Proposition 2. *If $m_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$, then $p_t \geq m_0$ for all t. If $m_0 \in \mathbf{R}_3$, then $m_t \in \mathbf{R}_3$ for all t.*

The first part of this proposition shows that, if the initial minimum price m_0 is not too high ($m_0 \leq S$), the optimal price path stays above m_0, i.e. the minimum price does not change over time, and so the state path remains within the region. On the other hand, if the initial minimum price m_0 is relatively high ($m_0 > S$), the minimum price decreases over time, but it never drops below S, so the state path remains in the same region. This result leads us to identify the possible convergence points of the optimal price paths, starting at any initial state.

Proposition 2 implies that, if the optimal price path of Problem (7) converges, it converges to a steady state in the same region as the initial state (m_0, p_0). These steady states, identified in Proposition 1, are: $(m_0, p_\lambda^{**}(m_0))$ for $m_0 \in \mathbf{R}_1$, (m_0, m_0) for $m_0 \in \mathbf{R}_2$, and (S, S) for $m_0 \in \mathbf{R}_3$.

We now turn to characterize the optimal price paths of Problem (7). For $m_0 \in \mathbf{R}_1 \cup \mathbf{R}_2$, $m_t = m_0$ by Proposition 2, so Problem (7) can be written (with m_0 as a parameter) as follows:

$$J_{m_0}(p_{t-1}) = \max_{p_t \geq m_0} \left\{ \pi(p_t, r_t) + \beta J_{m_0}(p_t) \right\}, \tag{13}$$

where $r_t = \theta m_0 + (1 - \theta)p_{t-1}$. That is, $J(m, p) = J_m(p)$ for $m \in \mathbf{R}_1 \cup \mathbf{R}_2$, and $m \leq p$. Because π is supermodular (Lemma 2), the optimal policy in Problem (13) is monotone, so $p_t^*(m_0, p_{t-1})$ is increasing in p_{t-1}. Therefore, the optimal price path is monotonic in a bounded interval, and hence converges to a steady state. This must be $(m_0, p_\lambda^{**}(m_0))$, by Proposition 1.

For $m_0 \in \mathbf{R}_3$, we show in the Appendix that optimal prices decrease, approaching S. This is done by observing that prices must eventually fall below m_0 (but not below S, by Proposition 2), at a certain time T. Until time T, a finite horizon version of Problem (13) is solved (and the same structural results hold). After time T, we show that optimal prices $p_t = m_t$ solve the problem:

$$\tilde{J}(p_{t-1}) = \max_{p_t \in \mathbf{P}} \{\pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t)\}. \tag{14}$$

Starting at $p_T = m_T \geq S$, the optimal path is decreasing to S, by supermodularity of π.

The following proposition characterizes the optimal price paths for Problem (7).

Proposition 3. *Given (m_0, p_0), the optimal price path $\{p_t\}$ of Problem (7) converges monotonically to a steady state, which is: (a) $p_\lambda^{**}(m_0)$, if $m_0 \in \mathbf{R}_1$, (b) m_0, if $m_0 \in \mathbf{R}_2$, and (c) S, if $m_0 \in \mathbf{R}_3$.*

Corollary 1. *The optimal pricing policy, $p^*(m, p)$, is increasing in both m and p.*

The corollary follows from the proof of Proposition 3 (see Appendix). Proposition 3 ensures that, starting at (m_0, p_0), the price path monotonically converges to a unique steady state. Monotonicity is driven by supermodularity of short-term profits, $\pi(p, r)$ (Lemma 2), within the relevant regions, via Proposition 2. The steady state of Problem (7) depends only on m_0; p_0 influences the price path, but not the steady state.

Proposition 3 shows that there exists a threshold, S (defined by (10)), such that for high initial minimum prices, $m_0 \geq S$, the firm will decrease prices down to S. Otherwise, the firm will always charge prices above m_0. Prices converge to m_0 for intermediate values of the minimum price, $s \leq m_0 \leq S$, and to a higher (local) steady state, $p_\lambda^{**}(m_0) > m_0$, for low values, $m_0 < s$. In the latter case, for relatively low initial prices, $p_0 < p_\lambda^{**}(m_0)$ (Region $\overline{\mathbf{R}}_{1a}$), the optimal price is increasing and thus induces a consistent perception of loss. Otherwise, the gain/loss perception may alternate (see e.g. Figure 2). A constant steady state price path either stays equal to the reference price (for $m \geq s$), or induces a consistent perception of loss ($p_\lambda^{**}(m) > m$ for $m < s$). This is because the steady state price either equals the minimum price, or stays above it (see Proposition 3), and hence also above the reference price.

3.3. Comparison with Literature

We now compare our results with those for exponentially smoothed memory models obtained in the literature, in particular in PW. The dynamic pricing model under exponential smoothing is:

$$V(r_t) = \max_{p_t \in \mathbf{P}} \{\pi(p_t, r_t) + \beta V(r_{t+1})\}, \qquad r_{t+1} = \alpha r_t + (1 - \alpha) p_t, \tag{15}$$

where $\alpha \in [0, 1]$. For linear reference effects (4), the main results (Proposition 2 in Fibich et al. 2003, see also Corollary 2 in PW) can be summarized as follows: The set of steady states for Problem (15) with linear reference effects (4) is $[\tilde{p}_\lambda, \tilde{p}_\gamma]$, which solve:

$$\pi_0'(\tilde{p}_k) - k\frac{1-\beta}{1-\alpha\beta}\tilde{p}_k = 0, \qquad \text{for } k \in \{\lambda, \gamma\}. \tag{16}$$

Moreover, starting from an initial reference price r_0, the optimal price and reference price paths: (a) increase to $\tilde{p}_\lambda$, if $r_0 < \tilde{p}_\lambda$; (b) decrease to $\tilde{p}_\gamma$, if $r_0 > \tilde{p}_\gamma$; (c) remains constant, $p_t = r_0$, otherwise. Consumers either experience a systematic perception of loss (a), or gain (b). Except for price monotonicity, these results extend under non-linear reference effects (Theorem 4 in PW). For future reference, for $\alpha = 0$, we denote the corresponding steady states $p_k^{**} = \tilde{p}_k(\alpha = 0), k \in \{\lambda, \gamma\}$.

This suggests that for both peak-end anchoring and exponential smoothing under linear reference effects, prices and reference prices are monotone and converge to a range of steady states. The same sensitivity insights hold, in particular, the range of steady states is wider, the more loss averse consumers are. In Section 6 we will show that all these common insights are preserved for non-linear reference effects, except price monotonicity.

The predictions of peak-end and exponential smoothing models differ, nevertheless, in several respects under linear (and non-linear, previewing Section 6) reference effects. First, as argued above and illustrated in Figure 2, consumers gain/loss perception may alternate under peak-end anchoring, whereas it is always consistent under exponential smoothing. Second, as detailed below, a range of steady states obtains for loss neutral buyers under peak-end anchoring, whereas exponential smoothing leads to a unique global steady state.

Consumers are *loss neutral* when they are equally sensitive to gains and losses, captured here by $\lambda = \gamma$. There is ample empirical evidence on loss neutrality, captured by a so-called "symmetric sticker shock" in the pricing context (see e.g. Table 2 in Mazumdar et al. 2005 for a summary of references), as well as in more general choice contexts (Schmidt and Traub 2002). In our model, a range of steady states obtains even for loss neutral buyers, as specified in Proposition 1 ($s < S$, even for $\gamma = \lambda$). Below a threshold (S), the firm cannot profitably manipulate the minimum price
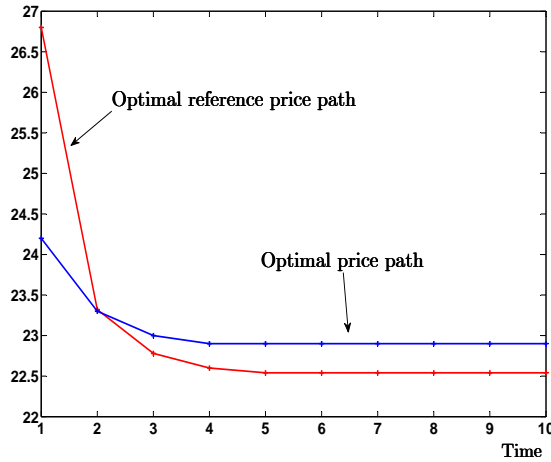
Figure 2   Alternating price perception. $d(p,r) = 500 - 10p - 3(p-r)^+ + 2(r-p)^+, \theta = 0.4, \beta = 0.7, s = 23.00, S = 24.93, m_0 = 22, p_0 = 30$. On this optimal price path, consumers initially perceive a discount $r > p$, and then a surcharge $r < p$.

anchor, which is reflected in the steady state. This is in contrast with the result for exponentially smoothed memory models, where loss neutrality leads to a unique global steady state $\tilde{p}_\lambda = \tilde{p}_\gamma$ (and loss aversion leads to a range). The reason to have a range of steady states for loss neutral buyers (i.e. smooth linear reference effects) under peak-end anchoring, is the asymmetric memory structure (loss aversion widens this range). So overall, the range of steady states in our model is due to both loss aversion and the asymmetric peak-end memory.

Intuitively, there is overall less opportunity value to manipulating prices under peak-end anchoring. On one hand, offering deep discounts can permanently erode demand in the future, as lowest prices remain salient in the memory anchoring process. On the other hand, the future benefit of increasing prices is short lived, as these high prices only affect the reference price in the next period. This is unlike the adaptive expectations framework, where the effect of all past prices lingers in memory, but eventually vanishes (even for lowest prices).

We infer that capturing typical behavioral asymmetries in either consumer anchoring or decision processes leads to a range of steady states. This also suggests that, for a range of initial price expectations, EDLP (everyday low prices) is more likely to be an optimal policy when consumers

exhibit such behavioral asymmetries (Lal and Rao 1997, Kopalle et al. 1996, Fibich et al. 2003).

4. Heterogeneity and Sensitivity to Behavioral Parameters

This section extends our results to allow consumers to be heterogeneous in their anchoring behavior and gain-loss response. As a preliminary, we first investigate the impact of behavioral parameters, in particular the memory anchoring parameter θ, on the firm's prices and profits.

4.1. Sensitivity to the Memory Anchoring Parameter θ

This section analyzes the sensitivity of the firm's optimal policy structure and profits with respect to the parameter $\theta \in (0, 1]$, which measures how strongly consumers anchor on the lowest price. Because $r = \theta m + (1 - \theta)p$, as θ increases, consumers anchor more on the lowest price, and hence have lower price expectations. In the extreme case when $\theta = 1$, consumers anchor only on the minimum price, $r = m$.

Our results in Section 3 show that there exists a global threshold, S given by (10), such that for large enough initial low-price anchors ($m_0 \geq S$), and independent of how much weight θ consumers put on the minimum price, the price path decreases to a global steady state, S. Indeed, S is independent of θ because it solves $\pi_0'(S) = \gamma(1 - \beta)S$ by (10). Therefore, as long as the initial price perception is sufficiently high, m_0 and θ do not affect the long run optimal policy of the firm.

In contrast, a minimum price $m_0 \leq S$, affects the firm's long term strategy (steady state price). If m_0 is sufficiently low ($m_0 \leq s$), the firm's optimal long run policy is $p_\lambda^{**}(m_0) = p_\lambda^{**}(m_0, \theta) \geq m_0$, which solves (12). As consumers anchor more on minimum prices, long run optimal prices are lower, i.e. $p_\lambda^{**}(m; \theta)$ is decreasing in θ. This is because the LHS in (12) is decreasing in θ for $m \leq p$. As $\theta \to 0$, i.e. as consumers anchor more on the last price, $p_\lambda^{**}(m_0; \theta)$ converges to $p_\lambda^{**} = \tilde{p}_\lambda(\alpha = 0)$, the unique solution of $\pi_0'(p) - \lambda(1 - \beta)p = 0$; this is the steady state under exponential smoothing for $\alpha = 0$, see (16). By (9), $s = s(\theta)$ is decreasing in θ, and $s(\theta) \leq s(0) = p_\lambda^{**}$. This shows that the range $[s(\theta), S]$ of steady states of the type (m, m) gets wider with θ, i.e. the more consumers pay attention to the minimum price.

Our results for $\theta \in (0, 1]$ extend to the extreme case $\theta = 0$, when consumers anchor only on the last price (so the minimum price anchor is irrelevant) by setting $m_0 = p_0$, i.e. assuming that the lowest price recalled by consumers is p_0. In this case, we recover the results of PW for $\alpha = 0$. Specifically, the range of steady states is $[s = p_\lambda^{**}, S = p_\gamma^{**}]$, and the price paths converge monotonically to p_λ^{**}, whenever $p_0 \leq p_\lambda^{**}$, or to p_γ^{**}, if $p_0 \geq p_\gamma^{**}$.

Proposition 4 summarizes the above results, and shows that the optimal prices and profits decrease the more consumers account for lowest prices.

Proposition 4. *(a) The optimal prices $p^*(p, m; \theta)$ and profits $J(m, p; \theta)$ in Problem (7) decrease in θ, $\theta \in (0, 1]$. (b) $s(\theta)$ and $p_\lambda^{**}(m; \theta)$ decrease in θ, and $S(\theta) \equiv S$ is independent of θ.*

Our results so far suggest the following sequential estimation procedure for determining the optimal long run policy of the firm:

1. Compute the global threshold S based on (10). Among behavioral parameters, only consumers' sensitivity to discounts, γ, needs to be estimated for this.

2. Assess if lower prices than S were charged in the past (or recalled by consumers). If not, S is the optimal long term price.

3. If the lowest historic price is $m \in [p_\lambda^{**}, S]$, where p_λ^{**} is given by (12), then this price is the optimal long term price. Consumers' sensitivity to surcharges, λ, needs to be estimated for this.

4. If $m_0 < p_\lambda^{**}$, it is necessary to assess the anchoring parameter θ, and calculate $s = s(\theta)$, based on (9). Comparing the minimum price with s determines the equilibrium price, via Proposition 3.

4.2. Heterogeneous Market

This section studies the effect of consumer heterogeneity on firm's optimal pricing policies. Specifically, we show that our insights extend when consumers' anchoring process follows the peak-end rule, but consumers differ in the memory parameter θ_i, and their sensitivities to gains, γ_i, and losses, $\lambda_i, i \in I$, where I is the set of consumer types.

4.2.1. Segmented Market. If the firm can perfectly segment the market, Proposition 1 determines the optimal long run price that should be charged to each segment, depending on their memory anchoring parameter and sensitivities to discounts and surcharges. Segments who anchor more on the minimum price and are more sensitive to discounts and surcharges see lower steady state prices, and are less profitable, as argued next. We further identify conditions when, despite consumer heterogeneity, the firm does not segment the market on the long run.

If consumers differ in their memory parameter θ_i, but share the same sensitivity to gains γ and losses λ, our results in Section 4.1 indicate that, as long as $m_0 \geq p_\lambda^{**}$, heterogeneity in θ is irrelevant on the long run, and the firm charges the same steady state price to all segments (S if $m_0 \geq S$, and m_0 if $m_0 \in [p_\lambda^{**}, S]$). Otherwise, for $m_0 < p_\lambda^{**}$, segments who anchor more on the minimum price, face lower optimal long run prices $p_\lambda^{**}(m_0; \theta)$, by Proposition 4.

For the same memory parameter θ, segments who are more sensitive to reference effects will be charged lower steady state prices. This is because $s = s(\lambda)$ and $S = S(\gamma)$ decrease with λ, respectively γ, by (9-10), and $p_\lambda^{**}(m)$ is decreasing in λ, by Proposition 1. If consumers' initial minimum price anchor is relatively high $m_0 \geq S(\gamma)$, the steady state price $S(\gamma)$ depends on consumer sensitivity to discounts (γ) but not surcharges (λ), whereas for $m_0 \leq s(\lambda)$, the steady state price $p_\lambda^{**}(m_0)$ depends on λ, but not γ. For a range of initial minimum price anchors ($m_0 \in [s(\lambda_{min}), S(\gamma_{max})]$, where $\lambda_{min} = \min_{i \in I} \lambda_i, \gamma_{max} = \max_{i \in I} \gamma_i$) the steady state is the same for all segments (equal to m_0). So in this case, heterogeneous gain/loss sensitivity is not relevant on the long run as the firms does not segment the market.

4.2.2. Unsegmented Market. If the firm cannot segment the market, i.e. offers the same price to all consumers, we show that the results of Section 3 generalize to a heterogeneous market. Suppose that consumers are heterogeneous in behavioral parameters $\theta_i, \lambda_i, \gamma_i$, as well as price sensitivity (i.e. the base demand). Denote $\pi_0^i(\cdot)$ the base profit function of segment $i \in I$, and $\pi_0(p) = \sum_{i \in I} \pi_0^i(p)$ the aggregate base profit. Although consumers are heterogeneous, the same state variables are sufficient to represent the dynamics. The corresponding Bellman Equation is:

$$J(m_{t-1}, p_{t-1}) = \max_{p \in \mathbf{P}} \Big\{ \sum_{i \in I} (\pi_0^i(p) - \lambda_i(p_t - r_t^i)^+ + \gamma_i(r_t^i - p_t)^+) + \beta J(\min(p_t, m_{t-1}), p_t) \Big\}, \qquad (17)$$

where $r_t^i = \theta_i m_{t-1} + (1 - \theta_i) p_{t-1}$ is the reference price for segment $i \in I$. The solution technique developed in Section 3 for a homogeneous market extends under heterogeneity. The thresholds s^H and S^H are defined as in (9-10), and solve, respectively:

$$\pi_0'(p) - p \sum_{i \in I} \lambda_i(1 - \beta(1 - \theta_i)) = 0, \qquad (18)$$

$$\pi_0'(p) - p(1 - \beta) \sum_{i \in I} \gamma_i = 0. \qquad (19)$$

Proposition 5. *Suppose that either (1) $\lambda_i \geq \gamma_i$, for all $i \in I$, or (2) $\theta_i \equiv \theta$, and $\sum_{i \in I} \lambda_i \geq \sum_{i \in I} \gamma_i$. The optimal pricing path of Problem (17) converges monotonically to a steady state, which is: $(m, p_\lambda^{H**}(m))$ for $m \in [0, s^H]$, and (m, m) for $m \in [s^H, S^H]$, where $p_\lambda^{H**}(m)$ solves:*

$$\pi_0'(p) - p \sum_i \lambda_i(2 - (1 - \theta_i)(1 + \beta)) + m \sum_i \lambda_i \theta_i = 0. \qquad (20)$$

Proposition 5 shows that the results and insights of Section 3 extend for heterogeneous markets if (1) each market segment is loss averse, or (2) the aggregate market is loss averse, i.e. $\sum_{i \in I} \lambda_i \geq \sum_{i \in I} \gamma_i$, and consumers are homogeneous in θ. In the latter case, some segments can be 'gain seeking', i.e. more sensitive to discounts than surcharges of the same magnitude captured by $\lambda_i < \gamma_i$ ('loss seeking' in prospect theory terms).

5. The Cost of Ignorance. Comparisons with a Mis-Calibrated Firm.

This section compares the optimal policy and profits, as identified in the previous sections, with those of a mis-calibrated firm. We consider cases where the firm ignores future profits or reference effects, or misjudges consumers' behavioral processes, specifically the effect of the minimum price on memory (peak-end rule), or the asymmetric sensitivity to discounts and surcharges (loss aversion). We explain when and how such mis-calibration affects pricing and projected profits, and identify which effects are relevant for the firm to capture under what conditions.

5.1. Ignoring Future Profits. Myopic policies.

A myopic firm is one that fails to consider the effect of its current pricing policy on future demand, and profits, and focuses on short term profit maximization. Such a firm estimates demand correctly, but uses a myopic pricing policy, $p^M(r) = \text{argmax}_{p \in \mathbf{P}} \pi(p, r)$.

Lemma 6. *The optimal pricing policy of the myopic firm is increasing in r, and equals:*

$$p^M(r) = \begin{cases} p_\lambda(r), & \text{if } r \leq r_\lambda \\ r, & \text{if } r_\lambda \leq r \leq r_\gamma \\ p_\gamma(r), & \text{if } r \geq r_\gamma \end{cases},$$

where r_k is the unique fixed point of $p_k(r) = argmax_{p \in \mathbf{P}} \pi_k(p, r)$, solving $\pi_0'(r_k) = k \, r_k$ for $k \in \{\lambda, \gamma\}$.

A myopic firm adjusts prices over time, as loss averse consumers update their reference price in response to the firm's policy according to equation (2); this amounts to setting $\beta = 0$ in (7). We benchmark this against the policy of a strategic firm, facing the same customer pool. In each period, the state of the myopic firm is (m_t^M, p_{t-1}^M), where $p_t^M = p_t^M(m_{t-1}, p_{t-1}^M) = \arg\max\{\pi(p_t^M, r_t)\}$, resulting in the reference point $r_t^M = \theta m_t^M + (1 - \theta)p_{t-1}^M$. The corresponding state of the strategic firm is (m_{t-1}^*, p_{t-1}^*), with $r_t^* = \theta m_t^* + (1 - \theta)p_{t-1}^*$. We show that the myopic firm underprices the product and follows a monotonic pricing strategy.

Proposition 6. *For any initial state (m_0, p_0), the price charged by a myopic firm at any point in time is less than the optimal price, i.e. $p_t^M \leq p_t$, for all t. Furthermore, the myopic price paths converge monotonically to a constant price, which is: (a) $(m_0, p_\lambda^{M*}(m_0))$ for $m_0 \leq r_\lambda$, where $p_\lambda^{M*}(m_0)$ solves: $\pi_0'(p) - \lambda(1 + \theta)p + \lambda\theta m_0 = 0$, (b) (m_0, m_0) for $r_\lambda \leq m_0 \leq r_\gamma$, and (c) (r_γ, r_γ) for $m_0 \geq r_\gamma$.*

A myopic firm erodes future demand by charging low prices, which decrease consumers' price expectations. In contrast, a strategic firm maintains higher prices to leverage higher reference prices on the long term. A constant myopic pricing policy can nevertheless be optimal for a range of initial price expectations, whenever the firm's discount factor satisfies $\beta \leq \frac{1 - \frac{\gamma}{\lambda}}{1 - \theta}$. In particular, this is always true if consumers anchor sufficiently on low prices, i.e. if $\theta \geq \gamma/\lambda$; practical evidence suggests that $\gamma/\lambda \simeq 0.5$, see e.g. Kahneman et al. (1990), and Ho and Zhang (2008). More specifically, if $\theta \geq 1 - \frac{1}{\beta}(1 - \frac{\gamma}{\lambda})$, the myopic constant pricing policy is optimal for all initial states $p_0 = m_0 \in [s, r_\gamma]$.

Otherwise, a constant myopic pricing policy can never be optimal. It is therefore important for a firm to assess the magnitude of behavioral parameters, such as θ, λ and γ, in order to quantify their impact on profits and decisions.

5.2. Ignoring Reference Effects

A firm who ignores reference effects on demand, assumes $h(r-p,r) \equiv 0$ and charges a constant price $p^0 = \text{argmax}_{p \in \mathbf{P}} \pi_0(p)$ in each period. In this case, consumers are exposed solely and systematically to this price, so it is natural to assume that they form a reference price $r_0 = m_0 = p_0 = p^0$. Observe that $p^0 > S$, from (10), $\pi_0'(p^0) = 0$ and concavity of π_0 (Assumption 1c). Proposition 3 indicates that, starting with initial conditions $m_0 = p_0 = p^0 > S$, it is sub-optimal for the firm to charge a constant price $p^0 > S$. To maximize long term profits, the firm should decrease prices over time, down to S. So the firm who ignores reference effects over-charges at p^0, when in fact it could gain additional profit by offering increasingly steeper discounts to exploit consumer reference dependence.

5.3. Ignoring Peak-End Anchoring

Consider a firm which bases its pricing policy on an inaccurate understanding of consumers' learning and anchoring process. Specifically, suppose that the strategic firm devises its pricing policy assuming that consumers update reference prices based on exponential smoothing $r_{t+1} = \alpha r_t + (1-\alpha)p_t$ (as in PW), while in fact the actual anchoring process follows the peak-end rule $r_{t+1} = \theta m_t + (1-\theta)p_t$. For benchmarking purposes, we consider two relevant cases: (1) $\alpha = 0$, i.e. the mis-calibrated firm simply ignores the lowest price anchor, and (2) $\theta = \alpha$, i.e. the mis-calibrated firm estimates correctly the weight of the last price in the anchoring process.

Consider first the case $\alpha = 0$, when the firm ignores the salience of the lowest price in consumer memory. In this case, for $m_0 = p_0 \geq S$, the mis-calibrated firm prices optimally in each period. Indeed, the price path under peak-end anchoring decreases monotonically to $S = \tilde{p}_\gamma(\alpha = 0) = p_\gamma^{**}$, by (10) and (16). Moreover, this price path coincides with the optimal price path under exponential smoothing with $\alpha = 0$ (see the proof of Proposition 3, part b). In other words, if consumers recall

sufficiently high historical prices $m_0 = p_0$, peak-end anchoring is irrelevant even in a transient regime. In general, the mis-calibrated firm charges higher transient prices, but prices optimally in steady state if $m_0 \geq p_\lambda^{**}$. This is because for $p_0 > m_0 \geq S$, the steady state is $S = p_\gamma^{**}$, and for $m_0 \in [p_\lambda^{**}, p_\gamma^{**}]$, m_0 is the steady state price. Otherwise, if $m_0 < p_\lambda^{**}$, the firm overcharges p_λ^{**} in steady state, whereas the optimal long run price is m_0, for $m_0 \in [s, p_\lambda^{**})$, and $p_\lambda^{**}(m_0)$, for $m_0 \leq s$.

Consider now the case $\theta = \alpha$. By assuming adaptive expectations, the mis-calibrated firm over-estimates the consumer's reference price r_t, and hence the resulting profits. Theorem 4 in PW indicates that the mis-calibrated firm will charge a constant price equal to consumer's initial reference price if the latter $r_0 \in [\tilde{p}_\lambda, \tilde{p}_\gamma]$, given by (16). From (9-10) and (16), $s \leq \tilde{p}_\lambda \leq \tilde{p}_\gamma \leq S$. It follows that, for an intermediate range of initial reference prices, and provided that no historically lower price is recalled, specifically for $m_0 = p_0 \in [\tilde{p}_\lambda, \tilde{p}_\gamma]$, the mis-calibrated firm charges optimally in steady state. Otherwise, by ignoring the fact that consumers anchor on the minimum price, the mis-calibrated firm either overprices or underprices on the long run, depending on consumers' initial price exposure. It overprices if consumers recall a relatively low historic price $m_0 < \tilde{p}_\lambda$, and underprices if the minimum price anchor is relatively high $m_0 > \tilde{p}_\gamma$.

5.4. Ignoring Loss Aversion

We briefly analyze the consequences of ignoring consumers' asymmetric response to discounts and surcharges, i.e. loss aversion. Among pricing practices, discounts are typically more prevalent than surcharges, so it is plausible that demand estimation is biased towards sensitivity to discounts. For benchmarking purposes, suppose that the firm estimates sensitivity to discounts γ correctly, and incorrectly assumes the same sensitivity to surcharges $\lambda = \gamma$, when in fact consumers are loss averse, $\lambda > \gamma$. For completeness, we will also consider the opposite case.

In the first case, it is easy to see from (9-10) that the mis-calibrated firm estimates S correctly (because S only depends on γ), but overestimates $s(\gamma) > s(\lambda)$, because $\lambda > \gamma$. For $m_0 \leq s(\gamma)$, the mis-calibrated firm overprices in steady state at $p_\gamma^{**}(m_0)$, which solves (12) with γ instead of λ (by Proposition 3). Indeed, the actual steady state for the well-calibrated firm is $m_0 \leq p_\gamma^{**}(m_0)$,

for $m_0 \in [s(\lambda), s(\gamma)]$, and $p_\lambda^{**}(m_0) \leq p_\gamma^{**}(m_0)$, for $m_0 \leq s(\lambda)$. For $m_0 \geq s(\gamma)$, the mis-calibrated firm behaves optimally in steady state, which equals m_0 if $m_0 \in [s(\gamma), S]$, and S for $m_0 \geq S$. In general, facing identical customer pools, the mis-calibrated firm systematically charges higher than optimal transient prices (even if long run prices might be equal), and over-estimates profits.

Consider for completeness the opposite case when the firm correctly estimates λ, and wrongly assumes $\gamma = \lambda$. In this case, by overestimating γ, the firms generally under-prices in transience and steady state, and overestimates profits. However, the firm reaches optimal steady-state prices for sufficiently low price memory $m_0 \leq S(\lambda) < S(\gamma)$ (from equation (10)). Moreover, transient prices are also optimal for initial states in $\overline{\mathbf{R}}_{1a}$. This is because the optimal price path starting from an initial state in this region creates a consistent perception of loss (see proof of Proposition 3), and thus is independent of γ. Gain/loss perception alternates starting in $\overline{\mathbf{R}}_{1b}$ or $\overline{\mathbf{R}}_2$ (see Figure 2), leading to suboptimal (too low) transient prices.

To summarize, the results in this section assess the relevance of identifying consumers' actual behavioral processes for pricing purposes. Ignoring reference effects is shown to lead to systematic over-pricing, whereas ignoring future profits leads to under-pricing. Depending on consumer's initial price exposure, we find that mis-calibrated firms may actually price optimally, at least on the long term, even though they ignore behavioral regularities or future profits. Our results emphasize the importance of estimating behavioral parameters $(\theta, \lambda$ and $\gamma)$, and consumers' initial price perception, in order to characterize the optimal pricing policy.

6. General Reference Effects

This section extends the results obtained in Section 3 under linear reference effects, to general, non-linear, reference effects, under behavioral assumptions motivated by prospect theory (Tversky and Kahneman 1991).

6.1. The Model and Assumptions

The demand function with general reference effects is given by:

$$d(p, r) = d_0(p) + h^K(r - p, r) = d_0(p) + \begin{cases} h^L(r - p, r), & \text{if } p \geq r \\ h^G(r - p, r), & \text{if } p \leq r \end{cases}. \tag{21}$$

Let $x = r - p$ denotes the reference gap, i.e. the perceived discount or surcharge relative to the reference point. The smooth functions $h^G(x, r)$ and $h^L(x, r)$ satisfy Assumption 1 over the entire domain $\bar{\mathbf{P}} = [-\bar{p}, \bar{p}] \times [0, \bar{p}]$. Specifically, for $k \in \{L, G\}$, $h^k(0, r) = 0$, $h^k(x, r)$ is increasing in x (reference dependence) and $h^k(r - p, r)$ is increasing in r (because $d(p, r)$ is increasing in r). The following additional assumptions on the reference effect are supported in part by behavioral considerations:

Assumption 2. *(a) $(h^L - h^G)(x, r)$ single crosses the x-axis from below; (b) $(h^L - h^G)(x, r)$ is increasing in x, for $x \leq 0$; (c) $(h^L - h^G)(r - p, r)$ is increasing in r for $x \geq 0$; (d) $h_1^L(0, r) > h_1^G(0, r)$ for all r; (e) $h^G(r - p, r)$ is supermodular in (p, r).*

The first three assumptions restrict the extensions of h^G, respectively h^L, over the loss, respectively gain domains. These extensions do not affect the value function and solution, but are useful for our line of argument. In particular, Assumption 2b implies 2a under diminishing sensitivity, i.e. if h^L is convex and h^G is concave in x, as posited by prospect theory. Assumption 2d captures weak loss aversion. Assumption 2e can be formalized as $h_{11}^G + h_{12}^G \leq 0$. Over gains ($x \geq 0$), this is implied by two behavioral assumptions of prospect theory: diminishing sensitivity, in particular concavity of the value function on the gain domain (implying $h_{11}^G \leq 0$), and decreasing curvature (implying $h_{12}^G \leq 0$). The assumption further restricts the degree of convexity of the extension of h^G over the loss domain. For details on prospect theory assumptions, see Section 2.1 in PW.

The single-crossing property (Assumption 2a) allows to write the kinked reference effect as: $h^K(x, r) = \min\left(h^L(x, r), h^G(x, r)\right)$, and the short-term profit function as:

$$\pi^K(p, r) = \min\left(\pi^L(x, r), \pi^G(x, r)\right). \tag{22}$$

The following technical assumption is made on the smooth profit functions π^L and π^G.

Assumption 3. *(a) $\pi^k(p, r)$ is strictly concave in p, for $k \in \{L, G\}$. (b) $\pi^L(p, r)$ is supermodular in (p, r). (c) $\pi^L(p, \theta m + (1 - \theta)p)$ is strictly concave in p. (d) $\pi^L(p, \theta m + (1 - \theta)p)$ is supermodular in (m, p). (e) $\pi_1^k(p, p) = \pi_0'^k(p) - p h_1^k(0, p)$ is strictly decreasing in p, for $k \in \{L, G\}$*

Concavity of short term profits (Assumption 3a) is a typical economic assumption. Supermodularity (Assumption 3b) supports the intuition that a higher reference price enables the myopic firm to charge higher prices. Supermodularity of π^G immediately follows from the supermodularity of h^G (Assumption 2e), and the fact that $h^G(r-p,r)$ is increasing in r (Assumption 1a). Supermodularity of π^L restricts the degree of convexity of h^L in the reference gap, x. Assumptions 3c and 3d ensure uniqueness of steady states, and validate the intuition that higher price expectations enable the firm to charge higher prices. Assumption 3c essentially means that the marginal profit is more sensitive to changes in price than changes in the reference price in the direction of $(p, \theta m + (1-\theta)p)$. Defining $\tilde{\pi}^L(x,r) = \pi^L(p,r)$, for $x = r-p$, Assumption 3c is equivalent to $\tilde{\pi}^L(x,r)$ being strictly concave in r, i.e. keeping the reference gap, x, constant, there are decreasing marginal (short term) profits with respect to r. Under Assumption 3a, Assumption 3d holds trivially if the reference effect, h^L (i.e. demand) is convex in r. Effectively, Assumption 3d restricts the degree of concavity of h^L in r. Assumption 3e, means that if the firm charges a price equal to consumers' reference price, profit is more sensitive to changes in price at lower reference prices. That is, marginal profit diminishes as price is set to match the reference price (this is the same as Assumption 3b in PW).

Assumptions 2 and 3 are satisfied by common absolute and relative difference reference effect models (see PW). Absolute and relative difference models (AD, respectively RD) are defined by a reference effect h where $h(x,r) = f(x)$, $x = r-p$ for AD models and $x = \frac{r-p}{r}$ for RD models. Following prospect theory, (1) f is increasing in x and $f(0) = 0$ (reference dependence); (2) $f(x)$ is concave for $x \geq 0$ and convex for $x \leq 0$ (diminishing sensitivity), and (3) $\lambda = f'(0^-) \geq f'(0^+) = \gamma$ (weak loss aversion). For both models, we extend the gain and loss components of the reference effect $f(x)$ linearly, by defining smooth functions g, and l as follows: (1) $g(x) = f(x)$ for $x \geq 0$ and $g(x) = \gamma x$ for $x \leq 0$, and (2) $l(x) = f(x)$ for $x \leq 0$ and $l(x) = \lambda x$ for $x \geq 0$.

Remark 1. *Assumptions 1, 2, and 3 hold for (a) AD models satisfying (i) $f'(-x) \geq \gamma$ for $x \in \mathbf{P}$, (ii) $f''(0) = 0$, and (iii) $\frac{f''(x)}{f'(x)} \leq \frac{1}{p}$ for all $|x| \in \mathbf{P}$, and for (b) RD models satisfying (i) $f'(x) \geq \gamma$ for $x \leq 0$, (ii) $f''(0) = 0$, and (iii) $(1-x)\frac{f''(x)}{f'(x)} \leq 2$ for all x.*

In particular, the conditions of Remark 1 are satisfied for piecewise linear AD and RD reference effects. Condition (iii) essentially limits the convexity of reference effects over losses; this is supported by empirical evidence which finds the reference effect to be nearly linear on the loss domain (see e.g. Abdellaoui et al. 2007 for a review). Conditions (i) and (ii) are obtained from constructing linear extensions of the reference effect on the loss, respectively gain domains. Weaker conditions can be obtained, at the cost of mathematical and expository complexity, by constructing non-linear extensions of the reference effect, which satisfy Assumptions 1, 2, and 3.

6.2. General Results

Throughout this section, we assume that the demand model (21) satisfies Assumptions 1,2 and 3. The following critical result extends Lemma 2 for general reference effects:

Lemma 7. *The short-term profit, $\pi^K(p,r) = \min\left(\pi^L(p,r), \pi^G(p,r)\right)$, is supermodular in (p,r).*

The Bellman Equation for this section can now be written as:

$$J(m_{t-1}, p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ \pi^K(p_t, r_t) + \beta J\left(\min(p_t, m_{t-1}), p_t \right) \right\}, \ r_t = \theta m_{t-1} + (1-\theta)p_{t-1}. \tag{23}$$

The same approach as in Section 3 obtains the steady states of Problem (23). The auxiliary upper bound problem, J_m^ν, used to identify the steady states of Problem (23) has the same structure as in (8). The Euler Equation (11) characterizing the steady states of J_m^ν becomes:

$$(1-\nu)(\pi_1^L + \beta(1-\theta)\pi_2^L)(p, \theta m + (1-\theta)p) + \nu(\pi_1^G + \beta \pi_2^G)(p,p) = 0. \tag{24}$$

For simplicity, we use the same notation as in Section 3 for the thresholds s and S, defined as the unique solutions of the respective equations:

$$\pi_0'(p) - (1 - \beta(1-\theta))\, p\, h_1^L(0,p) = 0, \tag{25}$$

$$\pi_0'(p) - (1 - \beta)\, p\, h_1^G(0,p) = 0. \tag{26}$$

In particular, for linear reference effects $h^L(x,r) = \lambda x$ and $h^G(x,r) = \gamma x$, we recover equations (9) and (10). As in Section 3, these thresholds partition the price domain $\mathbf{P}$ into three regions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. The following lemma identifies common steady states of J_m^ν and Problem (23).

Lemma 8. *(a) For $m \in \mathbf{R}_1$, $p_L^{**}(m)$ is a steady state of J_m^ν for $\nu = 0$, where $p_L^{**}(m)$ solves:*

$$\pi_1^L(p, \theta m + (1-\theta)p) + \beta(1-\theta)\pi_2^L(p, \theta m + (1-\theta)p) = 0. \tag{27}$$

In this case $(m, p_L^{**}(m))$ is a steady state of Problem (23).

(b) There exists $\nu \in [0,1]$ such that, for $m \in \mathbf{R}_2$, m is a steady state for J_m^ν. In this case, (m,m) is a steady state of Problem (23).

Assumption 3c ensures the uniqueness of $p_L^{**}(m)$ as the solution to equation (27). Assumption 3d guarantees that $p_L^{**}(m)$ is increasing in m, i.e. exposure to a higher initial minimum price enables the firm to charge higher long run prices.

A similar proof as for Proposition 2 shows that, if the initial state is in any of the regions $\mathbf{R}_i$, $i = 1, 2, 3$, then the state path remains in that region over time. Thus, if the price path converges, it must converge to a point in the same region. The following result extends Proposition 3.

Proposition 7. *The optimal pricing policy $p^*(m,p)$ of Problem (23) is increasing in m and p. Starting at an initial state (m_0, p_0), the optimal price path $\{p_t\}$ converges monotonically to a steady state price which equals: (a) $p_L^{**}(m_0)$, if $m_0 \in \mathbf{R}_1$, (b) m_0, if $m_0 \in \mathbf{R}_2$, and (c) S, if $m_0 \in \mathbf{R}_3$.*

Therefore, under the additional Assumptions 2 and 3, the results in Section 3 extend to general reference effects. Specifically, the memory structure leads to a range of steady states. This range is affected by the slopes of h^L and h^G at $x = 0$ (see equations (25) and (26)).

Interestingly, price monotonicity is a robust effect in our model, holding under general, non-linear reference effects. Under the adaptive expectations framework, the optimal price paths are monotonic for linear (or convex) reference effects, but not in general (see e.g. Figure 4 in PW). Intuitively, under exponential smoothing, the firm has more flexibility to manipulate reference prices upwards by charging a high price, and then exploits these high reference prices by decreasing prices. In contrast, the benefit of raising reference prices by increasing prices is only momentary under peak-end anchoring, as high prices do not linger in memory (only the lowest price is salient). Technically, price monotonicity is affected by the composite effect of: (1) the interaction between

price and reference price in memory and (2) the shape of the value function, and reference effects in particular. Our results suggests that high-low pricing cannot be generally attributed to the (non-linear) shape of reference effects, but may be triggered by the persistence and interaction of past prices in memory, in an adaptive expectation framework. Overall, we conclude that the memory process is an important factor affecting the structure of optimal pricing policies.

7. Extensions and Future Work

This section, highlights the robustness and limitations of our work, by suggesting some possible extensions and directions for future research.

7.1. Estimation and Alternative Memory Models.

Current empirical research in the pricing context has focused on validating exponentially smoothed reference price models. Our work highlights the importance of testing alternative memory models, such as the peak-end rule, and determining the relevant parameters, θ, λ, γ, in a practical setting. Section 5 provides conditions when it is relevant to identify the accurate behavioral processes and estimate the corresponding behavioral parameters, for pricing purposes. In particular, our results in Section 4.1 suggest that estimating θ is only relevant for the firm's long term pricing strategy when consumers have been exposed to sufficiently low prices. Overall, it is important to understand what type of anchoring model is appropriate for which practical conditions (e.g. type of product or service, industry), and to test other possible anchoring models, some of which we propose below.

7.1.1. A Hybrid Model. As a possible direction for future research, we propose to estimate the validity of a memory model which combines exponential smoothing and the peak-end rule. Albeit not motivated by a stand alone behavioral theory, a hybrid model such as $r_t = \theta m_{t-1} + (1 - \theta)(\alpha r_{t-1} + (1 - \alpha)p_{t-1})$, encompasses both exponential smoothing or peak-end models. In particular, for $\theta = 0$ this model reduces to the exponential smoothing model in PW, whereas $\alpha = 0$ yields the peak-end model in this paper.

One can prove analytically that our convergence and steady state insights remain valid for this hybrid memory model. In particular, a range of steady states obtains even for loss neutral

buyers, due to the kink in the memory structure (this collapses to only one for $\theta = 0$). Further, loss aversion leads to a wider range of steady states. In general, prices for this model are not necessarily monotonic, although reference prices are. Price monotonicity extends to this general model under the sufficient conditions identified in PW, i.e. when short term profit $\pi(p, r)$ is convex in the reference price, in particular for linear reference effects, or when $\alpha = 0$ for which the peak-end model is obtained. This suggests that price monotonicity is a special feature of the peak-end model, which does not extend when intermediate prices linger and interact in the memory process. These results isolate some of the robust insights in pricing with reference effects. Yet, they should be interpreted with caution, as long as the proposed memory model awaits empirical validation.

7.1.2. Loss Averse Memory. Other types of memory asymmetries, such as the kinked exponential smoothing model proposed in Gaur and Park (2007), may affect the optimal pricing policy of the firm. Preliminary results (Nasiry and Popescu 2008) indicate that the same insights remain valid when consumers exhibit loss aversion in their memory structure (corresponding to loss aversion in experienced utility).

7.1.3. Updating the Low Price Anchor. The model presented in this paper assumes that consumers remember the lowest price indefinitely. This is a reasonable proxy for a product or service where the frequency of transactions is very high relative to the horizon length. We next argue that our results are relatively robust when accounting for the possibility of occasionally forgetting and updating the minimum price anchor.

Consider an alternative model where consumers anchor on the lowest price in the last τ periods, i.e. $m_t = \min(p_{t-\tau+1}, \cdots, p_t)$ and $r_t = \theta m_{t-1} + (1-\theta)p_{t-1}$. We refer to τ as the length of consumers' memory window. The Bellman equation for this problem is:

$$J(p_{t-\tau}, \cdots, p_{t-1}) = \max_{p \in \mathbf{P}} \{\pi(p_t, r_t) + \beta J(p_{t-\tau+1}, \cdots, p_t)\}. \tag{28}$$

In particular, for $\tau = 1$, the reference price is the last period price, so this problem coincides with

the limiting case $\theta = 0$ in our model, or $\alpha = 0$ in PW (see Section 4.1). At the other extreme, as $\tau \to \infty$ we recover our peak-end model. Under linear reference effects (4), define $s(\theta; \tau)$ to solve:

$$\pi'_0(p) - \lambda(1 - \beta(1 - \theta(1 - \beta^{\tau-1})))p = 0. \tag{29}$$

A similar technique as in Section 3 is used to characterize the steady states of Problem (28). We say that p is a steady state price of Problem (28) if, starting with initial memory $p_0 = \ldots = p_{-\tau+1} = p$, the optimal price to charge is again p.

Proposition 8. *The set of steady state prices for Problem (28) is* $[s(\theta; \tau), S]$.

Like in our original model, there is a range of steady states for loss neutral consumers, which becomes wider under loss aversion. The largest steady state price S, given by (10), is independent of θ, τ (and λ), and coincides with that of our original model. The range of steady state prices expands (downwards) with the memory window, in other words, the shorter consumers' memory the more the firm can push long term prices up. In particular $s = \lim_{\tau \to \infty} s(\theta; \tau) < s(\theta; \tau)$, where s is as defined in (9) for our original model. The steady states of Problem (28) lie only on the diagonal, as no matter how low the initial minimum price is, it will eventually be forgotten as the memory window moves forward. The firm can exploit this by charging higher prices than in the original model, if consumers have relatively low price expectations.

By numerical investigation, we conjecture that, starting at any initial state $(p_{-\tau+1}, \cdots, p_{-1}, p_0)$, the optimal price and reference price paths converge monotonically as of time $t = \tau + 1$ to a steady state price characterized by Proposition 8, as illustrated in Figure 3. Specifically, prices increase to $s(\theta; \tau)$ if $r_{\tau+1} \le s(\theta; \tau)$, they decrease to S if this exceeds $r_{\tau+1}$, whereas for $r_{\tau+1} \in [s(\theta; \tau), S]$ prices stay constant at $p_{\tau+1}$. This suggests that our main insights from model (3) remain valid when consumers have a bounded memory window. Only the local steady states $p^{**}_\lambda(m_0)$ in region $\mathbf{R_1}$ (see Proposition 3) appear to be a result of indefinite salience of the minimum price m_0 in our original model, and vanish if the minimum price is ever updated.
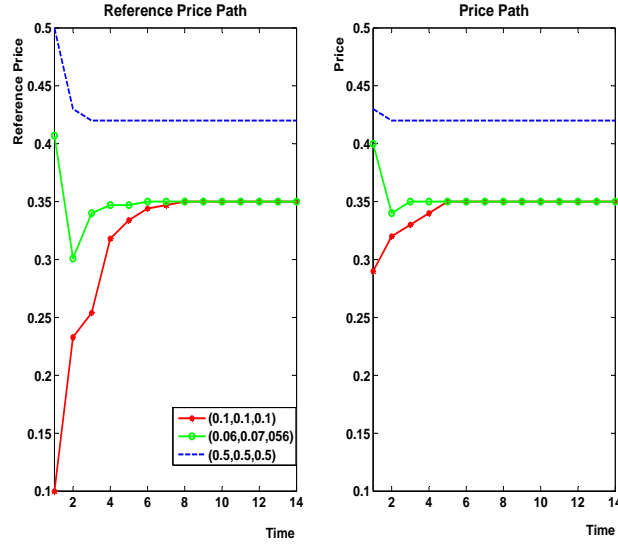
Figure 3 Optimal reference price and price paths for Problem (28) are monotone for $t > \tau = 3$. The demand function is:
$d(p) = 1 - p - (p - r)^+ + 0.5(r - p)^+$, $p \in [0, 1]$. $\theta = 0.3$, $\beta = 0.2$.

7.1.4. Maximum Price Anchoring. This paper assumed that consumers dwell on lowest historical prices when making purchase decisions, consistent with the peak-end rule for positive experiences. Arguably, in a negative experience context (such as paying late fees, fines or taxes), the peak-end rule could be interpreted to predict anchoring on the highest price instead of the lowest. We briefly investigate the case when loss averse consumers anchor on the highest (instead of lowest) price in the price history, $M_t = \max(M_{t-1}, p_t)$ and the reference price is $r_t = \theta M_{t-1} + (1 - \theta)p_{t-1}$. The Bellman Equation is:

$$J(M_{t-1}, p_{t-1}) = \max_{p \in \mathbf{P}} \{\pi(p_t, r_t) + J(\max(M_{t-1}, p_t), p_t)\}. \tag{30}$$

Numerical investigation indicates that some of the structural results obtained in this paper continue to hold under maximum price anchoring. In particular, the optimal price path always converges and there is a range of steady states, even for loss neutral buyers. On the other hand, the transient behavior is different, and depends on how much the firm discounts future profits β, relative to the loss aversion index $\frac{\gamma}{\lambda}$. Behavioral evidence suggests that $\gamma/\lambda \simeq 0.5$, see e.g. Kahneman et al. (1990), and Ho and Zhang (2008). If $\beta < 1 - \frac{\gamma}{\lambda}$, i.e. the firm's discount factor is low, or consumers are

sufficiently loss averse, then the optimal price (and reference price) paths are monotonic (similar to the peak-end model).

In the more plausible case when $\beta \geq 1 - \frac{\gamma}{\lambda}$, which includes loss neutral buyers, price monotonicity need not hold. In this case, unlike previous research, higher initial reference prices do not necessarily result in a higher steady state price (see Figure 4). As illustrated in Figure 5, the firm charges a high price $\hat{p}$ at an early stage (not necessarily the first period), and then exploits this high price anchor for the rest of the horizon by offering lower prices to boost the demand. So, for low initial reference prices, the price path is non-monotonic. For high initial reference prices, the price path is decreasing, as the firm offers discounts without eroding the reference price. The value of $\hat{p}$ is higher, the more patient the firm is (i.e. the higher β), or the higher the loss aversion index $\frac{\gamma}{\lambda}$.

As expected, the firm obtains higher profits when consumers anchor on maximum, rather than minimum, prices. Further analytical investigation and empirical validation for this model are left for future research.
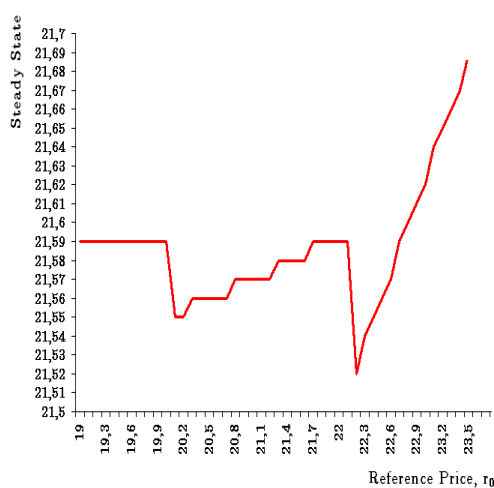

Figure 4 Steady states of Problem (30) as a function of the initial reference price for $d_0(p) = 300 - 6p$, $p \in [0, 50]$, $\lambda = 3$, $\gamma = 2$, $\theta = 1$, $\beta = 0.5$.
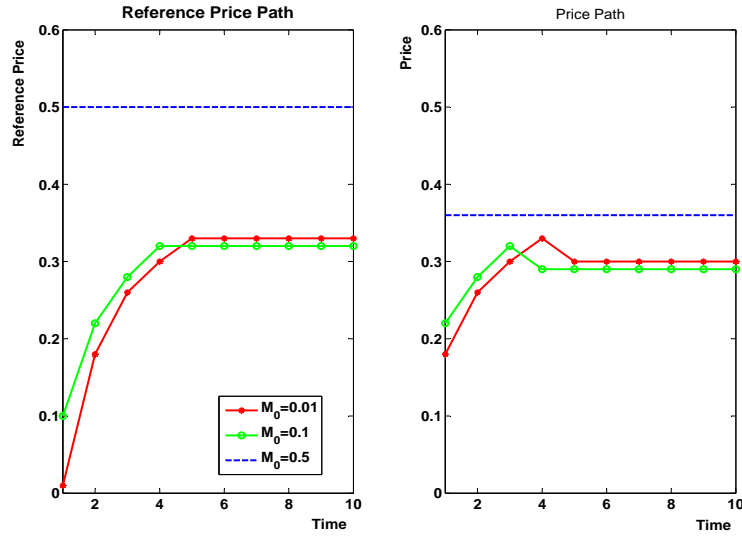
Figure 5   Optimal reference price path and price paths of Problem (30), with $d_0(p) = e^{-p}$, $p \in [0, 1]$. $\lambda = 3$, $\gamma = 2$, $\theta = 1$, $\beta = 0.5$.

7.2. Gain Seeking Consumers.

Our results in this paper focused on consumer loss aversion, a widely validated behavioral regularity in empirical research (Kahneman et al. 1990, Tversky and Kahneman 1991), and in particular in the pricing context (Kalyanaram and Winer 1995, Mazumdar et al. 2005). The opposite 'gain seeking' behavior (also known as 'loss seeking'), whereby consumers are more sensitive to discounts than surcharges of the same magnitude, has found very limited evidence in the pricing context (Krishnamurthi, et al. 1992, Mazumdar and Papatla 2000). Previous literature on pricing with reference effects has considered this case for completeness. Under exponential smoothing, previous results show that gain seeking behavior leads to a cycling optimal pricing policy, in particular no constant pricing policy can be optimal (see PW and references therein).

For linear reference effects, gain seeking behavior translates into $\lambda < \gamma$ in the demand model (4). The short term profit can be written as $\pi(p_t, r_t) = \max(\pi_\lambda, \pi_\gamma)(p_t, r_t)$, which is not supermodular, in contrast with Lemma 2. This significantly changes the structure of the problem; in particular, it implies the counter-intuitive fact that a myopic firm does not necessarily charge higher prices to consumers with higher price expectations. Our numerical investigation of the optimal pricing

policy for gain seeking buyers under peak-end anchoring, for a variety of parameter values and base demand models $d_0(p)$, suggests the following conjectures:

1. For any $\theta \in (0, 1]$ there exists a threshold m_θ so that the optimal policy converges for all buyers with low initial price exposures $m_0 \leq m_\theta$. Convergence always holds for $\theta = 1$, i.e. if consumers anchor only on the lowest price.

2. There exists a threshold $\bar{\theta}$, such that: (a) if consumers put a relatively low weight on the minimum price, $\theta < \bar{\theta}$, the optimal policy oscillates for all $m_0 > m_\theta$; (b) if $\theta > \bar{\theta}$, the optimal policy converges for all $m_0 > S$, given by (10).

3. In general, the structure of the results depends on how strongly the firm discounts future profits relative to the index of loss aversion: (a) if $\beta < 1 - \frac{\gamma}{\lambda}$ the optimal policy appears to always converge for $\theta > \bar{\theta}$, (b) if $\beta < 1 - \frac{\gamma}{\lambda}$ (in particular for loss neutral buyers), for any θ, there is a middle range of initial price exposures m_0 for which the optimal policy oscillates, and outside of which it converges.

Further analytical and empirical validation for the gain seeking model is left for future research.

7.3. Strategic Interaction and Competition

The current model ignores the possible strategic reaction of consumers to the pricing policy of the firm. For example, consumers may postpone their purchase in response to a skimming type strategy, or advance purchase in anticipation of increasing prices (see e.g. Ovchinnikov and Milner 2005, Liu and van Ryzin 2007, Su 2007, 2008). Future research should guide firms to respond to consumer strategic behavior, combined with reference dependence, and ask which, if any, of these behavioral effects is dominant.

Another type of strategic interaction ignored here is between firms. Under quantity competition, Fibich et al. (2003) obtain convergence for loss averse buyers under exponential smoothing. It would be interesting to explore how our results extend to competitive settings.

8. Summary and Conclusions

The literature on dynamic pricing with reference effects generally assumes that price anchors are formed by an exponential smoothing process, as a weighted average of all past prices. In this paper, we motivated the peak-end rule, as a consumer anchoring and memory mechanism. First proposed by Kahneman et al. (1993), the peak-end rule suggests (in the pricing context) that consumers' price judgements are based on the lowest and the most recent price, and the reference price is formed as a weighted average of these two prices. In this context, we studied the optimal pricing strategies of a monopolist, in response to consumers that exhibit behavioral regularities. Specifically, consumers anchor on past prices, by forming a reference price based on the peak-end rule, and their purchase decisions are influenced by these anchors, in the spirit of prospect theory.

Our results showed how the peak-end anchoring process interacts with loss aversion to affect the structure of the optimal pricing strategies. Under the peak-end rule, a range of constant pricing strategies is optimal even with loss neutral buyers, unlike the current literature. In our case, this range is due to the asymmetry in memory structure, in addition to loss aversion. The range of steady states is wider the more loss averse consumers are, and the more they anchor on the lowest price. Consistent with the literature, the value of the steady state prices decreases with consumers' sensitivity to gains and losses. In addition, the more consumers anchor on the minimum price, the lower the optimal prices, and the firm's profits.

In contrast with previous literature, we found that under peak-end anchoring, the optimal price paths are always monotonic, following a traditional skimming or penetration strategy even under general reference effects (Section 6). Table 1 summarizes our results for loss averse (and in particular loss neutral) buyers under peak-end anchoring processes, and compares them to PW under exponential smoothing. In contrast with the literature, numerical investigation suggests that gain seeking behavior (the opposite of loss aversion) does not necessarily lead to high-low pricing under peak-end anchoring (see Section 7.2).

The robustness of these insights is supported when consumers are heterogeneous (Section 4.2),

Table 1 Optimal policy structure for loss averse buyers under different anchoring processes.

Main Results	Peak-End Anchoring	Exponential Smoothing (PW)
Price path	monotone	possibly non-monotone
Reference price path	monotone	monotone
Gain/Loss Perception	possibly alternating	consistent
Steady States	a range	a range (unique for loss neutral)

as well as by alternative memory models (Section 7.1). In particular, the latter allow the possibility of forgetting or updating the minimum price anchor, or combining peak-end and exponential smoothing memory processes.

Our results suggested which behavioral effects and parameters are important for a firm to measure, under what conditions and in which sequence, in order to determine its optimal pricing strategy. For example, we found that sensitivity to the minimum price anchor (θ) is only relevant if consumers' initial perception is relatively low. Otherwise, a firm that ignores peak-end anchoring and relies on an exponential smoothing model (with $\alpha = 0$), still prices optimally. More generally, we explained (in Section 5) how mis-estimating behavioral processes, such as peak-end anchoring or loss aversion translates into mis-pricing. We found that ignoring reference effects results in systematic over-pricing, whereas ignoring future profits leads to under-pricing. Finally, we identified conditions (depending on consumer's initial price exposure), when firms may actually price optimally, at least on the long run, even though they ignore behavioral regularities or future profits. Our results highlight the relevance of identifying consumers' actual behavioral processes for pricing purposes, and reveal future research directions in this context.

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Appendix: Proofs

Proof of Lemma 1: By Assumption 1a, $d(p_t, r_t)$ is increasing in $r_t = \theta m_{t-1} + (1-\theta)p_{t-1}$. Hence $\pi(p_t, r_t) = p_t\, d(p_t, r_t)$ is increasing in p_{t-1} and m_{t-1}. Moreover the transition in the Bellman Equation (3) is increasing in m_{t-1} (and independent of p_{t-1}). So the value function is increasing in its arguments (Stokey et al. 1989, Theorem 4.7).

Proof of Lemma 2: For $p^l \leq p^h$ and $r^l \leq r^h$, we need to show that:

$$\pi(p^h, r^h) - \pi(p^l, r^h) \geq \pi(p^h, r^l) - \pi(p^l, r^l). \tag{31}$$

We show this by considering all possible cases: (1) $p^l \leq p^h \leq r^l \leq r^h$, (2) $p^l \leq r^l \leq p^h \leq r^h$, (3) $p^l \leq r^l \leq r^h \leq p^h$, (4) $r^l \leq p^l \leq p^h \leq r^h$, (5) $r^l \leq p^l \leq r^h \leq p^h$, (6) $r^l \leq r^h \leq p^l \leq p^h$. Cases 1 and 6 follow immediately because all (p, r) pairs fall on π_γ or π_λ, respectively, which are supermodular in (p, r).

Case 2: For $p^l \leq r^l \leq p^h \leq r^h$, (31) simplifies to:

$$\gamma(r^h - p^h)p^h - \gamma(r^h - p^l)p^l \geq -\lambda(p^h - r^l)p^h - \gamma(r^l - p^l)p^l, \text{ or}$$

$$\gamma(r^h - p^h)p^h + \lambda(p^h - r^l)p^h \geq \gamma(r^h - r^l)p^l.$$

Because $\lambda > \gamma$ and $p^h \geq r^l$, it is sufficient to show $\gamma(r^h - p^h)p^h + \gamma(p^h - r^l)p^h \geq \gamma(r^h - r^l)p^l$, which holds because $\gamma p^h(r^h - r^l) \geq \gamma p^l(r^h - r^l)$.

Case 3: For $p^l \leq r^l \leq r^h \leq p^h$, (31) simplifies to:

$$-\lambda(p^h - r^h)p^h - \gamma(r^h - p^l)p^l \geq -\lambda(p^h - r^l)p^h - \gamma(r^l - p^l)p^l,$$

or, $\lambda p^h(r^h - r^l) \geq \gamma p^l(r^h - r^l)$, which is true.

Case 4: For $r^l \leq p^l \leq p^h \leq r^h$, (31) simplifies to:

$$\gamma(r^h - p^h)p^h - \gamma(r^h - p^l)p^l \geq -\lambda(p^h - r^l)p^h + \lambda(p^l - r^l)p^l, \text{ or}$$

$$p^h\Big(\gamma(r^h - p^h) + \lambda(p^h - r^l)\Big) \geq p^l\Big(\lambda(p^l - r^l) + \gamma(r^h - p^l)\Big).$$

It is enough to show that $\gamma(r^h - p^h) + \lambda(p^h - r^l) \geq \lambda(p^l - r^l) + \gamma(r^h - p^l)$, which holds because $\lambda(p^h - p^l) \geq \gamma(p^h - p^l)$.

Case 5: For $r^l \leq p^l \leq r^h \leq p^h$, (31) simplifies to:

$$-\lambda(p^h - r^h)p^h - \gamma(r^h - p^l)p^l \geq -\lambda(p^h - r^l)p^h + \lambda(p^l - r^l)p^l, \text{ or}$$

$$\lambda p^h(r^h - r^l) \geq \lambda p^l(p^l - r^l) + \gamma(r^h - p^l)p^l.$$

Because $\lambda \geq \gamma$ and $r^h \geq p^l$, it is sufficient to show that:

$$p^h(r^h - r^l) \geq p^l(p^l - r^l) + (r^h - p^l)p^l = p^l(r^h - r^l),$$

which obviously holds.

Proof of Lemma 3: Dynamic programming theory (e.g. Stokey and Lucas, 1989, Theorem 4.6) indicates that, for all $p \geq m$, the function $J_m^\nu(p)$ uniquely solves $TJ_m^\nu(p) = J_m^\nu(p)$, where the operator T is defined for any continuous function f over $[m, \bar{p}]$ by:

$$Tf(p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ (1 - \nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, p_{t-1}) + \beta f(p_t) \right\}. \tag{32}$$

Moreover, $\lim_{n \to \infty} T^n f(p) = J_m^\nu(p)$.

We argue below that, for all $p \geq m$, $J(m, p) \leq TJ(m, p)$. This further implies $J(m, p) \leq T^n J(m, p) \leq \lim_{n \to \infty} T^n J(m, p) = J_m^\nu(p)$, concluding the proof. Indeed, for $m_{t-1} \leq p_{t-1}$, we have:

$$\begin{aligned}
J(m_{t-1}, p_{t-1}) &= \max_{p_t \in \mathbf{P}} \left\{ \pi(p_t, r_t) + \beta J(\min(m_{t-1}, p_t), p_t) \right\} \\
&\leq \max_{p_t \in \mathbf{P}} \left\{ (1 - \nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, r_t) + \beta J(m_{t-1}, p_t) \right\} \\
&\leq \max_{p_t \in \mathbf{P}} \left\{ (1 - \nu)\pi_\lambda(p_t, r_t) + \nu\pi_\gamma(p_t, p_{t-1}) + \beta J(m_{t-1}, p_t) \right\} \\
&= TJ(m_{t-1}, p_{t-1}).
\end{aligned}$$

The first inequality above holds because $\pi = \min(\pi_\lambda, \pi_\gamma) \leq (1 - \nu)\pi_\lambda + \nu\pi_\gamma$, and the value function is increasing in its arguments (Lemma 1). The second inequality holds because $p_{t-1} \geq r_t = \theta m_{t-1} + (1 - \theta)p_{t-1}$, for $p_{t-1} \geq m_{t-1}$. Finally, the last equality follows by the definition of T in (32) applied to $f(p) = J(m, p)$.

Proof of Lemma 4: (a) It is easy to check that, if Problem (8) admits an interior steady state, this solves the Euler Equation (11). Moreover, this equation admits a unique solution, because $\pi_0'(p) < 0$ and the coefficient of p in (11) can be written as $-(1 - \beta)((1 - \nu)\lambda(1 + \theta) + \nu\gamma) \leq 0$. It remains to verify that a steady state exists, and must be interior. Existence of a steady state follows from supermodularity of the objective function, because π_λ and π_γ are supermodular in (p, r). By Topkis Theorem (Topkis 1998, Theorem 2.8.2), this implies that the pricing paths of Problem (8) are monotonic on the bounded domain $\mathbf{P}$, hence converge to a steady state p^{**}.

We further argue that a steady state must be interior. First, $p^{**} = 0$ cannot be a steady state because any non-zero pricing strategy achieves positive profits. Second, Assumption 1c ensures that $p^{**} < \bar{p}$ for any steady state of Problem (8). This is because $\pi_0(p)$ is non-monotone, hence, its largest maximizer $\hat{p}$ is interior, i.e. $\hat{p} < \bar{p}$. Moreover, concavity of π_0 implies: $J^\nu(\hat{p}) \geq \frac{\pi(\hat{p})}{1 - \beta} \geq \frac{\pi(p^{**})}{1 - \beta} = J^\nu(p^{**})$.

Finally, because J^ν is increasing, we conclude that $p^{**} \leq \hat{p} < \bar{p}$, so p^{**} is interior and solves the Euler Equation (11).

Finally, by definition, $p_\lambda^{**}(m)$ solves (11) for $\nu = 0$. This has a unique solution because the LHS is strictly decreasing in p, positive at $p = 0$ and negative at $p = \bar{p}$.

(b) Substituting $p = m$ in (11), we have $L(m, \nu) = \pi_0'(m) - \lambda[(1-\nu)(1-\beta(1-\theta)) + \nu(1-\beta)]m = 0$. Equations (9) and (10) translate to $L(s, 0) = 0$ and $L(S, 1) = 0$. Because $L(m, \nu)$ is decreasing in m, it follows that for all $m \in [s, S]$, $L(m, 0) \leq 0$ and $L(m, 1) \geq 0$. The result follows because $L(m, \nu)$ is continuous in ν.

Proof of Lemma 5: We first show that $p_\lambda^{**}(m)$, as defined by (12), is feasible, i.e. $p_\lambda^{**}(m) \geq m$ for $m \in [0, s]$. Note that $p_\lambda^{**}(m)$ is increasing in m and single crosses the identity line from above at s, defined by (9). Feasibility follows because, at $m = 0$, (12) has a unique positive solution, $p_\lambda^{**}(0)$.

For $m \in [0, s]$, the constant pricing policy $p_t \equiv p_\lambda^{**}(m)$ is optimal for Problem (8) with $\nu = 0$, and feasible for Problem (7). Because $m \leq s$, $\min(m, p_\lambda^{**}(m)) = m$, and $r = \theta m + (1-\theta)p_\lambda^{**}(m) \leq p_\lambda^{**}(m)$, which implies $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. This constant pricing policy yields the same value in both problems, so it is also optimal for Problem (7), and $(m, p_\lambda^{**}(m))$ is a steady state of Problem (7).

For $m \in [s, S]$, the constant pricing policy $p_t \equiv m$ is optimal for Problem (8), feasible for Problem (7) ($\pi_\lambda = \pi_\gamma$ along this path), and yields the same value in both problems. Therefore (m, m) is a steady state of Problem (7).

Proof of Proposition 1: For any steady state (m, p), two cases are possible, either $m = p$, or $m < p$. In the second case, $r < p$ and starting at (m, p), the price path gives a consistent perception of loss. Therefore, the steady state price must be the same as the steady state of Problem (8), with $\nu = 0$, i.e. $p = p_\lambda^{**}(m)$. Thus Problem (7) has only two types of steady states. It remains to identify the regions where each type of steady state is relevant.

First assume $m < s$. We show by contradiction that (m, m) cannot be a steady state of Problem (7). If (m, m) is a steady state, the profit from charging a constant price $p_t \equiv m$ exceeds the profit on path $p_t = m + \delta$, for all t, i.e.:

$$\frac{\pi_0(m)}{1-\beta} \geq \pi_0(m+\delta) - \lambda\delta(m+\delta) + \frac{\beta}{1-\beta}\Big(\pi_0(m+\delta) - \lambda(m+\delta-r)(m+\delta)\Big),$$

where $r = \theta m + (1-\theta)(m+\delta)$. This reduces to: $\pi_0(m+\delta) - \pi_0(m) \leq \lambda\delta(m+\delta)(1-\beta(1-\theta))$. Dividing both sides by δ and letting δ go to zero, we have:

$$\pi_0'(m) \leq \lambda(1-\beta(1-\theta))m. \tag{33}$$

This holds with equality for $m = s$ (see (9)), and because the LHS is strictly decreasing in m, (33)

implies $m \geq s$, a contradiction. We conclude that, for $m < s$, the only possible steady state for Problem (7) is $(m, p_\lambda^{**}(m))$.

Moreover, because $p_\lambda^{**}(m) < m$ for $m > s$, it follows that, for $m \geq s$, the only possible steady state is (m, m). We prove by contradiction that (m, m) cannot be a steady state for $m > S$. If (m, m) is a steady state, the profit from charging a constant price $p_t \equiv m$ exceeds the profit along the alternative path $p_t = m - \delta$, for all t, i.e.:

$$\frac{\pi_0(m)}{1 - \beta} \geq \pi_0(m - \delta) + \gamma\delta(m - \delta) + \frac{\beta}{1 - \beta}\pi_0(m - \delta).$$

This reduces to $\pi_0(m) - \pi_0(m - \delta) \geq \gamma\delta(1 - \beta)(m - \delta)$. Dividing by δ and letting δ go to zero, we have:

$$\pi_0'(m) \geq \gamma(1 - \beta)m, \tag{34}$$

which holds with equality for $m = S$ (see (10)). Because $\pi_0'(m)$ is strictly decreasing in m, (34) implies that $m \leq S$, a contradiction. We conclude that steady states of the form (m, m) can only be relevant when $s \leq m \leq S$.

Finally, $p_\lambda^{**}(m)$ is decreasing in λ. This is because $p_\lambda^{**}(m) \geq m$ (for $m \leq s$) solves equation (12) the LHS of which is decreasing in p, and λ (for $p \geq m$).

Proof of Proposition 2: (a) We prove this in two parts, depending if $m_0 \in \mathbf{R}_1$, or $m_0 \in \mathbf{R}_2$. For $m_0 \in \mathbf{R}_1$, we consider two cases: $(m_0, m_0) \in \overline{\mathbf{R}}_{1a}$ and $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$.

Claim 1. For $(m_0, p_0) \in \overline{\mathbf{R}}_{1a}$, then $p_t \geq m_0$ for any t.

Proof: Denote $J^{\nu=0}$ the objective function in Problem (8), with $\nu = 0$. $J^{\nu=0}$ is supermodular in (p, r), and thus the price path converges monotonically to the steady state price, $p_\lambda^{**}(m_0)$. Because $p_0 < p_\lambda^{**}(m_0)$, the optimal price path for $J^{\nu=0}$, increases to this steady state, and $p_t^*(r_t) \leq p^{**}(m_0)$ for all t. This implies that $m_t = m_0$ along this path (the minimum price does not change over time), and thus $r_t = \theta m_0 + (1 - \theta)p_{t-1} \leq p_{t-1} \leq p_t$. Therefore, $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$, and this path is feasible for (7), and yields the same value which leads us to conclude that the same path is also optimal for Problem (7).

This result is stronger than stated in the claim, because it guarantees also the existence of the steady state, and the monotonicity of the price path.

Claim 2. Given $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$, then $p_t \geq p^{**}(m_0) \geq m_0$ for any t.

Proof: For $m_0 \leq s$, $(m_0, p_\lambda^{**}(m_0))$ is a steady state of Problem (7) (Proposition 1). We show that if at any time it is optimal for the price to be below $p_\lambda^{**}(m_0)$, then $(m_0, p_\lambda^{**}(m_0))$ cannot be a steady state of Problem (7) which is a contradiction.

Let $r_1^* = \theta m_0 + (1-\theta) p_\lambda^{**}(m_0)$. We show that $p_1 = p^*(m_0, p_0) \geq p_\lambda^{**}(m_0)$, and then by induction we conclude that $p_t \geq p_\lambda^{**}(m_0)$. Assume by contradiction, $p_1 < p_\lambda^{**}(m_0)$. Then:

$$\pi(p_1, r_1) + \beta J(\min(m_0, p_1), p_1) > \pi(p_\lambda^{**}(m_0), r_1) + \beta J(m_0, p_\lambda^{**}(m_0)),$$

or equivalently by defining $\Delta J = J(m_0, p_\lambda^{**}(m_0)) - J(\min(p_1, m_0), p_1)$,

$$\pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1) > \beta \Delta J. \tag{35}$$

Because $p_0 > p_\lambda^{**}(m_0) > p_1$, we have $r_1 > r_1^*$. Supermodularity of $\pi(p, r)$ (Lemma 2), then implies:

$$\pi(p_1, r_1^*) - \pi(p_\lambda^{**}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1). \tag{36}$$

Because $p_\lambda^{**}(m_0) > r_1^*$, it follows that $\pi_\lambda(p_\lambda^{**}(m_0), r_1^*) \leq \pi_\gamma(p_\lambda^{**}(m_0), r_1^*)$, and $\pi = \min(\pi_\lambda, \pi_\gamma) = \pi_\lambda$. Therefore equation (36) can be written as:

$$\pi(p_1, r_1^*) - \pi_\lambda(p_\lambda^{**}(m_0), r_1^*) \geq \pi(p_1, r_1) - \pi(p_\lambda^{**}(m_0), r_1).$$

Combining with equation (35), we have: $\pi(p_1, r_1^*) - \pi_\lambda(p_\lambda^{**}(m_0), r_1^*) > \beta \Delta J$. Or equivalently:

$$\pi_\lambda(p_\lambda^{**}(m_0), r_1^*) + \beta J(m_0, p_\lambda^{**}(m_0)) < \pi(p_1, r_1^*) + \beta J(\min(p_1, m_0), p_1).$$

This contradicts the fact that $(m_0, p_\lambda^{**}(m_0))$ is a steady state of Problem (7). We conclude that $p_t \geq p_\lambda^{**}(m_0)$ for all t.

Claim 3. For $m_0 \in \mathbf{R}_2$, then $p_t \geq m_0$ for any t.

Proof: We show that $p_1 = p^*(m_0, p_0) \geq m_0$. By induction, this shows that $p_t^* \geq m_0$, $\forall t$. Suppose by contradiction, $p_1 < m_0$. Then, because $p_1 \leq \theta m_0 + (1-\theta) p_0 = r_1$, we have $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. Now, (7) can be written as:

$$J(m_0, p_0) = \max_{p_1 < m_0} \left\{ \pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) \right\}. \tag{37}$$

We show that, in this case, (m_0, m_0) cannot be a steady state of Problem (7), a contradiction. Because $p_1 < m_0$, we have $\pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) > \pi_\gamma(m_0, r_1) + \beta J(m_0, m_0)$. Equivalently, by defining $\Delta J = J(m_0, m_0) - J(p_1, p_1)$, we obtain:

$$\pi(p_1, r_1) - \pi(m_0, r_1) > \beta \Delta J. \tag{38}$$

Because $\pi_\gamma(p, r)$ is supermodular, and $r_1 > m_0$, it follows that:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(m_0, r_1) \leq \pi_\gamma(p_1, m_0) - \pi_\gamma(m_0, m_0). \tag{39}$$

Combining equations (39) and (38), we have: $\pi_\gamma(p_1, m_0) - \pi_0(m_0) > \beta \Delta J$, or equivalently: $\pi_0(m_0) - \beta J(m_0, m_0) < \pi_\gamma(p_1, m_0) + \beta J(p_1, p_1)$. This contradicts the fact that (m_0, m_0) is a steady state of Problem (7). We conclude if the initial state is such that $s \le m_0 \le S$, then $p_t \ge m_0$ for all t.

(b) From Proposition 1, we know that (S, S) is a steady state of Problem (7). We consider two cases: $p_0 = m_0$ and $p_0 > m_0$.

Assume $m_0 = p_0$. Consider the problem:

$$J^\phi(p_{t-1}, p_{t-1}) = \max_{p_t} \left\{ \pi(p_t, p_{t-1}) + \beta J^\phi(p_t, p_t) \right\}. \tag{40}$$

Because the value function in Problem (7) is increasing in its arguments (Lemma 1), it follows that: $J(m_{t-1}, p_{t-1}) \le J^\phi(p_{t-1}, p_{t-1})$. Equality happens if $m_{t-1} = p_{t-1}$ for all t, i.e. starting from $m_0 = p_0$, the price path is decreasing. By construction, the steady state of Problem (40) is the same as the steady state of the following problem:

$$\tilde{J}(p_{t-1}) = \max_{p_t} \left\{ \pi(p_t, p_{t-1}) + \beta \tilde{J}(p_t) \right\}, \tag{41}$$

which is S. Therefore (S, S) is the unique steady state of Problem (40). Because $\pi(p_t, p_{t-1})$ is supermodular, and starting at $p_0 > S$, the optimal price path of Problem (40) is decreasing and converges to S. Starting at an initial state (m_0, p_0) such that $p_0 = m_0 > S$, the optimal path of Problem (40) is feasible for Problem (7) and yields the same value. This is because at each stage $m_{t-1} = p_{t-1}$, which implies $\min(p_{t-1}, p_t) = p_t$ and $r_t = p_{t-1}$. Therefore for such initial states, this price path is optimal for Problem (7) and converges to S. This also implies $m_{t-1} \ge S$, as desired.

Now assume that $p_0 > m_0$. The following claim proves the desired result.

Claim 4. For $p_0 > m_0 > S$, if the optimal price p_t is such that $p_t \le m_0$, then $p_t \ge S$.

Proof: We show that if p_1 is such that $p_1 \le m_0$, then $p_1 > S$. By induction, this implies $p_t \ge S$.

Suppose by contradiction that $p_1 = p^*(m_0, p_0) < S$. Then $p_1 \le \theta m_0 + (1 - \theta)p_0 = r_1$, and hence $\min(\pi_\lambda, \pi_\gamma) = \pi_\gamma$. This allows to write (7) as:

$$J(m_0, p_0) = \max_{p_1} \left\{ \pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) \right\}.$$

Because $p_1 < S$, we have $\pi_\gamma(p_1, r_1) + \beta J(p_1, p_1) > \pi_\gamma(S, r_1) + \beta J(S, S)$. Equivalently, defining $\Delta J = J(S, S) - J(p_1, p_1)$, we have:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(S, r_1) > \beta \Delta J. \tag{42}$$

Because $\theta m_0 + (1 - \theta)p_0 = r_1 > S$ and $p_1 < S$, and $\pi_\gamma(p, r)$ is supermodular, it follows that:

$$\pi_\gamma(p_1, r_1) - \pi_\gamma(S, r_1) \le \pi_\gamma(p_1, S) - \pi_\gamma(S, S). \tag{43}$$

Combining equations (43) and (42), we have $\pi_\gamma(p_1, S) - \pi_\gamma(S, S) > \beta \Delta J$, or equivalently, $\pi_0(S) + \beta J(S, S) < \pi_\gamma(p_1, S) + \beta J(p_1, p_1)$. This implies (S, S) cannot be a steady state of Problem (7), a contradiction. We conclude that if the initial state (p_0, m_0) is such that $m_0 > S$, we have $p_t \geq S$.

Proof of Proposition 3: (a) Consider two possible cases: 1) $(m_0, p_0) \in \overline{\mathbf{R}}_{1a}$, and 2) $(m_0, p_0) \in \overline{\mathbf{R}}_{1b}$. The first case is proved in Proposition 2. For the second case the argument in Section 3.2, shows that the price path is monotonic. Moreover, because $p_0 \geq p_\lambda^{**}(m_0)$, it follows that, in this case, the price path is decreasing and converges to $p_\lambda^{**}(m_0)$.

(b) The argument in Section 3.2 shows that the price path in this region is monotonic. Because $p_0 \geq m_0$, the price path is decreasing to its steady state m_0.

(c) In the proof of Proposition 2b, we showed that for initial states $p_0 = m_0 \geq S$, the price path decreases monotonically to the steady state (S, S). Now, we focus on the case where the initial state is such that $p_0 > m_0$. The next claim ensures the price path eventually falls below m_0.

Claim 5. *Starting at (m_0, p_0), where $p_0 > m_0 \geq S$, at some point in time, T, the optimal price falls below m_0, i.e $p_T \leq m_0$.*

Proof: Suppose by contradiction that the optimal price path is such that $\{p_t\} > m_0$. Thus the value function in Problem (7), with m_0 as a parameter, can be written as:

$$J_{m_0}(p_0) = \max_{p_1} \left\{ \pi(p, r) + \beta J_{m_0}(p_1) \right\}.$$

The objective function is supermodular in (p, r) (Lemma 2), so the price path is monotonic, and converges to a steady state. By Proposition 1, this must be (S, S), which contradicts $p_t > m_0 \geq S$. We conclude that at some point in time, T, the optimal price is such that $p_T \leq m_0$. $\square$

Let T be the first time that the optimal price falls below m_0. Thus at time T, the value function in Problem (7) can be written as:

$$J(m_0, p_{T-1}) = \max_{p_T \leq m_0} \left\{ \pi(p, r) + \beta J(p_T, p_T) \right\}.$$

$p_T > S$ (Claim 4). For an initial steady state (m_0, p_0) such that $p_0 = m_0 > S$, the price path decreases monotonically to S, and (S, S) is the corresponding steady state. The value function for $t < T$ is given by the finite horizon model:

$$J_{t-1}(p_{t-1}) = \max_{p_t} \left\{ \pi(p, r) + \beta J_t(p_t) \right\}, \quad t < T,$$

where $J_T(p_T) = J(m_0, p_T)$. Because the objective function is supermodular in (p, r), the price path is monotonic, and decreases to p_{T-1}. Note that it cannot be increasing, because then it would have

to converge to a steady state above m_0, which is impossible (Proposition 1). In summary, starting at an initial state (m_0, p_0) such that $p_0 > m_0 > S$, the price path decreases until it falls below m_0 and then converges decreasingly to S.

Proof of Proposition 4: $\pi(p, r)$ is supermodular in (p, r) by Lemma 2, and $r = r(\theta) = p + \theta(m - p)$ is decreasing in θ for $m \leq p$. Therefore π is submodular in (p, θ) and $p^*(p, m; \theta)$ is decreasing in θ. Moreover, because π is increasing in r, and r is decreasing in θ, we conclude that the value function, $J(m, p; \theta)$, is decreasing in θ (Stokey and Lucas, Theorem 4.7).

Proof of Proposition 5: First, we prove that conditions (1) and (2) are sufficient for supermodularity of the short-term profit $\pi(p, r^1, \cdots, r^N)$ in $(p, r^1, \cdots, r^N)$.

(1) If $\lambda_i \geq \gamma_i$, $\forall i \in \mathbf{I}$, then the short-term profit function in (17) can be written as: $\pi(p, r^1, \cdots, r^N) = \sum_{i \in \mathbf{I}} \min(\pi_\lambda^i(p, r^i), \pi_\gamma^i(p, r^i))$. Because $\lambda_i \geq \gamma_i$, Lemma 2 implies that $\min(\pi_\lambda^i, \pi_\gamma^i)$ is supermodular in (p, r^i), and hence $\pi(p, r^1, \cdots, r^N)$ is supermodular in $(p, r^1, \cdots, r^N)$.

(2) If $\theta_i = \theta$, and $\sum_i \lambda_i \geq \sum_i \gamma_i$, then at any time t, all the segments have the same reference price $r_t = \theta m_{t-1} + (1 - \theta) p_{t-1}$. This helps to write the short-term profit function simply as $\pi(p, r) = \min(\pi_\lambda, \pi_\gamma)$. π_λ and π_γ are the sum of supermodular functions, and so are supermodular. Supermodularity of $\pi(p, r)$ in (p, r) then follows by verifying Lemma 2.

The upper bound problem that helps identify the steady states of Problem (17) is:

$$J_m^\nu(p_{t-1}) = \max_{p_t \in \mathbf{P}} \left\{ (1 - \nu) \sum_i \pi_\lambda^i(p_t, \theta_i m + (1 - \theta_i) p_t) + \nu \sum_i \pi_\gamma(p_t, p_{t-1}) + \beta J_m^\nu(p_t) \right\}. \qquad (44)$$

Problem (44) is a smooth problem, and its steady state solves the Euler equation:

$$\pi_0'(p) - (1 - \nu) p \sum_i \lambda_i (2 - (1 - \theta_i)(1 + \beta)) - \nu p (1 - \beta) \sum_i \gamma_i + (1 - \nu) m \sum_i \lambda_i \theta_i = 0. \qquad (45)$$

s^H and S^H, as in equations (18-19), correspond to those values of m for which the steady state of Problem (44) equals m, for $\nu = 0$, respectively, $\nu = 1$. Specifically, for $\nu = 0$, the steady state of Problem (44) solves equation (20). Substituting $p = m$ in this equation, we retrieve equation (18).

The identification of the steady states of Problem (17) follows the same lines as Proposition 1. Moreover, a proof similar to that of Proposition 2 shows that starting at a state in regions $\overline{\mathbf{R}}_1$ and $\overline{\mathbf{R}}_2$, the minimum price does not change over time. Supermodularity of the short-term profit, proved above, is again critical for the proofs, which follow the same steps as for homogeneous buyers. Claim 4 also extends to conclude that for $m_0 \in \overline{\mathbf{R}}_3$, $p_t \geq S^H$. Finally, monotonicity of the price paths follows as Proposition 3 extends to heterogeneous markets under both conditions specified in the proposition.

Proof of Lemma 6: The proof uses the following intermediary result:

Claim 6. *Let $k \in \{\lambda, \gamma\}$. (a) There exists a unique fixed point r_k of $p_k(\cdot)$, i.e. $r_k = p_k(r_k)$, and r_k solves $\pi_0'(r_k) = k \, r_k$. Moreover, $r_\lambda \leq r_\gamma$. (b) For $r \leq r_k$, we have $r \leq p_k(r) \leq r_k$, and for $r \geq r_k$, we have $r \geq p_k(r) \geq r_k$.*

Proof: (a) By definition, $p_\lambda(r)$ solves the first order condition:

$$\frac{\partial \pi_\lambda(p, r)}{\partial p} = \pi_0'(p) - 2\lambda p + \lambda r = 0.$$

The equation $p_\lambda(r) = r$ results in:

$$\pi_0'(r) - \lambda r = 0. \tag{46}$$

The LHS of (46) is strictly decreasing in r, strictly positive for $r = 0$ and negative for a sufficiently large $r \in \mathbf{P}$. Therefore the above equation has a unique solution, r_λ.

The same argument guarantees the existence of r_γ, the unique solution of the equation:

$$\pi_0'(r_\gamma) - \gamma r_\gamma = 0. \tag{47}$$

Because r_λ and r_γ solve (46) and (47) respectively, and $\lambda \geq \gamma$, if follows that $r_\lambda \leq r_\gamma$.

(b) The result follows because $p_\gamma(0) \geq p_\lambda(0) > 0$ and $p_\lambda(\cdot)$ and $p_\gamma(\cdot)$ are increasing functions of r, and thus single cross the identity line from above. These crossing points exist and are unique as shown in part (a) above. $\square$

We now proceed with the proof of Lemma 6. First observe that $p_M(r) \in \{p_\lambda(r), p_\gamma(r), r\}$. By (5), $p_\lambda(r)$ is only feasible for $r \leq p_\lambda(r)$, i.e. $r \leq r_\lambda$ (by Claim 6b). Also $p_\gamma(r)$ is only feasible for $r \geq p_\gamma(r)$, i.e. $r \geq r_\gamma$ (by Claim 6b). Hence $p_\lambda(\cdot)$ is optimal for $r \leq r_\lambda$ because $\pi = \pi_\lambda$ and $p_\gamma(\cdot)$ is optimal for $r \geq r_\gamma$ because $\pi = \pi_\gamma$. For $r_\lambda \leq r \leq r_\gamma$, $p^M(r) = r$.

Proof of Proposition 6: The value function is increasing, which implies:

$$p^*(m, p) = \arg \max_{p \in P} \{\pi(p, r) + \beta J(\min(m, p), p)\} \geq \arg \max_{p \in P} \pi(p, r) = p^M(r).$$

Forward induction proves the other part. Starting at the same initial state, r_1, we have $p_1^* = p^*(r_1) \geq p_1^M(r_1)$. This implies that $m_1^* \geq m_1^M$, i.e. the minimum price for the strategic firm in the next period is larger than the one for the myopic firm. Now as the induction assumption, suppose that $p_{t-1}^* \geq p_{t-1}^M$ and $m_{t-1}^* \geq m_{t-1}^M$. This implies $\min(p_{t-1}^*, m_{t-1}^*) \geq \min(p_{t-1}^M, m_{t-1}^M)$. Then the reference price in period t is such that $r_{t-1} \geq r_{t-1}^M$. By Claim 6, $p^M(r)$ is increasing in r, and thus $p_t^M(r_t^M) \leq p_t^M(r_t^*) \leq p^*(r_t^*)$. Therefore, the myopic policy underprices the product.

Moreover, because π is supermodular (Lemma 2), it follows that all the price paths are monotonic and converge to a constant price, which depends on the initial state.

Proof of Remark 1: (a) The proof follows by verifying the Assumptions 1, 2, and 3. Convexity of $f(x)$ and part (i) insure the single crossing property (Assumption 2a) over the loss domain. Part (iii) ensures the supermodularity of π^L over the entire domain (Assumption 3b). The Other assumptions are easily verified.

(b) The proof follows by verifying the Assumptions 1, 2, and 3. Supermodularity of $g(x)$ is immediate because g is increasing in r and $-g'(x) + (1-x)g''(x) \leq 0$. Part (iii) ensures the supermodularity of π^L over the entire domain. Further, this assumption guarantees that Assumptions 3c and 3d hold. Others are also simply verified. $\square$

Proof of Lemma 7: For $p^l \leq p^h$ and $r^l \leq r^h$ we show that:

$$\pi^K(p^h, r^h) - \pi^K(p^l, r^h) \geq \pi^K(p^h, r^l) - \pi^K(p^l, r^l). \tag{48}$$

Similar to the case of linear reference effects (Lemma 2), we consider the following exhaustive cases: (1) $p^l \leq p^h \leq r^l \leq r^h$, (2) $p^l \leq r^l \leq p^h \leq r^h$, (3) $p^l \leq r^l \leq r^h \leq p^h$, (4) $r^l \leq p^l \leq p^h \leq r^h$, (5) $r^l \leq p^l \leq r^h \leq p^h$, (6) $r^l \leq r^h \leq p^l \leq p^h$.

Cases (1) and (6) follow because all (p, r) fall on either loss or gain domains, and π^G and π^L are supermodular (Assumption 3b).

Case 2: For $p^l \leq r^l \leq p^h \leq r^h$, (48) becomes:

$$p^h h^G(r^h - p^h, r^h) - p^l h^G(r^h - p^l, r^h) \geq p^h h^L(r^l - p^h, r^l) - p^l h^G(r^l - p^l, r^l).$$

Because $h^L(r^l - p^h, r^l) < h^G(r^l - p^h, r^l)$, a sufficient condition for the above inequality to hold is:

$$p^h h^G(r^h - p^h, r^h) - p^l h^G(r^h - p^l, r^h) \geq p^h h^G(r^l - p^h, r^l) - p^l h^G(r^l - p^l, r^l),$$

which follows because π^G is supermodular in (p, r).

Case 3: For $p^l \leq r^l \leq r^h \leq p^h$, (48) becomes:

$$p^h h^L(r^h - p^h, r^h) - p^l h^G(r^h - p^l, r^h) \geq p^h h^L(r^l - p^h, r^l) - p^l h^G(r^l - p^l, r^l). \tag{49}$$

Supermodularity of $\pi^L(p, r) = p h^L(r - p, r)$ implies:

$$p^h[h^L(r^h - p^h, r^h) - h^L(r^l - p^h, r^l)] \geq p^l[h^L(r^h - p^l, r^h) - h^L(r^l - p^l, r^l)]. \tag{50}$$

On the other hand, Assumption 2c implies that

$$h^L(r^h - p^l, r^h) - h^L(r^l - p^l, r^l) \geq h^G(r^h - p^l, r^h) - h^G(r^l - p^l, r^l), \tag{51}$$

which together with (50) yield the desired result (49).

Case 4: For $r^l \leq p^l \leq p^h \leq r^h$, (48) becomes:

$$p^h h^G(r^h - p^h, r^h) - p^l h^G(r^h - p^l, r^h) \geq p^h h^L(r^l - p^h, r^l) - p^l h^L(r^l - p^l, r^l). \tag{52}$$

Supermodularity of $h^G(r - p, r)$ (Assumption 2e) implies that:

$$h^G(r^h - p^h, r^h) - h^G(r^h - p^l, r^h) \geq h^G(r^l - p^h, r^l) - h^G(r^l - p^l, r^l).$$

On the other hand, Assumption 2b implies

$$h^G(r^l - p^h, r^l) - h^G(r^l - p^l, r^l) \geq h^L(r^l - p^h, r^l) - h^L(r^l - p^l, r^l).$$

Putting these two together, we obtain:

$$h^G(r^h - p^h, r^h) - h^G(r^h - p^l, r^h) \geq h^L(r^l - p^h, r^l) - h^L(r^l - p^l, r^l).$$

Because $h^G(r^h - p^h, r^h) \geq 0$ and $h^L(r^l - p^h, r^l) < 0$, we obtain the desired result (52).

Case 5: This follows the same approach as Case (2), driven by supermodularity of π^L.

Proof of Lemma 8: Assumptions 1c and 3e insure that (25) and (26) have unique solutions. This is because the LHS of both equations are strictly decreasing, positive at $p = 0$ and negative for a high enough $p \in \mathbf{P}$. Moreover, the solutions of these equations are such that $s \leq S$ (because $1 - \beta(1 - \theta) \geq 1 - \beta$), and $h_1^L(0, p) \geq h_1^G(0, p)$.

(a) By definition (27), p_L^{**} solves (24) for $\nu = 0$. This has a unique solution because the LHS of (27) is strictly decreasing in p. The derivative of LHS with respect to p is:

$$\pi_{11}^L + (1 - \theta)\pi_{12}^L + \beta(1 - \theta)(\pi_{12}^L + (1 - \theta)\pi_{22}^L), \tag{53}$$

which is negative because of Assumption 3c. Moreover the LHS of (27) is positive for $p = 0$, and negative for a high enough $p \in \mathbf{P}$, so this equation has a unique solution.

(b) Substituting $p = m$ in (24), we obtain:

$$\pi_0'(m) - [1 - \beta(1 - \theta(1 - \nu))] \, m \, h_1(0, m) = 0. \tag{54}$$

The LHS of (54) is strictly decreasing in m (Assumptions 3e). For $\nu = 0$, $m = s$ solves (54). It follows that for $m > s$ and $\nu = 0$ the LHS is negative. On the other hand, for $\nu = 1$, $m = S$ solves (54). Therefore, for $m < S$ and $\nu = 1$, the LHS is positive. The result follows because the LHS is strictly increasing in ν, negative at $\nu = 0$ and positive at $\nu = 1$.

(c) The proof is similar to that of Lemma 5.

Proof of Proposition 7: The proof follows the same lines as Proposition 1. For the same alternative price paths, equations (33) and (34) become $\pi'_0(m) \leq (1 - \beta(1 - \theta))mh_1^L(0, m)$, respectively $\pi'_0(m) \geq (1 - \beta)mh_1^G(0, m)$. The same reasoning as in Proposition 1 shows that steady states of the form $(m_0, p_L^{**}(m_0))$, respectively (m_0, m_0), are relevant only if $m_0 \in \mathbf{R}_1$, respectively $m_0 \in \mathbf{R}_2$. Moreover, an analogous proof to that of Lemma 5 shows that these are steady states of Problem (23). Feasibility of the constant pricing policy, $p_L^{**}(m)$ for $m \in \mathbf{R}_1$ follows because $p_L^{**}(m)$ is increasing in m. To see this, write:

$$\frac{dp_L^{**}(m)}{dm} = -\frac{\theta\pi_{12}^L + \beta\theta(1 - \theta)\pi_{22}^L}{\pi_{11}^L + (1 - \theta)(1 + \beta)\pi_{12}^L + \beta(1 - \theta)^2\pi_{22}^L}.$$

This is non-negative because $\pi_{11}^L + 2(1 - \theta)\pi_{12}^L + (1 - \theta)^2\pi_{22}^L < 0$ (Assumption 3c), $\theta\pi_{12}^L + \theta(1 - \theta)\pi_{22}^L \geq 0$ (Assumption 3d), and $0 \leq \beta \leq 1$. Moreover, p_L^{**} single crosses the identity line from above, because at $m = 0$ the LHS of (27) is positive. Indeed, the crossing point solves (25), i.e. it is given by s, as defined in Section 6. Moreover, at $m = 0$, equation (27) has a unique positive solution. Therefore, for all $m \in \mathbf{R}_1$, we have $p_L^{**}(m) \geq m$.

The proof for the price path is analogous to that of Proposition 3. □

Proof of Proposition 8: For any $\phi \in [0, 1]$, consider the problem:

$$V(p_{t-\tau}, \cdots, p_{t-1}) = \max_{p_t \in \mathbf{P}}\{\phi\pi_\lambda(p_t, \theta p_{t-\tau} + (1 - \theta)p_{t-1}) + (1 - \phi)\pi_\gamma(p_t, p_{t-1}) + \beta V(p_{t-\tau+1}, \cdots, p_t)\}. \quad (55)$$

Problem (55) is a upper bound for Problem (28), because $p_{t-1}, p_{t-\tau} \geq \min(p_{t-\tau}, \cdots, p_{t-1})$, and demand (hence the profit function) is increasing in the reference price. Problem (55) is a smooth problem, and admits a unique steady state, $p \cdot \mathbf{1}_\tau$, where $\mathbf{1}_\tau$ is the τ-vector of ones, and p solves the following Euler equation:

$$\pi'_0(p) - \phi\lambda(1 - \beta(1 - \theta(1 - \beta^{\tau-1})))p - (1 - \phi)\gamma(1 - \beta)p = 0. \quad (56)$$

To see this, denote $\mathbf{s}_t = (p_{t-\tau}, \cdots, p_{t-1})$, the state vector at time t, and the transition function $\mathbf{s}_{t+1} = g(\mathbf{s}_t, p_t) = (p_{t-\tau+1}, \cdots, p_t)$. Further, let $\pi^\phi(\mathbf{s_t}, p_t)$ denote the one-stage profit function in Problem (55). The steady state of Problem (55), if it exists, solves the equilibrium conditions (see Chapter 4 in Stokey and Lucas 1989):

$$\pi_p^\phi(\mathbf{s}, p) + \beta\mathbf{u}g_p(\mathbf{s}, p) = 0, \quad (57)$$

$$\mathbf{u} = \pi_\mathbf{s}^\phi(\mathbf{s}, p) + \beta\mathbf{u}g_\mathbf{s}(\mathbf{s}, p), \quad (58)$$

$$\mathbf{s} = g(\mathbf{s}, p), \quad (59)$$

where $g_p = (0, \cdots, 1)'_\tau$, $g_{\mathbf{s}} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(\tau-1)\times(\tau-1)} \\ \hline 0 & \mathbf{0} \end{bmatrix}_{\tau\times\tau}$, and $\mathbf{u} = (u_\tau, \cdots, u_1)$ is the vector of shadow prices.

From (59), $p_{t-\tau} = \cdots = p_t = p$, so steady states must be of the form $p \cdot \mathbf{1}_\tau$. Using this in (58),

$$u_\tau = \phi\lambda\theta p, \quad u_{\tau-i} = \beta u_{\tau-i+1}, \, i = 1,\ldots,\tau-2, \quad u_1 = \phi\lambda(1-\theta)p + (1-\phi)\gamma p + \beta u_2,$$

which implies $u_1 = \phi\lambda(1-\theta)p + (1-\phi)\gamma p + \beta^{\tau-1}\phi\lambda\theta p$. Finally, from (57), it follows that the steady state price p must indeed solve (56).

It is straightforward to observe that for any $p \in [s(\theta;\tau), S]$, there is a $\phi \in [0,1]$ such that p is the solution to equation (56). To prove the steady state structure of Problem (28), first observe that for all $p \in [s(\theta;\tau), S]$, the constant pricing path starting at $p \cdot \mathbf{1}_\tau$, is optimal for Problem (55), and feasible for Problem (28), hence also optimal for Problem (28). We conclude that for all $p \in [s(\theta;\tau), S]$, $p \cdot \mathbf{1}_\tau$ is a steady state of Problem (28).

It remains to prove that Problem (28) cannot admit other steady state prices p.

First assume that there is a steady state price p of Problem (28) such that $p > S$. Consider an alternative price path $p_t = p - \delta$, $\forall t$. For p to be a steady state price, it must be that:

$$\frac{\pi_0(p)}{1-\beta} \geq \pi_0(p-\delta) + \gamma(p-\delta)\delta + \frac{\beta}{1-\beta}\pi_0(p-\delta). \tag{60}$$

Note that as soon as a price $p - \delta < p$ is charged, the minimum price becomes $p - \delta$, and stays there for the rest of the horizon. From equation (60), it follows that $\pi'_0(p) \geq \gamma(1-\beta)p$, or $p \leq S$, a contradiction. Thus Problem (28) cannot have a steady state $p > S$.

Now assume that Problem (28) admits a steady state price $p < s(\theta;\tau)$. To obtain a contradiction, consider an alternative price path $p_t = p + \delta$, $\forall t$. For p to be a steady state price, it must be that:

$$\frac{\pi_0(p)}{1-\beta} \geq \pi_0(p+\delta) - \lambda(p+\delta)\delta + (\beta + \cdots + \beta^{\tau-1})\pi_\lambda(p+\delta, \theta p + (1-\theta)(p+\delta)) + \frac{\beta^\tau}{1-\beta}\pi_0(p+\delta) \tag{61}$$

where $\pi_\lambda(p+\delta, \theta p + (1-\theta)(p+\delta)) = \pi_0(p+\delta) - \lambda(p+\delta)\delta\theta$. Note that for τ periods the minimum price remains p, and starting from period $\tau + 1$, the minimum price is updated to $p + \delta$, and stays there for the rest of the horizon. Simplifying (61), dividing by δ and letting $\delta \to 0$, we have:

$$\pi'_0(p) \leq \lambda(1 - \beta(1 - \theta(1-\beta^\tau)))p. \tag{62}$$

Equation (62) implies that $p \geq s(\theta, \tau)$, contradicting our initial assumption. We conclude that the only steady state prices of Problem (28) are $[s(\theta;\tau), S]$.

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