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A Theorem of the Maximin and Applications to Bayesian Zero-Sum Games

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Consider a family of zero-sum games indexed by a parameter that determines each player's payoff function and feasible strategies. Our first main result characterizes continuity assumptions on the payoffs and the constraint correspondence such that the equilibrium value and strategies depend continuously and upper hemicontinuously (respectively) on the parameter. This characterization uses two topologies in order to overcome a topological tension that arises when players' strategy sets are infinite-dimensional. Our second main result is an application to Bayesian zero-sum games in which each player's information is viewed as a parameter. We model each player's information as a sub-σ-field, so that it determines her feasible strategies: those that are measurable with respect to the player's information. We thereby characterize conditions under which the equilibrium value and strategies depend continuously and upper hemicontinuously (respectively) on each player's information. This clarifies and extends related results of Einy et al. (2008).

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1. Introduction

We study zero-sum Bayesian games and characterize conditions under which the equilibrium value and strategies are a continuous function and an upper hemicontinuous correspondence (respectively) of the information structure.

We do this for an ex ante formulation of Bayesian games and Bayesian Nash equilibria. Uncertainty is represented by a fixed probability space, and information structures are represented by partitions or sub-σ-fields; this fits the interpretation of a Bayesian game as a game of imperfect information about moves by nature. In a normal-form reduction of this game, a player's strategy set is the set of strategies that are measurable with respect to his information; changing the information structure changes the strategy sets but not the payoffs.

We frame the exercise as an application of an analogue of the Theorem of the Maximum for zero-sum games.

- Recall that the Theorem of the *Maximum* provides conditions under which, for a single-person decision problem, the value and the solution to the problem are a continuous function and an upper hemicontinuous correspondence (respec-

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tively) of a parameter that affects the feasible set and the objective function.

- In our Theorem of the *Maximin*, we provide conditions under which, for a zero-sum game, the value and the equilibrium strategies are a continuous function and an upper hemicontinuous correspondence (respectively) of a parameter that affects the players' feasible strategy sets and the players' payoffs.

In the application to Bayesian zero-sum games, the parameter is the players' sub-σ-fields, which affect the players' feasible (measurable) strategies.

This paper was motivated by Einy et al. (2008), who use a direct approach to study continuity of the value with respect to the information structure for Bayesian zero-sum games. Our approach, which decomposes the problem into (a) continuity of the measurability constraint and (b) an abstract "Theorem of the Maximin", simplifies and clarifies the analysis. In addition, the Theorem of the Maximin is of independent interest beyond applications to the value of information. An unexpected benefit of our approach is that we have generalized most of the results in Einy et al. (2008).

We are interested in the case in which there are infinitely many states or types in the Bayesian game—which implies that the strategy sets are infinite-dimensional. This creates a tension for our Theorem of the Maximin between (a) the continuity assumptions, which benefit from a *strong* topology on the strategies, and (b) the compactness assumptions, which benefit from a *weak* topology on the strategies. The same topological tension arises in the single-person Theorem of the Maximum. Such tension is relaxed in Horsley, Van Zandt, and Wrobel (1998) by working with two topologies: a weak topology is used for the compactness, upper semicontinuity, and upper hemicontinuity assumptions; a strong topology is used for lower semicontinuity and lower hemicontinuity assumptions. We follow the same approach in this paper for our Theorem of the Maximin.

Continuity of actions and payoffs with respect to information, in non-zero-sum Bayesian games, has been studied in, for example, Milgrom and Weber (1985), Cotter (1994), Monderer and Samet (1996), and Kajii and Morris (1997, 1998). As in this paper, Milgrom and Weber (1985) and Cotter (1994) both consider upper hemicontinuity—loosely, equilibria should not suddenly disappear as a parameter approaches a limit. Milgrom and Weber (1985) models shifts in information by shifts in the common prior, whereas Cotter (1994) models shifts in information by shifts in sub-σ-fields (as in this paper) but considers correlated equilibria. In Section 11, we explain why it is difficult to obtain positive results on upper hemicontinuity for non-correlated equilibria of non-zero-sum games.

The other three papers consider lower hemicontinuity—loosely, equilibria should not suddenly appear as a parameter approaches a limit—of an approximate equilibrium correspondence, and are thus less related to the current paper. For example, an approximate equilibrium may be defined by "best responses" that have to achieve within ε of the highest possible payoff. (Lower hemicontinuity of the actual equilibrium correspondence generally cannot be expected to hold.) Monderer and Samet (1996) study this problem treating information as sub-σ-fields, as in the current paper; Kajii and Morris (1997, 1998) treat variations in information as variations in the common prior, as in Milgrom and Weber (1985).

2. An interlude on terminology

This paper is very much about continuity of real-valued functions and set-valued correspondences, divided into lower and upper semicontinuity for functions and into lower and upper hemicontinuity for correspondences.

Lower (resp., upper) semicontinuity of a function $f\colon X \to \mathbb{R}$ means that lower (resp., upper) contour sets—that is, sets of the form $\{x \in X \mid f(x) \leq \alpha\}$—are closed. We abbreviate lower semicontinuous and upper semicontinuous by lsc and usc, respectively.

Appendix A contains a brief summary of the definitions of lower hemicontinuous (lhc) and upper hemicontinuous (uhc) correspondences; this is provided for the convenience of the reader and because there are small variations in the literature regarding their definitions. The property of upper hemicontinuity will rarely interest us by itself; we are interested in the combination of being uhc and having compact values. We use the abbreviation uhc* for this combination.

As is common with an exercise such as this, we do not work directly with the raw definition of upper hemicontinuity. Instead we work with the simpler property of having a closed graph, which we use as an adjective as in "φ is closed". We are able to go back and forth between uhc* and closed because (a) an uhc* correspondence is closed and (b) conversely, a closed correspondence is uhc* if either it takes values in a compact set or (more generally) it is a subset of another uhc* correspondence. The reason for retaining the property uhc* is that the composition of two uhc* correspondences is uhc*, whereas the weaker condition of closedness is not preserved by compositions.

3. Road map

A strictly linear presentation of our analysis would run as follows.

1. Define parameterized zero-sum games and prove a two-topology Theorem of the Maximin.
2. Define our ex ante formulation of zero-sum Bayesian games, in which information sub-σ-fields are parameters, and apply the Theorem of the Maximin to conclude that the value depend continuously on the information.

However, in order to clarify the role of the two topologies in this approach, we go back and forth between these two steps. Overall, this paper has the following structure.

Section 4. We define parameterized zero-sum games and state a one-topology Theorem of the Maximin.

Section 5. We define our formulation of Bayesian games and illustrate the topological tension that makes it impossible to apply the one-topology Theorem of the Maximin.

Section 6. We explain how the same topological tension arises in single-person decision problems but can be resolved by a two-topology Theorem of the Maximum.

Section 7. We then prove a two-topology Theorem of the Maximin by iterative application of the two-topology Theorem of the Maximum.

Section 8. We can then apply this Theorem of the Maximin to the Bayesian zero-sum games to obtain the main continuity results.

Section 9 takes up a variation in which the information constraint is viewed as a function into the set of subsets of measurable functions endowed with the Hausdorff metric. This has various advantages and disadvantages, which are specifically enumerated. We then proceed linearly: a theorem for single-person decision problems; a theorem for two-player zero-sum games; application to Bayesian games with incomplete information.

We conclude with two observations: Section 10 compares our results to those of Einy et al. (2008), and Section 11 explains why the methods can only be used for zero-sum games.

4. Parameterized zero-sum games

Consider a two-player zero-sum game in which a parameter $p \in P$ affects the players' feasible strategy sets and their payoffs. Denote the players by 1 and 2. Let X_i be the set of player i's potential strategies and let $\varphi_i(p) \subset X_i$ be player i's nonempty set of feasible strategies given p. Let $u\colon X_1 \times X_2 \times P \to \mathbb{R}$ be player 1's p-dependent payoff function. Since this is a zero-sum game, player 2's payoff function is $-u$.

The value $v_1 : P \to \mathbb{R}$ and the solution $\psi_1 : P \twoheadrightarrow X_1$ for player 1 are defined by

$$v_1(p) = \sup_{x_1 \in \varphi_1(p)} \inf_{x_2 \in \varphi_2(p)} u(x_1, x_2, p), \tag{1}$$

$$\psi_1(p) = \operatorname*{arg\,max}_{x_1 \in \varphi_1(p)} \inf_{x_2 \in \varphi_2(p)} u(x_1, x_2, p). \tag{2}$$

A main goal of this paper is to establish conditions under which v_1 is continuous and ψ_1 is uhc.

One of our main results will be a generalization of the following proposition.

PROPOSITION 1 (Theorem of the Maximin). *Assume that u is continuous and that φ_1 and φ_2 are lhc and uhc*. Then v_1 is continuous and ψ_1 is uhc*.*

Proposition 1 is an analogue of and a corollary to Berge's Theorem of the Maximum (Berge, 1963); based on this parallel, we call it a "Theorem of the Maximin". However, it is not adequate for the application to Bayesian games that motivates this paper, as we explain in Section 5.

REMARK 1. The value v_2 and the solution ψ_2 for player 2 are defined by equations analogous to (1) and (2). The Minimax Theorem provides conditions under which (a) $v_1(p) = -v_2(p)$, (b) ψ_1 and ψ_2 are nonempty, and (c) the set of Nash equilibria of the game is $\psi_1(p) \times \psi_2(p)$. (One example is Sion, 1958, Thm. 3.4.) These three properties are considered fundamental for the value and solution to be meaningful—in the sense that they represent equilibrium payoffs and strategies—and hence these properties are fundamental for the results in this paper being of interest. However, they do not play a direct role in our analysis.

5. Imperfect-information formulation of a Bayesian game

5.1. Setup

Consider a two-player zero-sum Bayesian game with an ex ante formulation in which (a) payoff uncertainty with a common prior is represented by a probability space (Ω, Σ, μ) and (b) player i's information is represented by a sub-σ-field $\mathscr{F}_i$ of Σ. Denote player i's set of actions (or mixed strategies over actions) by A_i. We endow A_i with a metric d_i with respect to which we define continuity and measurability of functions. Denote player 1's state-dependent utility by $w\colon A_1 \times A_2 \times \Omega \to \mathbb{R}$. Since this is a zero-sum game, player 2's state-dependent utility is $-w$.

In the normal form of this game, player i chooses an $\mathscr{F}_i$-measurable strategy $x_i : \Omega \to A_i$. Given a strategy profile (x_1, x_2), player 1's expected payoff is

$$u(x_1, x_2) = \int_\Omega w(x_1(\omega), x_2(\omega), \omega)\, d\mu(\omega) \tag{3}$$

and player 2's expected payoff is $-u(x_1, x_2)$. So that this integral will be well-defined, we maintain the following assumption throughout.

ASSUMPTION 1. The function w is a Carathéodory function—that is, continuous in (a_1, a_2) and measurable in ω. Its absolute value is bounded by an integrable function of ω.

We treat each player's information as a parameter that restricts her set of feasible strategies. Let $\mathfrak{F}$ be the set of sub-σ-fields of Σ. Let X_i be player i's potential strategies (i.e., the set of Σ-measurable functions from Ω to A_i). Let $\varphi_i(\mathscr{F}_i) \subset X_i$ be the set of $\mathscr{F}_i$-measurable strategies given $\mathscr{F}_i \in \mathfrak{F}$.

We have transformed this game into the class of parameterized zero-sum games outlined in Section 4 by defining:

1. a parameter set $P = \mathfrak{F} \times \mathfrak{F}$;
2. potential strategy sets X_i;
3. a payoff function $u\colon X_1 \times X_2 \to \mathbb{R}$; and
4. constraint correspondences $\varphi_i : \mathfrak{F} \to X_i$.

We can thus attempt to apply Proposition 1 to determine conditions under which equilibrium payoffs and strategies depend continuously on the information structure.

5.2. Topological concerns: Metric spaces of action

Determining these conditions requires that we define a topology on information in order to give meaning to the continuity conditions in the assumptions and the conclusions, but we defer this to Section 8. In the rest of this section, we examine the following two assumptions that would be part of applying Proposition 1:

1. X_i is compact;
2. u is continuous.

We explain why these two assumptions are difficult to satisfy simultaneously, making Proposition 1 of limited interest for Bayesian games.

One topology on X_i that we work with is the topology of convergence in measure. It has the following metric:

$$\eta_i(x_i, x_i') \equiv \inf\{\varepsilon > 0 \mid \mu\{\omega \in \Omega \mid d_i(x_i(\omega), x_i'(\omega)) > \varepsilon\} < \varepsilon\}.$$

We first state a positive result on the continuity of u. Recall that, throughout this paper: (a) A_i is assumed to be a metric space with respect to which the metrizable topology on X_i of convergence in measure is defined; and (b) Assumption 1 is maintained.

PROPOSITION 2. *The payoff function u is continuous with respect to the topology of convergence in measure. If A_1 and A_2 are compact, then u is uniformly continuous.*

Proof. See Proposition C.1 in Appendix C. That result shows (uniform) continuity viewing $X_1 \times X_2$ as the set of measurable function into $A_1 \times A_2$, endowed with the (a) the metric of convergence in measure. Proposition 2 is about (uniform) continuity endowing $X_1 \times X_2$ with (b) the metric $\max\{\eta_1(x_1, x_1'), \eta_2(x_2, x_2')\}$ of the product topology when each X_i is endowed with the metric η_i of convergence in measure. These two sets of results are equivalent because the metrics (a) and (b) are uniformly equivalent. □

However, the topology of convergence in measure is too strong for X_i to be compact if (Ω, Σ, μ) is nonatomic and A_i contains at least two elements. This is illustrated in the following well-known example.

EXAMPLE 1. Suppose that (Ω, Σ, μ) is the usual unit interval with the Lebesgue measure and denote two of the elements of A_i by 0 and 1. For $n \in \mathbb{N}$, define x_{in} by dividing $[0,1]$ into $2n$ equal-length intervals and then setting $x_{in}(\omega)$ to 0 on the odd intervals and to 1 on the even intervals (e.g., $x_{in}(\omega) = 0$ for $\omega \in [0, 1/(2n))$ and $x_{in}(\omega) = 1$ for $\omega \in [1/(2n), 2/(2n))$, etc.). Any two elements of this sequence coincide on a set of measure $1/2$ and differ on its complement; therefore, any two elements are the same distance apart in the metric for the topology of convergence in measure. The sequence $\{x_{in}\}$ thus has no convergent subsequence.

5.3. Topological concerns: Euclidean spaces of action

In order to understand more precisely the conflict between the compactness and continuity assumptions, suppose in this subsection that A_i is a compact convex subset of $\mathbb{R}^n$ containing more than one element. Then X_i is a convex subset of $L_1(\Omega, \Sigma, \mu; \mathbb{R}^n)$ or of $L_\infty(\Omega, \Sigma, \mu; \mathbb{R}^n)$. There are two salient topologies on X_i:

(a) the *weak* topology $\sigma(L_1, L_\infty)$, which coincides with the weak* topology $\sigma(L_1, L_\infty)$;
(b) the *norm* topology $\|\cdot\|_1$, which coincides with the Mackey topology $\tau(L_\infty, L_1)$ and the topology of convergence in measure. (Furthermore, the L_1-norm and the metric of convergence in measure are uniformly equivalent on X_i.)

We know from Section 5.2 that, in the norm topology, u is uniformly continuous but X_i is not compact. Switching to the weak topology makes X_i compact but introduces another problem: u is no longer continuous. More specifically, we have the following.

1. The set X_i is weakly compact but not norm compact (Example 1).
2. On the other hand, suppose that (Ω, Σ, μ) is nonatomic, w is strictly concave in a_1, and A_1 contains more than one element. Then $u(\cdot, x_2)$, as a function of x_1, is not weakly lsc (Bewley, 1972, p. 529; Balder and Yannelis, 1993, Thm 2.6).
3. If w were linear in a_1 and in a_2—an assumption that could be justified by letting A_i be a set of mixed strategies—then u would be weakly continuous in x_1 and x_2 independently. However, u would still fail joint weak continuity in (x_1, x_2).

Example 2 illustrate this third point and reminds us how weak the weak topology is.

EXAMPLE 2. Extend Example 1 by assuming that $A_1 = A_2 = [0,1]$. Suppose that each player has two pure actions, high and low, and that A_i is the probability that player i chooses high. Assume also that player 1 is an expected utility maximizer, with utility equal to 1 when both players choose high and to 0 otherwise. Thus, $w(a_1, a_2, \omega) = a_1 a_2$ and $u(x_1, x_2) = \int_\Omega x_1(\omega) x_2(\omega)\, d\mu$.

Define the sequence x_{1n} as in Example 1 and let $x_{2n} = 1 - x_{1n}$. The sequence $\{x_{in}\}$ converges weakly to the constant function $x_{i,\infty} = 1/2$ and so $u(x_{1,\infty}, x_{2,\infty}) = 1/4$. However, anywhere in the sequence $\{x_{1n}, x_{2n}\}$, in each state one player chooses the low action for sure; therefore, $u(x_{1n}, x_{2n}) = 0$.

However, we can leverage an assumption of concavity in own action to obtain weak upper semicontinuity of payoffs in own strategy: the definition of quasiconcavity is that upper contour sets are convex; the definition of upper semicontinuity is that upper contour sets are closed; and convex norm-closed sets are weakly closed. This is summarized in Proposition 3. In Section 7 we show that such continuity is sufficient.

PROPOSITION 3. *Assume that A_1 and A_2 are compact convex subsets of $\mathbb{R}^n$. Assume also that w is weakly concave in a_1 a.e. and weakly convex in a_2 a.e. Then the following statements hold:*

1. *u is usc when X_1 is endowed with the weak topology and X_2 is endowed with the norm topology;*
2. *u is lsc when X_1 is endowed with the norm topology and X_2 is endowed with the weak topology.*

Proof. We prove the first conclusion; the second follows by a symmetric argument.

1. Proposition 2 states that u is uniformly norm continuous.
2. A quasiconcave function that is norm usc is also weakly usc (see the paragraph preceding the proposition); hence $u(\cdot, x_2)$ is weakly usc for each x_2.
3. Uniform norm continuity implies that $\{u(x_1, \cdot) \mid x_1 \in X_1\}$ is equi-usc for the norm topology on X_2.
4. As stated in Appendix B, if a functional u is such that $u(\cdot, x_2)$ is usc for all x_2 and

if $\{u(x_1,\cdot) \mid x_1 \in X_1\}$ is equi-usc, then u is jointly usc.

□

6. Two-topology Theorem of the Maximum

The lack of weak lower semicontinuity in a single action arises even for a single-person decision problem when the choice set is an infinite-dimensional space. Horsley, Van Zandt, and Wrobel (1998) provide a solution to this topological tension for the Theorem of the Maximum by using two topologies, one for the "upper" conditions and another for the "lower" conditions. Similarly, Aliprantis and Border (1999) break down the continuity of the value function in a single-person optimization problem into a "lower" half and an "upper" half, which allows one to approach each half with different topologies. We summarize these results and observe how they overcome the topological tension for single-person decision problems. In Section 7, we use these results to prove a two-topology Theorem of the Maximin.

In this section, we consider a single-agent optimization problem. Let X be the choice set, let P again be a set of parameters, let $u\colon X \times P \to \mathbb{R}$ be the agent's utility function, and let $\varphi\colon P \twoheadrightarrow X$ be a correspondence with nonempty values that defines the agent's parameter-specific set of feasible alternatives. The optimization problem is $\sup_{x\in\varphi(p)} u(x,p)$. Let $v\colon P \to \mathbb{R}$ be the value function and let $\psi\colon P \twoheadrightarrow X$ be the solution correspondence. That is,

$$v(p) = \sup_{x\in\varphi(p)} u(x,p),$$
$$\psi(p) = \{x \in \varphi(p) \mid u(x,p) = v(p)\}.$$

The following statement is the classic Theorem of the Maximum.

PROPOSITION 4 (Berge's Theorem of the Maximum). *Assume that φ is continuous and has compact values and that u is continuous. Then v is continuous and ψ is uhc and has compact and nonempty values.*

As noted, application of this theorem is problematic when X is a subset of an infinite-dimensional vector space, such as the single-person optimization problems that occur in Bayesian games, because it may be difficult to satisfy both compactness and lower semicontinuity for the same topology on X. In this case, the following variant may be useful. In it, $\mathscr{W}$ and $\mathscr{S}$ are two (not necessarily related) topologies on X.

PROPOSITION 5 (Two-Topology Theorem of the Maximum—Horsley, Van Zandt, and Wrobel (1998)). *Assume:*

1. *φ is $\mathscr{S}$-lhc and u is $\mathscr{S}$-lsc.*
2. *φ is $\mathscr{W}$-uhc* and u is $\mathscr{W}$-usc.*

Then ψ is $\mathscr{W}$-uhc and has nonempty values.*

In Proposition 5, there is no relationship between the topologies $\mathscr{W}$ and $\mathscr{S}$. Furthermore, the theorem is agnostic about whether X is even a vector space. However, the intended application is to settings as in Section 5.3, where:

(a) X is a convex subset of an infinite-dimensional vector space $\mathfrak{X}$ with dual $\mathfrak{X}^*$;
(b) $\mathscr{W}$ is the weak topology $\sigma(\mathfrak{X}, \mathfrak{X}^*)$ and $\mathscr{S}$ is the strong topology $\tau(\mathfrak{X}_i, \mathfrak{X}_i^*)$, so that convex $\mathscr{S}$-closed sets are also $\mathscr{W}$-closed;
(c) X is $\mathscr{W}$-compact;
(d) u is uniformly $\mathscr{S}$-continuous and quasiconcave and hence is $\mathscr{W}$-usc.

Proposition 5 does not characterize the value function v. For this, it is useful to use Lemmas 14.28 and 14.29 of Aliprantis and Border (1999).

LEMMA 1 (Aliprantis and Border, 1999, Lemmas 14.28, 14.29).

(L) *Assume that φ is lhc and u is lsc. Then v is lsc.*
(U) *Assume that φ is uhc* and u is usc. Then v is usc.*

Observe that Lemma 1(L) and Lemma 1(U) are separate sublemmas; each can be applied to a problem using a distinct topology.

7. Application to zero-sum games

We now return to the setting of parameterized zero-sum games from Section 4.

Define $t_2 \colon X_1 \times P \to \mathbb{R}$ by the value of player 2's problem of choosing a best response:

$$t_2(x_1, p) = \sup_{x_2 \in \varphi_2(p)} -u(x_1, x_2, p). \tag{4}$$

We then have

$$v_1(p) = \sup_{x_1 \in \varphi_1(p)} -t_2(x_1, p). \tag{5}$$

We establish this notation so that we can apply Lemma 1 iteratively, first to characterize continuity of t_2 from the maximization problem in equation (4) and then to characterize continuity of v_1 from the maximization problem in equation (5).

Let $\mathscr{W}_i$ and $\mathscr{S}_i$ be topologies on X_i.

THEOREM 1 (Two-Topology Theorem of the Maximin). *Assume:*

1. *φ_i is $\mathscr{S}_i$-lhc and $\mathscr{W}_i$-uhc* for $i = 1, 2$;*
2. *u is $(\mathscr{S}_1, \mathscr{W}_2)$-lsc and $(\mathscr{W}_1, \mathscr{S}_2)$-usc.*

Then v_1 is a continuous function and ψ_1 is a $\mathscr{W}_1$-uhc correspondence.*

Proof. We first apply Lemma 1 to characterize $t_2 \colon X_1 \times P \to \mathbb{R}$. In this application, the parameters of the maximization problem are (x_1, p). The constraint correspondence $\varphi_2 \colon P \to X_2$ depends only on p. The objective function $-u \to X_1 \times X_2 \times P$ depends on (x_1, p).

(2L) By assumption, φ_2 is $\mathscr{S}_2$-lhc and $-u$ is $(\mathscr{W}_1, \mathscr{S}_2)$-lsc. According to Lemma 1(L), t_2 is $\mathscr{W}_1$-lsc.

(2U) By assumption, φ_2 is $\mathscr{W}_2$-uhc* and $-u$ is $(\mathscr{S}_1, \mathscr{W}_2)$-usc. According to Lemma 1(U), t_2 is $\mathscr{S}_1$-usc.

Next we apply Lemma 1 and Proposition 5 to the maximization problem in equation (5), which defines v_1 and ψ_1.

(1L) By assumption, φ_1 is $\mathscr{S}_1$-lhc. By (2U), $-t_2$ is $\mathscr{S}_1$-lsc. According to Lemma 1(L), v_1 is lsc.

(1U) By assumption, φ_1 is $\mathscr{W}_1$-uhc*. By (2L), $-t_2$ is $\mathscr{W}_1$-usc. According to Lemma 1(U), v_1 is usc.

Furthermore, according to Proposition 5, ψ_1 is $\mathscr{W}_1$-usc*. □

Much as in the discussion following Proposition 5, we note that Theorem 1 presumes no relationship between the topologies $\mathscr{W}_i$ and $\mathscr{S}_i$ nor any particular structure on X_i. Yet the intended application is to settings such as in Section 5.3.

8. Application to continuity of the value of information

We return to the setting of zero-sum Bayesian games from Section 5, in which the parameters are the information structure. As in the second half of that section, we assume that A_i is a convex compact subset of $\mathbb{R}^n$. In order to apply Theorem 1, we must first define the respective topologies on X_i. As suggested already, we use the weak and norm topologies and denote them by $\mathscr{W}_i$ and $\mathscr{S}_i$, respectively.

We must then define a topology on information sub-σ-fields. Allen (1983) was the first to introduce such a topology into economics. She used the Boylan metric, which we shall also employ in Section 9.4. For the current framework, however, we can use a weaker topology introduced by Cotter (1986, 1987), called the *pointwise convergence* topology and that we denote by $\mathscr{P}$. It is the weakest topology such that the mapping $\mathscr{F} \mapsto E[f \mid \mathscr{F}]$ from $\mathfrak{F}$ to L_1 is continuous in the L_1-norm for each $f \in L_1(\mathbb{R})$.

Van Zandt (2002) shows that the information measurability constraint φ_i satisfies the assumptions of Theorem 1.

PROPOSITION 6 (Van Zandt, 2002, Cor. 1). *Endow $\mathfrak{F}$ with the topology $\mathscr{P}$. Then φ_i is $\mathscr{S}_i$-lhc and $\mathscr{W}_i$-uhc*.*

We thus have our first result on continuity of the value and the solution with respect to the information structure.

THEOREM 2. *Assume the following.*

1. *A_i is a compact convex subset of $\mathbb{R}^n$.*
2. *Each player's payoff is concave in own action: w is concave in a_1 and convex in a_2.*
3. *Assumption 1 holds.*

Then v_1 is continuous and ψ_1 is $\mathscr{W}_1$-uhc when $\mathfrak{F}$ is endowed with the topology $\mathscr{P}$.*

Proof. According to Proposition 3, u is $(\mathscr{W}_1, \mathscr{S}_2)$-usc and $(\mathscr{S}_1, \mathscr{W}_2)$-lsc. Therefore, assumption 2 of Theorem 1 is satisfied. According to Proposition 6, assumption 1 of Theorem 1 is satisfied. □

9. An alternate approach: Hausdorff metric

9.1. Overview

If a multivalued mapping into a metric space has closed values, then one can treat it as a function into the set of nonempty closed subsets of that metric space endowed with the Hausdorff metric. Lower and upper hemicontinuity of a correspondence are replaced by continuity of this function.

There are several advantages and disadvantages to this approach, whether the goal is to study a single-person decision problem or the value of a zero-sum game. The advantages include the following.

1. Compactness is not required: one can work with a single strong topology, the vector space structure does not play a role in applications, and the entire analysis is simpler.
2. It is possible to derive uniform and Lipschitz continuity of the value function.

The disadvantages include the following.

1. The choice set must be a metric space (which is not a problem for most applications, but the Theorem of the Maximum applies to general topological spaces).
2. One characterizes continuity of the value function but not of the solution correspondence.
3. In the context of our application to the value of information, we must use a stronger topology on information.

At this point, the reader should have a basic idea of how the characterization of a single-person decision problem will feed into a theorem for the value of abstract zero-sum games, which in turn will be applied to continuity of the value with respect to information structures in Bayesian zero-sum games. Therefore, we now proceed in that order.

9.2. Single-person decision problem

We begin, then, with a single-person decision problem as in Section 6. Both the set P of parameters and the set X of alternatives must be metric spaces; denote their metrics by ρ and η, respectively. Let $\mathscr{X}$ denote the set of nonempty closed subsets of X and let h be the Hausdorff metric on $\mathscr{X}$:

$$h(Y,Z) = \sup_{y\in Y}\inf_{z\in Z} \eta(y,z) + \sup_{z\in Z}\inf_{y\in Y} \eta(y,z)\,.$$

Building in an assumption that the constraint correspondence $\varphi\colon P \twoheadrightarrow X$ has closed values, we write it as a function $\varphi\colon P \to \mathscr{X}$.

LEMMA 2.

(a) *If φ and u are uniformly continuous, then v is uniformly continuous.*
(b) *If φ is k-Lipschitz and if u is k_x-Lipschitz in x and k_p-Lipschitz in p, that is, if*

$$|u(x,p) - u(x',p')| \leq k_x \eta(x,x') + k_p \rho(p,p'),$$

then v is $(k_x k + k_p)$-Lipschitz.

Proof. (a) *Uniform continuity.* Let $\varepsilon > 0$. Since u is uniformly continuous, there is a $\delta > 0$ such that, if $\eta(x,x') < \delta$ and $\rho(p,p') < \delta$, then $|u(x,p) - u(x',p')| < \varepsilon$. Since φ is uniformly continuous, there is a $\delta' > 0$ such that, if $\eta(p,p') < \delta'$, then $h(\varphi(p),\varphi(p')) < \delta$. Assume then that $\rho(p,p') < \min\{\delta,\delta'\}$.

Let $x \in \varphi(p)$. Since $\rho(p,p') < \delta'$, it follows that $h(\varphi(p),\varphi(p')) < \delta$ and hence (by definition of the Hausdorff metric) there is an $x' \in \varphi(p')$ such that $\eta(x,x') < \delta$. Since also $\rho(p,p') < \delta$, we have $u(x,p) - u(x',p') < \varepsilon$; therefore, $v(p) - v(p') < \varepsilon$. An analogous argument shows that $v(p') - v(p) < \varepsilon$ and hence that $|v(p) - v(p')| < \varepsilon$.

(b) *Lipschitz continuity.* Let $p, p' \in P$, and let $x \in \varphi(p)$. By definition of the Hausdorff metric, for each $\varepsilon > 0$ there is an $x' \in \varphi(p')$ such that $\eta(x,x') < h(\varphi(p),\varphi(p')) + \varepsilon$ and hence

$$\begin{aligned} u(x,p) - u(x',p') &\leq k_x(h(\varphi(p),\varphi(p')) + \varepsilon) + k_p \rho(p,p') \\ &\leq k_x k \rho(p,p') + k_x \varepsilon + k_p \rho(p,p'). \end{aligned}$$

Therefore, letting $\varepsilon \downarrow 0$, we have $v(p) - v(p') \leq (k_x k + k_p)\rho(p,p')$. Reversing the roles of p and p' shows that $v(p') - v(p) \leq (k_x k + k_p)\rho(p,p')$. □

9.3. Zero-sum games

Consider a zero-sum game such as in Section 4, but now include topological assumptions like those in Section 9.2. For example, let η_1 be the metric on X_1, $\mathscr{X}_1$ the set of nonempty closed subsets of of X_1, and h_1 the Hausdorff metric on $\mathscr{X}_1$. Player 1's constraint is $\varphi_1 \colon P \to \mathscr{X}_1$.

THEOREM 3.

(a) *If φ_i and u are uniformly continuous for $i = 1,2$, then v_1 is uniformly continuous.*
(b) *If φ_i is k_i-Lipschitz for $i = 1,2$ and if u is k_{x1}-Lipschitz in x_1, k_{x2}-Lipschitz in x_2, and k_p-Lipschitz in p, then v_1 is $(k_{x1}k_1 + k_{x2}k_2 + k_p)$-Lipschitz.*

Proof. (a) *Uniform continuity.* By Lemma 2, $t_2 \colon X_1 \times P \to \mathbb{R}$ is uniformly continuous. Therefore, again by Lemma 2, v_1 is uniformly continuous.

(b) *Lipschitz continuity.* By Lemma 2, t_2 is k_{x1}-Lipschitz in x_1 and $(k_{x2}k_2 + k_p)$-Lipschitz in p. Therefore, again by Lemma 2, v is $(k_{x1}k_1 + k_{x2}k_2 + k_p)$-Lipschitz. □

9.4. Application to the value of information in Bayesian zero-sum games

Consider a Bayesian zero-sum game, as in the first half of Section 5. That is, each player's action set A_i is any separable metric space with metric d_i. Endow X_i with the metric η_i of the topology of convergence in measure. The set of $\mathscr{F}_i$-measurable functions is a closed subset of (X_i, η_i); hence the measurability constraint is a mapping $\varphi_i : \mathfrak{F} \to \mathscr{X}_i$, where $\mathscr{X}_i$ is the set of nonempty and closed subsets of X_i.

The Boylan metric on $\mathfrak{F}$ is defined by

$$\rho(\mathscr{F},\mathscr{G}) \equiv \sup_{F\in\mathscr{F}} \inf_{G\in\mathscr{G}} \mu(F \,\Delta\, G) + \sup_{G\in\mathscr{G}} \inf_{F\in\mathscr{F}} \mu(F \,\Delta\, G),$$

where $F \,\Delta\, G = (F \setminus G) \cup (G \setminus F)$ is the symmetric difference.

The Boylan metric was introduced by Boylan (1971), characterized further by Rogge (1974) and Landers and Rogge (1986), and introduced to economics by Allen (1983). It is stronger than the pointwise convergence topology and is equivalent to the topology of uniform convergence when each sub-σ-field $\mathscr{F}$ is viewed as a linear operator that maps $f \in L_1([0,1])$ to $E[f \mid \mathscr{F}] \in L_1([0,1])$.

Van Zandt (1993) shows that the Boylan metric is also equivalent to measuring the distance between sub-σ-fields by the Hausdorff distance between the corresponding sets of measurable functions. In particular, φ_i is a Lipschitz-continuous function.

LEMMA 3 (Van Zandt, 1993, Thm. 1). *For $\mathscr{F},\mathscr{G} \in \mathfrak{F}$,*

$$h_i(\varphi_i(\mathscr{F}), \varphi_i(\mathscr{G})) \leq 4\rho(\mathscr{F},\mathscr{G}).$$

Combining Theorem 3 and Lemma 3 yields the following.

COROLLARY 1.

(a) *If u is uniformly continuous, then v_1 is uniformly continuous.*
(b) *If u is k_{xi}-Lipschitz in x_i, then v_1 is $4k_{xi}$-Lipschitz in $\mathscr{F}_i$. That is, for $\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}_1, \mathscr{G}_2 \in \mathfrak{F}$,*

$$|v(\mathscr{F}_1,\mathscr{F}_2) - v(\mathscr{G}_1,\mathscr{G}_2)| \leq 4k_{x1}\rho(\mathscr{F}_1,\mathscr{G}_1) + 4k_{x2}\rho(\mathscr{F}_2,\mathscr{G}_2).$$

In order to apply this theorem, we make the following observations about the continuity of u.

PROPOSITION 7.

(a) *If A_i is compact, then u is uniformly continuous.*
(b) *The function u is $(1+\Delta_i)k_{ai}$-Lipschitz in x_i if w is k_{ai}-Lipschitz in a_i and if Δ_i is the diameter of the metric space A_i.*

Proof. (a) This is an application of Proposition 2.

(b) The proof is similar to that of Proposition B.1. For $i = 1,2$, let $x_i, x_i' \in X_i$ and let $F_i = \{\omega \in \Omega \mid d_i(x_i(\omega), x_i'(\omega)) \leq \eta_i(x_i, x_i')\}$. Thus, by definition of the metric η_i, we have $\mu(F_i^c) \leq \eta_i(x_i, x_i')$. The Lipschitz condition on w then implies

$$|u(x_1,x_2) - u(x_1',x_2')| \leq k_{a1}\int_\Omega d_1(x_1(\omega), x_1'(\omega))\, d\mu + k_{a2}\int_\Omega d_2(x_2(\omega), x_2'(\omega))\, d\mu.$$

For the first term, we have

$$\int_\Omega d_1(x_1(\omega), x_1'(\omega))\, d\mu = \int_{F_1} d_1(x_1(\omega), x_1'(\omega))\, d\mu + \int_{F_1^c} d_1(x_1(\omega), x_1'(\omega))\, d\mu.$$

By the definition of F_1,

$$\int_{F_1} d_1(x_1(\omega), x_1'(\omega))\, d\mu \leq \int_{F_1} \eta(x_1, x_1')\, d\mu \leq \eta(x_1, x_1'),$$
$$\int_{F_1^c} d_1(x_1(\omega), x_1'(\omega))\, d\mu \leq \Delta_1 \mu(F_1^c) \leq \Delta_1 \eta_1(x_1, x_1').$$

We have similar inequalities for the terms containing x_2. Therefore,

$$|u(x_1, x_2) - u(x_1', x_2')| \leq (1+\Delta_1) k_{a1} \eta_1(x_1, x_1') + (1+\Delta_2) k_{a2} \eta_2(x_2, x_2'),$$

□

10. Comparison with Einy et al. (2008)

This paper was motivated by Einy et al. (2008), who use a direct approach for Bayesian zero-sum games to study continuity of the value with respect to the information structure. We believed that our approach, which decomposes the problem into (a) continuity of the measurability constraint and (b) an abstract "Theorem of the Maximin", would simplify and clarify the analysis. In addition, we felt that the Theorem of the Maximin would be of independent interest beyond applications to the value of information.

An unexpected benefit of this approach is that we have generalized most of the results in Einy et al. (2008). In this section, we briefly outline their main results and compare them to ours. All results (theirs and ours) are based on the model of Bayesian zero-sum games outlined in Section 5.1, including Assumption 1 (u is an integrably bounded Carathéodory function) and the assumption that A_i is a compact metric space. Einy et al. (2008) also assume that Σ is countably generated. Differences in additional assumptions are outlined as follows for each result.

1. Their Theorem 1 shows that v_1 is uniformly continuous with respect to the Boylan metric on information when: (a) A_i is a convex compact subset of Euclidean space and w is concave in a_1 and convex in a_2 a.e.; and (b) $w(\cdot, \omega)$ is $k(\omega)$-Lipschitz for a q-integrable function $k(\cdot)$.

   We obtain the same conclusion without assumption (a) or (b). See Corollary 1(a) and Proposition 7(a). (Note, however, that their Lipschitz condition allows them also to show Hölder continuity of order $(q-1)/q$.)
2. Their Corollary 2 shows that v_1 is Lipschitz-continuous with respect to the Boylan metric on information when: (a) A_i is a convex compact subset of Euclidean space and w is concave in a_1 and convex in a_2 a.e.; and (b) $w(\cdot, \omega)$ is k-Lipschitz a.e.

   We obtain the same conclusion with the same assumption (b) but without assumption (a). See Corollary 1(b) and Proposition 7(b).
3. Their Theorem 2 shows that v_1 is continuous with respect to the Boylan metric on information when: (a) A_i is a convex compact subset of Euclidean space and w is concave in a_1 and convex in a_2 a.e.; and (b) $w(\cdot, \omega)$ is $k(\omega)$-Lipschitz for an integrable function $k(\cdot)$.

   Our Corollary 1(a) and Proposition 7(a) obtain the stronger conclusion of uniform continuity without assumption (a) or (b). Our Theorem 2 obtains the stronger conclusion of continuity with respect to the topology of pointwise convergence on information; it uses the same assumption (a) but not assumption (b).

4. Their Theorem 3(1) shows that ψ_1 is weakly uhc* for the Boylan metric on information when: (a) A_i is a convex compact subset of Euclidean space and w is concave in a_1 and convex in a_2 a.e.; and (b) $w(\cdot,\omega)$ is $k(\omega)$-Lipschitz for an integrable function $k(\cdot)$.

   Our Theorem 2 obtains the stronger conclusion that ψ_1 is weakly uhc* for the topology of pointwise convergence on information; it uses the same assumption (a) but not assumption (b).

11. Why these techniques can only be used for zero-sum games

One might also be interested in whether the Nash equilibrium correspondence is uhc* in games that are not zero-sum. In such games, equilibrium payoffs would not be a function, but one could also study whether the Nash equilibrium payoffs correspondence is uhc*.

Consider, then, a two-player normal-form game. A parameter $p \in P$ affects the feasible strategy sets and payoffs of players 1 and 2. Let X_i be the player i's potential strategy set; let $\varphi_i(p) \subset X_i$ be her nonempty set of feasible strategies given p; and let $u_i\colon X_1 \times X_2 \times P \to \mathbb{R}$ be her p-dependent payoff function. Let $N(p)$ be the set of Nash equilibrium strategy profiles and let $v(p)$ be the set of Nash equilibrium payoffs given p. We have thus defined the correspondences $N\colon P \twoheadrightarrow X_1 \times X_2$ and $v\colon P \twoheadrightarrow \mathbb{R}^2$.

Player 1's best-response correspondence $b_1\colon X_2 \times P \twoheadrightarrow X_1$ is defined by

$$b_1(x_2, p) = \operatorname*{arg\,max}_{x_1 \in \varphi_1(p)} \; u_1(x_1, x_2, p);$$

player 2's best-response correspondence $b_2\colon X_1 \times P \twoheadrightarrow X_2$ is defined analogously. The graph of the Nash correspondence N is merely the intersection of the graphs of b_1 and b_2.[1] For example, $(p, x_1, x_2) \in \mathrm{Gr}(b_1)$ means that x_1 is a best response by player 1 to x_2 given p (and hence is also a feasible action for player 1 given p). We can therefore establish that N has a closed graph by assuming that φ_i and u_i satisfy the assumptions of the Theorem of the Maximum (Proposition 4), because then each b_i has a closed graph.

THEOREM 4. *For $i = 1, 2$, assume that φ_i is continuous and has compact values and assume that u_i is continuous. Then N and v are uhc and have compact values.*

The result about v follows because $v(p) = (u_1(N(p), p), u_2(N(p), p))$ and because the composition of two uhc* functions is uhc*. In this application, v is the composition of (a) the continuous function and hence uhc* correspondence $(x_1, x_2, p) \mapsto (u_1(x_1, x_2, p), u_2(x_1, x_2, p))$, and (b) the uhc* correspondence $p \mapsto N(p) \times \{p\}$.

However, once we take up Bayesian games with nonatomic state spaces, we again have the problem that the norm topology is too strong for compactness whereas the

1. After reordering the triples for each graph to match each other—that is, after treating b_1 and b_2 as subsets of $P \times X_1 \times X_2$ rather than of $X_2 \times P \times X_1$ and $X_1 \times P \times X_2$, respectively.

weak topology is too weak for upper semicontinuity of payoffs. Unfortunately, the two-topology Theorem of the Maximum does not save the situation. The problem is that, when applying Proposition 5 to the best responses of each player, we can use the weaker topology $\mathscr{W}$ only on that player's own action. We will thus conclude: (a) that b_1 has a closed graph when X_1 has the $\mathscr{W}$ topology and X_2 has the $\mathscr{S}$ topology; and (b) that b_2 has a closed graph when X_2 has the $\mathscr{W}$ topology and X_1 has the $\mathscr{S}$ topology. However, we need b_1 and b_2 to be closed under the weaker topology $\mathscr{W}$ on X_1 and X_2 so that their intersection—the graph of the Nash equilibrium correspondence—is also closed under this topology.

Shifting to correlated strategies circumvents the problem of joint-continuity. Cotter (1994) shows that the set of type-correlated equilibria (an extension of correlated equilibria to Bayesian games) depends upper hemicontinuously on information in the topology of pointwise convergence.

Appendix A: Continuity definitions

The following definitions of continuity for a correspondence $\varphi\colon P \twoheadrightarrow X$ between two topological spaces are standard, except that they are independent of whether φ has nonempty values.

DEFINITION A.1. Let $\varphi\colon P \twoheadrightarrow X$ be a correspondence. Then:

1. φ is *closed* if and only if its graph, $\mathrm{Gr}(\varphi)$, is closed in $P \times X$;
2. φ is *upper hemicontinuous* (uhc) if and only if $\{p \in P \mid \varphi(p) \subset U\}$ is open for every open $U \subset X$;
3. φ is *lower hemicontinuous* (lhc) if and only if—for all $p \in P$, all $x \in \varphi(p)$, and all neighborhoods V of x—there exists a neighborhood U of p such that $\varphi(p') \cap V \neq \emptyset$ for all $p' \in U$.

We reserve the term "semicontinuity"—which is used by some authors to mean what we call "hemicontinuity"—for real-valued functions and preorders. A real-valued function or preorder is upper semicontinuous (usc) if the upper-contour (weakly preferred-to) sets are closed, and it is lower semicontinuous (lsc) if the lower-contour sets are closed.

Appendix B: A joint continuity result

Let X and Y be topological spaces.

Let $f : X \to \mathbb{R}$ be a functional on X. The following is an equivalent definition of upper semicontinuity: f is usc at $x \in X$ if for all $\varepsilon > 0$ there is a neighborhood N of x such that, for $x' \in N$, $f(x') - f(x) < \varepsilon$; f is usc if it is usc at x for all $x \in X$.

Let F be a family of functionals on X and let $x \in X$. Then F is equi-usc at x if for all $\varepsilon > 0$ there is a neighborhood N of x such that, for $x' \in N$ and $f \in F$, $f(x') - f(x) < \varepsilon$; F is equi-usc if it is equi-usc at x for all $x \in X$.

The following is likely to be a standard result, but we provide a proof for completeness because we do not have a clear citation.

LEMMA B.1. *Let* $f\colon X\times Y\to\mathbb{R}$ *be a functional on* $X\times Y$. *Assume:*

1. *for all* $y\in Y$, $f(\cdot,y)\colon X\to\mathbb{R}$ *is usc;*
2. $\{f(x,\cdot)\colon Y\to\mathbb{R}\mid x\in X\}$ *is equi-usc.*

Then f *is (jointly) usc.*

Proof. Let $(x,y)\in X\times Y$ and let $\varepsilon>0$. Since $f(\cdot,y)$ is usc, there is a neighborhood N_x of x such that, for $x'\in N_x$, we have $f(x',y)-f(x,y)<\varepsilon/2$. Since $\{f(x,\cdot)\colon Y\to\mathbb{R}\mid x\in X\}$ is equi-usc, there is a neighborhood N_y of y such that, for $x'\in X$ and $y'\in N_y$, we have $f(x',y')-f(x',y)<\varepsilon/2$. Then, for $(x',y')\in N_x\times N_y$, it follows that

$$f(x',y')-f(x,y)=\underbrace{f(x',y')-f(x',y)}_{<\varepsilon/2}+\underbrace{f(x',y)-f(x,y)}_{<\varepsilon/2}<\varepsilon.$$

□

Appendix C: Continuity of state-dependent expected utility

Consider a single-person decision problem under uncertainty. Uncertainty is represented by a probability space (Ω,Σ,μ). In each state, the set of outcomes is A, which is endowed with a metric d. An *act* is a measurable function $X\colon\Omega\to A$; let X be the set of acts. The agents' preferences over acts are represented by expected utility with a state-dependent utility function $w\colon A\times\Omega\to\mathbb{R}$. That is, the expected utility of an act x is

$$u(x)=\int_\Omega w(x(\omega),\omega)\,d\mu(\omega). \tag{C.1}$$

We are interested in assumptions on w such that $u\colon X\to\mathbb{R}$ is well-defined and is continuous with respect to the topology on X of convergence in measure. That topology has the following metric:

$$\eta(x,x')\equiv\inf\left\{\varepsilon>0\,\middle|\,\mu\{\omega\in\Omega\mid d(x(\omega),x'(\omega))>\varepsilon\}<\varepsilon\right\}. \tag{C.2}$$

Recall that w is called a *Carathéodory function* if it is both continuous in a and measurable in ω. We say that w is *integrably bounded* if there is an integrable function on (Ω,Σ,μ) that bounds the absolute value of w pointwise.

PROPOSITION C.1. *If* w *is an integrably bounded Carathéodory function, then* u *is continuous. If also* A *is compact, then* u *is uniformly continuous.*

REMARK C.1. We believe that the uniform continuity result is new. The uniformity is critical in our proof of joint weak-norm upper semicontinuity of u in Proposition 3.

On the other hand, the continuity result (without uniformity) has appeared in other forms. We provide a proof for the sake of completeness and because we are not aware of a result for general metric spaces X. For example, Balder and Yannelis (1993) give a very thorough treatment of continuity for the case in which the set of possible outcomes is state-dependent and takes values in a Banach space.

Proof. It is well known that the integral defining u is well-defined when w is an integrably bounded Carathéodory function. Let $\bar{w}\colon \Omega \to \mathbb{R}$ be an integrable function that bounds the absolute value of w. Then the absolute value of u is bounded by $\int_\Omega \bar{w}(\omega)\, d\mu$.

Continuity. Let $\{x_n\}$ be a sequence in X that converges in measure to x. Because u is bounded, if $\{u(x_n)\}$ does not converge to $u(x)$ then there exists a subsequence $\{u(x_{n_k})\}$ that converges to a number other than $u(x)$. We show that this is impossible because such a subsequence must have a subsubsequence that converges to $u(x)$.

To simplify notation, denote the subsequence by $\{x_n\}$; what is important is that, like the original sequence, the subsequence also converges in measure to x. Any sequence that converges in measure has a subsequence that converges almost everywhere (see e.g., Aliprantis and Border (1999, Thm. 13.38)); denote such a subsequence by $\{x_{n_k}\}$. Since w is continuous in x, it follows that $w(x_{n_k}(\omega),\omega)$ converges to $w(x(\omega),\omega)$ almost everywhere. Since the absolute value of $w(x_{n_k}(\omega),\omega)$ is bounded by $\bar{w}$, by the dominated converge theorem we have that $\int_\Omega w(x_{n_k}(\omega),\omega)\, d\mu$ converges to $\int_\Omega w(x(\omega),\omega)\, d\mu$; that is, $u(x_{n_k}) \to u(x)$.

Uniform continuity. Assume that A is compact. Any continuous function on a compact metric space is uniformly continuous. Hence, $w(\cdot,\omega)\colon A \to \mathbb{R}$ is uniformly continuous for each ω. We leverage this into uniform continuity of u.

Let $\delta > 0$. Define A^2_δ by

$$\{(a,a') \in A \times A \mid d(a,a') \le \delta\};$$

that is, $A^2_\delta = d^{-1}([0,\delta])$. Since d is continuous, A^2_δ is a closed and hence compact subset of $A \times A$. Then, for $\omega \in \Omega$, let

$$f_\delta(\omega) = \max_{(a,a')\in A^2_\delta} |w(a,\omega) - w(a',\omega)|.$$

Observe that $f_\delta(\omega)$ is well-defined because $(a,a') \mapsto |w(a,\omega) - w(a',\omega)|$ is continuous (because $w(\cdot,\omega)$ is continuous). Furthermore, $(a,a',\omega) \mapsto |w(a,\omega) - w(a',\omega)|$ is a Carathéodory function and hence, according to the Measurable Maximum Theorem (e.g., Aliprantis and Border, 1999, Thm. 17.18), f_δ is a measurable function.

For $\delta < \delta'$, we have $A^2_\delta \subset A^2_{\delta'}$ and hence $f_\delta \le f_{\delta'}$. Furthermore, $\lim_{\delta\downarrow 0} f_\delta = 0$ pointwise, as follows. Let $\omega \in \Omega$ and let $\varepsilon > 0$; then, since $w(\cdot,\omega)$ is uniformly continuous, there is a δ such that (a) $d(a,a') \le \delta$ implies $|w(a,\omega) - w(a',\omega)| \le \varepsilon$ and hence (b) $f_\delta(\omega) \le \varepsilon$. By the dominated convergence theorem, $\lim_{\delta\downarrow 0} \int_\Omega f_\delta(\omega)\, d\mu = 0$.

Let $\varepsilon > 0$. Pick $\delta_1 > 0$ such that $\int_\Omega f_{\delta_1}(\omega)\, d\mu < \varepsilon/2$. Since $\bar{w}$ is integrable, we can pick $\delta_2 > 0$ such that, if $F \in \Sigma$ and $\mu F < \delta_2$, $\int_F \bar{w}(\omega)\, d\mu < \varepsilon/4$. (See, for example, Dudley (2002, p. 122, problem 7, and p. 177, problem 4).) Let $x, x' \in X$ be such that $\eta(x,x') < \min\{\delta_1,\delta_2\}$. We show that $|u(x) - u(x')| < \varepsilon$.

Let $F = \{\omega \in \Omega \mid d(x(\omega),x'(\omega)) \le \eta(x,x')\}$. We will put bounds on the decomposition

$$|u(x) - u(x')| \le \int_F \left| w(x(\omega),\omega) - w(x'(\omega),\omega)\right| d\mu + \int_{F^c} \left| w(x(\omega),\omega) - w(x'(\omega),\omega)\right| d\mu,$$

establishing that each term is less than $\varepsilon/2$.

1. For $\omega \in F$, we have $d(x(\omega), x'(\omega)) \le \eta(x, x') \le \delta_1$. Therefore,

$$\left| w(x(\omega),\omega) - w(x'(\omega),\omega) \right| \le f_{\delta_1}(\omega).$$

   The first term is thus bounded by $\int_\Omega f_{\delta_1}(\omega)\, d\mu < \varepsilon/2$.

2. Since

$$\left| w(x(\omega),\omega) - w(x'(\omega),\omega) \right| \le 2\bar{w}(\omega),$$

   it follows that the second term is bounded by $2\int_{F^c} \bar{w}(\omega)\, d\mu$. From the definition of η in equation (C.2) and the definition of F, we have $\mu F^c \le \eta(x, x')$ and hence $\mu F^c \le \delta_2$. Therefore, $2\int_{F^c} \bar{w}(\omega)\, d\mu < \varepsilon/2$.

□

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