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ROBUSTNESS OF ADAPTIVE EXPECTATIONS AS AN EQUILIBRIUM SELECTION DEVICE

by

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Robustness of Adaptive Expectations as an Equilibrium Selection Device*

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Abstract

Dynamic models in which agents' behavior depends on expectations of future prices or other endogenous variables can have steady states that are stationary equilibria for a wide variety of expectations rules, including rational expectations. When there are multiple steady states, stability is a criterion for selecting among them as predictions of long-run outcomes. The purpose of this paper is to study how sensitive stability is to certain details of the expectations rules, in a simple OLG model with constant government debt that is financed through seigniorage. We compare simple recursive learning rules, learning rules with vanishing gain, and OLS learning, and also relate these to expectational stability. One finding is that two adaptive expectation rules that differ only in whether they use current information can have opposite stability properties.

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Contents

1 Introduction

Dynamic macroeconomic models often have many equilibria, which are parameterized by exogenous initial conditions or endogenous initial equilibrium values. Some of these equilibria may have the form of steady states, cycles, or other repeating patterns. One criterion for robustness of a steady state (or other pattern) as a long-run prediction is its stability. Roughly, a steady state is *stable* if, whenever the initial conditions or equilibrium values are close enough to the steady state, the equilibrium path both stays close to the steady state and eventually converges to it. (We are referring here to what is often called *local* or *asymptotic* stability. See Section 3 for the precise definition.)

It is possible to study stability in rational expectations models, but there are some objections to this exercise. First, in dynamic models of both the physical and human world, the usual interpretation of stability as a criterion of robustness is that the system would converge back to a stable steady state (without ever moving too far from it) after a small unmodeled perturbation. However, rational expectations models do not allow for unmodeled perturbations, and hence "stability" only indicates that there is an open set of equilibria converging to the steady state. Second, even some leading proponents of rational expectations in macroeconomics view rational expectations as a consistency condition that must be satisfied in the long run, when the economy settles into some kind of recognizable pattern such as a steady state or cycle, rather than a true description of the dynamics. For example, Lucas (1978, p. 1429) states: "[Rational expectations] do not describe the way agents think about their environment, how they learn, process information, and so forth. It is rather a property likely to be (approximately) possessed by the *outcome* of this unspecified process of learning and adapting. One would feel more comfortable, then, with rational expectations equilibria if these equilibria were accompanied by some form of 'stability theory' which illuminated the forces which move the economy towards equilibrium." (See also Grandmont (1988, 1998) and Sargent (1993, 1999) for more extensive arguments in this direction, and see Guesnerie and Woodford (1992, Section 7) for an overview of different learning criteria for selecting equilibria.) Rational and adaptive expectations can be viewed as complementary approaches: rational expectations allows one to identify the steady states, cycles, or other patterns that might be collectively learnable in the long-run, and then adaptive expectations allows one to test their stability and learnability. There has thus developed a large literature on stability in macroeconomic models of adaptive expectations and other forms of learning.

This paper studies stability in a discrete-time model of inflation and a government debt financed through seigniorage, as in Sargent and Wallace (1981), Marcet and Sargent (1989), and Arifovic (1995).[1] It is a simple model with a single state variable (inflation π_t), yet it is not trivial because its reduced form is $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$, i.e., the realized inflation factor depends on expectations for two periods.[2] It features both a low-inflation (π^L) and a high-inflation (π^H) steady state, and we analyze the stability of these steady states under a variety of expectations rules.

The value of this exercise is threefold:

1. A comparison of papers in the stability literature is often complicated by the fact that they differ both in the underlying macroeconomic model and in the specific type of

[1] It is similar to the continuous-time hyperinflation model of, e.g., Cagan (1956), Sargent and Wallace (1987), and Bruno and Fischer (1990).

[2] In contrast, many general treatments of stability, such as Guesnerie and Woodford (1991, 1992), study the reduced form $x_t = \varphi(x^e_{t+1})$ or $x_t = \varphi(x_{t-1}, x^e_{t+1})$.

expectations rules. In this paper, we instead study a single deterministic macroeconomic model and compare in detail the stability properties for several general classes of expectations rules. This pedagogical exercise clarifies the differences between these rules; in particular, we shed light on the stability of OLS learning in Marcet and Sargent (1989).

2. Our main methodological finding is that stability can depend crucially on whether the agents use current information when forming expectations. In the model, as in a large class of temporary equilibrium models, a Walrasian mechanism clears markets in each period. The current-period price, which is necessarily known to agents at the time of trading, could affect demand both through current terms of trade and because it affects expectations about the future. Yet taking these effects into account can lead to complications, such as multiple within-period Walrasian equilibria in models that would have a unique within-period equilibrium when expectations are fixed. Thus, it is common to make the simplifying assumption that agents ignore current information for the purpose of forming expectations. However, we find that this assumption is not innocuous.

3. The macroeconomic model of hyperinflation is also of intrinsic interest and we provide an extensive characterization of the stability of the steady states. Our results are not conclusive; it is a fact of life that the indeterminacy of equilibria is replaced by an indeterminacy in the specification of the expectations rules and hence such theoretical research cannot alone yield specific conclusions. However, it allows one to link the different equilibria to the different expectations rules. The gist of our results is that π^L tends to be stable and π^H unstable, as in previous literature, when not too much weight is placed on current information, whereas otherwise the stability properties may be reversed.

Although the results presented here are for more general expectations rules, our findings can be understood by considering the following examples. Suppose that, in period t, agents form expectations π^e_{t+1} of the inflation factor in the next period as a weighted average of the previous inflation expectations π^e_t and of an observed inflation factor—either π_{t-1} (lagged information) or π_t (current information). That is, either $\pi^e_{t+1} = \alpha\pi_{t-1} + (1-\alpha)\pi^e_t$ or $\pi^e_{t+1} = \alpha\pi_t + (1-\alpha)\pi^e_t$. The coefficient α is constant over time, and hence we call these "constant-gain" expectations rules. The two rules differ only in the timing of the observed inflation used to update expectations, but they lead to different stability properties of the steady states, as follows:

Constant-gain expectations rules

Information	π^L	π^H
Lagged	Stable ($\sim$)	Unstable
Current	Unstable ($\sim$)	Stable

(All results hold for sufficiently low government debt; results marked by $\sim$ hold only for some overlapping values of the other parameters.)

We consider also the "diminishing-gain" case, in which the weight α on new information decreases to zero over time. Not surprisingly, stability does not depend on the lag of the information and is the same as in the constant-gain case with lagged information:

Diminishing-gain expectations rules

Information	π^L	π^H
Lagged	Stable	Unstable
Current	Stable	Unstable

By applying and extending results in Evans and Honkapohja (1999), we also show that stability in this case is characterized by expectational stability. Expectational stability has been used most extensively to characterize stability in stochastic models with diminishing gain. For the case of current information, we are able to directly apply the main propositions in Evans and Honkapohja (1999). With lagged information, the resulting second-order system cannot be transformed to their framework, and hence we provide a new proof.

An expectations rule that has received much attention in the literature is OLS learning. Consider first the OLS estimate of $\bar{\pi}$ for the linear model $\pi_s = \bar{\pi} + \epsilon_s$. This estimate is just the unweighted average of past inflation factors, which is an example of a diminishing-gain expectations rule; thus, the timing of information does not affect stability. In Marcet and Sargent (1989), agents instead calculate π^e_{t+1} as the OLS estimate of $\bar{\pi}$ for the linear model $p_s = \bar{\pi} p_{s-1} + \epsilon_s$, using price data up through period $t-1$ (lagged information). We show that, with this rule, the timing of information *does affect* the stability of π^H but *not* the stability of π^L:

OLS estimates for $p_s = \bar{\pi} p_{s-1} + \epsilon_s$

Information	π^L	π^H
Lagged	Stable	Unstable
Current	Stable	Stable

These results can be understood as follows. The homoskedasticity assumption implicit in OLS means that, in the linear regression $p_s = \bar{\pi} p_{s-1} + \epsilon_s$, the ϵ_s have the same variance. Dividing this equation by p_{s-1}, we obtain (a) $\pi_s = \bar{\pi} + \epsilon_s / p_{s-1}$. This resembles the equation (b) $\pi_s = \bar{\pi} + \epsilon_s$, whose OLS estimates correspond to diminishing-gain expectations rules. However, the errors ϵ_s / p_{s-1} in (a) are no longer homoskedastic; instead, around the high-inflation steady state π^H in which prices are rising, the variance of recent errors is lower than the variance of older errors. This is why the OLS regression puts more weight on recent inflation factors in a neighborhood of π^H and the stability of π^H is qualitatively the same as for constant-gain expectations rules. In contrast, around the steady state π^L—which is close to unity and in which prices are nearly constant—the errors have approximately the same variance and stability is the same as for diminishing-gain expectations rules.

This shows how the results of, for example, Marcet and Sargent (1989) depend on seemingly minor details of the expectations rules. They conclude that, contrary to rational expectations dynamics, π^H is unstable with OLS learning. If they had instead assumed that agents formed expectations in period t using information up through period t, then they would have found that π^H was stable. If then they had changed to the OLS estimates of the linear model $\pi_s = \bar{\pi} + \epsilon_s$, whose implicit assumptions on errors are perhaps more plausible, they would have again found that π^H was unstable.

This paper does not advocate any one of these expectations rules over others or claim that one set of results is more accurate than others. Our point is to compare a variety of expectations rules in the context of a single simple macroeconomics model in order to better understand their properties. The exercise provides concrete examples of how seemingly innocuous changes in the expectations rules, which might be brushed aside as unimportant, actually do affect the stability results. Hence, it is hard to draw strong conclusions about equilibrium selection via a purely theoretical study of adaptive expectations. However, such exercises may prove useful when coupled with empirical or experimental tests, such as Marimon and Sunder (1993, 1994).

2 Model

The underlying economic model is one of inflation with financing of a government debt by seigniorage. Time is discrete, with periods $t \in \{0, 1, \ldots\}$. The expression "for all t" means "for all $t \in \{0, 1, \ldots\}$", and expressions such as "$\pi_t \to \hat{\pi}$" mean "$\lim_{t\to\infty} \pi_t = \hat{\pi}$".

For all t, $p_t \in \mathbb{R}_{++}$ is the period-t price level, $\pi_{t+1} := p_{t+1}/p_t$ is the period-$(t+1)$ inflation factor, and m_t is the period-t money supply. There is an initial money supply of m_{-1}, which is augmented in each period t by $p_t\delta$ in order to finance a constant real deficit $\delta > 0$. Hence, for all t, $m_t = m_{t-1} + p_t\delta$. A price path $\{p_t\}_{t=0}^{\infty}$ thus deterministically determines both an associated inflation path $\{\pi_{t+1}\}_{t=0}^{\infty}$ and a money supply path $\{m_t\}_{t=0}^{\infty}$.

The period-$(t+1)$ inflation factor expected in period t is denoted π_{t+1}^e; it is a function—called the "expectations rule"—of the history up through and including period t. Although we study rational expectations in Section 4, elsewhere the expectations rules are *adaptive* in the sense that they are history dependent and are not necessarily correct in equilibrium. They also have the flavor of predicting inflation from past inflation because the inflation factor expected in any period is in the convex hull of previous realized and expected inflation factors.

The period-t demand for real money balances depends on expected inflation and is denoted $S(\pi_{t+1}^e)$, where $S\colon \mathbb{R}_{++} \to \mathbb{R}_+$; the nominal demand is $p_t S(\pi_{t+1}^e)$. We impose the following assumption on S.

Assumption 2.1

(1) There exists $\pi^a \in (1, \infty)$ such that $S(\pi) = 0$ if and only if $\pi \geq \pi^a$.

(2) S is continuous everywhere and is continuously differentiable on $(0, \pi^a)$.

(3) $S'(\pi) < 0$ for $\pi \in (0, \pi^a)$ and $S'(\pi^a) := \lim_{\pi\uparrow\pi^a} S'(\pi) < 0$.

(4) $\lim_{\pi\downarrow 0} S(\pi) > \delta$.

Example 2.1 The affine case is $S(\pi) = a - b\pi$ for $\pi \in (0, \pi^a]$, where $a > b > 0$ and $\pi^a = a/b > 1$. This is the form assumed in Marcet and Sargent (1989). Assumption 2.1 is more general but is not consistent with the exponential real-balances demand curve $S(\pi) = ce^{-a\pi}$ introduced by Cagan (1956), for which the demand for real money balances is always strictly positive.

Remark 2.1 For instance, S might be derived from an overlapping generations model in which (a) the only form of savings is to hold money and (b) it is impossible to borrow against earnings in old age.[3] Then π_{t+1}^e is the expected price of period-$(t+1)$ consumption relative to period-t consumption; π^a is the relative price at which each generation prefers to consume its endowment; and S is equal to the younger generation's Walrasian net supply of period-t consumption, until the no-borrowing constraint is binding. This is illustrated in Figure 2.1. The assumption $\pi^a > 1$ holds, for example, if the utility function is monotone

[3]See Sargent (1987) and Azariadis (1993) for textbook treatments of the rational expectations equilibria of this model and for references to earlier work. See Lettau and Van Zandt (2000) for an outline of the model in the notation of this paper.

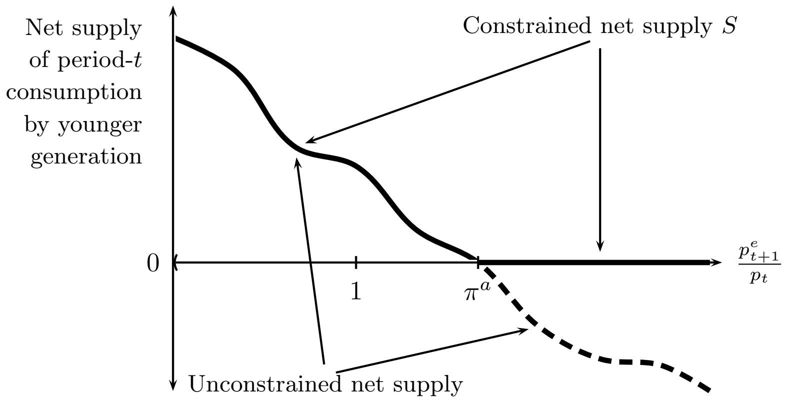

FIGURE 2.1. An illustration of Assumption 1. In an OLG model with two-period households, the inflation factor represents the terms of trade between consumption tomorrow and today and the demand for real money balances by youth is equal to their net supply of consumption, if positive. The wavy line (solid and dashed) might be the unconstrained net supply by youth as a function of relative prices, and the solid line is the actual supply curve S given that households cannot borrow.

and symmetric and the endowment in youth is greater than the endowment in old age. That S is strictly decreasing up to π^a holds if consumption in youth and old age are gross substitutes.

Given the period-$(t-1)$ history, the period-t market clearing condition for p_t is that the supply of and demand for money be equal:

$$p_t S(\pi^e_{t+1}) \;=\; m_{t-1} + p_t \delta \; . \tag{2.1}$$

The period-$(t-1)$ history determines m_{t-1} and affects π^e_{t+1}. The expectations π^e_{t+1} may also depend on p_t. Assuming otherwise represents an ad hoc restriction: inherent in this temporary equilibrium model, in which a Walrasian mechanism clears markets each period given the prior history, is that agents know p_t when trading in period t. One of the main themes of this paper is that excluding current information affects the stability of steady states.

Because a price path determines also the money supply path, the inflation path, and the expected inflation path, we can define an equilibrium in terms of the price path only. Then, for example, the *equilibrium inflation path* means the inflation path associated with an equilibrium price path.

Definition 2.1 An *equilibrium* is a price path $\{p_t\}_{t=0}^{\infty}$ that, together with its associated money supply path $\{m_t\}_{t=0}^{\infty}$, satisfy equation (2.1) for all t. An equilibrium is *stationary* if there is a $\hat{\pi} \in \mathbb{R}_{++}$ such that $\pi_t = \pi^e_t = \hat{\pi}$ for all $t \geq 1$; $\hat{\pi}$ is then called the *steady-state inflation factor.*

We can characterize the equilibria in terms of expected and realized inflation factors. For each $t \geq 1$, combining the equilibrium conditions

$$p_{t-1} S(\pi^e_t) \;=\; m_{t-1} \qquad \text{and} \qquad p_t S(\pi^e_{t+1}) \;=\; m_{t-1} + p_t \delta$$

yields $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$, where

$$W(\pi^e_{t+1}, \pi^e_t) \;:=\; \frac{S(\pi^e_t)}{S(\pi^e_{t+1}) - \delta} \; .$$

With a few more steps, which are given in Lettau and Van Zandt (2000, Section 2), we can thus show the following.

Proposition 2.1 *A price path* $\{p_t\}_{t=0}^{\infty} \in \mathbb{R}_{++}^{\infty}$ *is an equilibrium if and only if* $p_0(S(\pi_1^e) - \delta) = m_{-1}$ *and, for* $t \geq 1$, $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$.

A necessary condition for $\hat{\pi}$ to be a steady state is that $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$. We consider only classes of expectations rules for which this is also a sufficient condition. Using this condition, we now identify two steady states—one low (π^L) and one high (π^H)—which will be the focus of our stability analysis.

These two steady states depend on δ (though we usually denote them simply by π^L and π^H) and are identified as follows. Rewrite the steady-state condition as $\hat{\pi}(S(\hat{\pi}) - \delta)) = S(\hat{\pi})$. If we allowed $\delta = 0$, this condition would require either that $\hat{\pi} = 1$ or $S(\hat{\pi}) = 0$. The $S(\hat{\pi}) = 0$ case would correspond to an autarkic equilibrium in which money had no value. The inflation factor would not be well-defined, but think of it as being π^a because if π is close to π^a, then $S(\pi)$ is close to 0 and the economy is approximately in autarky. We use the implicit function theorem to find steady states π^L and π^H that are close to 1 and π^a, respectively, for δ close to 0.

Proposition 2.2 *There exist* $\hat{\delta} > 0$ *as well as continuously differentiable functions* $\pi^L(\cdot)$ *and* $\pi^H(\cdot)$ *defined on* $[0, \hat{\delta})$ *such that:*

(1) $\pi^L(\delta)$ *and* $\pi^H(\delta)$ *are steady-state inflation factors for* $\delta \in (0, \hat{\delta})$*;*

(2) $\pi^L(0) = 1$ *and* $\pi^H(0) = \pi^a$*;*

(3) $d\pi^L/d\delta > 0$ *and* $d\pi^H/d\delta < 0$ *for* $\delta \in [0, \hat{\delta})$.

Proof: We may rewrite $\pi = W(\pi, \pi)$ as

$$f(\pi, \delta) \;:=\; S(\pi) - S(\pi)/\pi - \delta \;=\; 0 \;.$$

Then $f(1, 0) = 0$ and $f(\pi^a, 0) = 0$. Furthermore,

$$\begin{aligned} \frac{\partial f}{\partial \pi} &= S'(\pi) - S'(\pi)/\pi + S(\pi)/\pi^2 \;, \\ \frac{\partial f}{\partial \pi}(1, 0) &= S(1) \;>\; 0 \;, \\ \frac{\partial f}{\partial \pi}(\pi^a, 0) &= S'(\pi^a)(1 - 1/\pi^a) \;<\; 0 \;. \end{aligned}$$

Hence, by the implicit function theorem, there is a neighborhood U of 0 and there are continuously differentiable functions π^L and π^H defined on U that satisfy the three properties in the proposition, where the signs of the derivatives depend also on $\partial f/\partial \delta = -1$. □

The functions π^L and π^H for the affine example $S(\pi) = 1 - \pi/2$ are shown in Figure 2.2.[4] Marcet and Sargent (1989) say that the comparative statics of π^L are "classical" because an increase in the budget deficit increases the steady-state inflation factor ($d\pi^L/d\delta > 0$), whereas the comparative statics of π^H are "perverse" because the opposite is true ($d\pi^H/d\delta < 0$). In the OLG model described in Remark 2.1, π^L Pareto dominates π^H.

[4]Their functional forms for $S(\pi) = a - b\pi$ are given in Lettau and Van Zandt (2000, Section 3).

FIGURE 2.2. The steady states π^L and π^H for $S(\pi) = 1 - \pi/2$.

Remark 2.2 Throughout this paper, we will denote the first derivatives of W by $W_1 := \partial W/\partial \pi^e_{t+1}$ and $W_2 := \partial W/\partial \pi^e_t$; these are

$$\begin{aligned} W_1(\pi^e_{t+1}, \pi^e_t) &= -\frac{S(\pi^e_t)S'(\pi^e_{t+1})}{(S(\pi^e_{t+1}) - \delta)^2} > 0 , \\ W_2(\pi^e_{t+1}, \pi^e_t) &= \frac{S'(\pi^e_t)}{S(\pi^e_{t+1}) - \delta} < 0 . \end{aligned}$$

For a steady state $\hat{\pi}$, we can use $W(\hat{\pi}, \hat{\pi}) = \hat{\pi}$ to obtain

$$\begin{aligned} W_1(\hat{\pi}, \hat{\pi}) &= -\hat{\pi}^2 S'(\hat{\pi})/S(\hat{\pi}) , \\ W_2(\hat{\pi}, \hat{\pi}) &= \hat{\pi} S'(\hat{\pi})/S(\hat{\pi}) = -W_1/\hat{\pi} . \end{aligned} \tag{2.2}$$

Because $S'(\pi^a) < 0$ and $S(\pi^a) = 0$,

$$\lim_{\delta \downarrow 0} W_1(\pi^H, \pi^H) = -\lim_{\delta \downarrow 0} W_2(\pi^H, \pi^H) = \infty .$$

On the other hand,

$$-\infty < \lim_{\delta \downarrow 0} W_2(\pi^L, \pi^L) < 0 < \lim_{\delta \downarrow 0} W_1(\pi^L, \pi^L) < \infty .$$

3 Stability

In subsequent sections, we study the stability of the steady states π^L and π^H for various expectations rules. In each case, we are able to obtain a reduced-form model in which there is an endogenous variable θ_t (which typically is π_t or π^e_t) and a law of motion (difference equation) $\{g_t\}_{t=k}^{\infty}$ characterizing the equilibrium paths such that $\{\theta_1, \ldots, \theta_k\}$ are exogenous parameters (initial conditions) and, for $t \geq k$, $\theta_{t+1} = g_t(\theta_t, \theta_{t-1}, \ldots, \theta_{t-k-1})$. We then use fairly standard definitions of stability and instability, which we restate here because of the variety of minor variations in the literature.

Definition 3.1 A steady state $\hat{\theta}$ is *stable* if:

(a) for any neighborhood U_1 of $\hat{\theta}$ there is a neighborhood $U_2 \subset U_1$ such that, if each of the initial conditions is in U_2, then the equilibrium path never leaves U_1; and

(b) there is a neighborhood U of $\hat{\theta}$ such that, if each of the initial conditions is in U, then the equilibrium path converges $\hat{\theta}$.

We refer to the two conditions as "stability(a)" and "stability(b)", respectively. The following table shows equivalent terms as commonly used in economics and mathematics:

Equivalent terminology

Ours	Economics	Mathematics
stability(a)		stability
stability(b)	local stability	
stability	local stability	asymptotic stability

Definition 3.2 A steady state $\hat{\theta}$ is *unstable* if there is a neighborhood U_1 of $\hat{\theta}$ such that every neighborhood $U_2 \subset U_1$ contains an open set of initial conditions for which the equilibrium path leaves U_1.

The "open set" qualification is not standard in such a definition; however, as long as the difference equation is continuous, the existence of any such initial conditions implies the existence of an open set of such initial conditions. Otherwise, this is a standard definition in mathematics. In economics, this is often called "local instability".[5]

When the law of motion is autonomous (time-invariant), an informal interpretation of stability may be that, whenever the system is initially at the steady state and then is subjected to small unmodeled perturbations, it does not diverge too far from the steady state and ultimately converges back to it. If instead the steady state is unstable, then such perturbations, no matter how small, may cause the system to diverge (at least temporarily) from the steady state.

In most cases the reduced form we obtain is autonomous, and we are able to use standard characterizations of stability and instability. Suppose that $g_t = g$ for all t and that g is a first-order difference equation; if it is of higher order, then we first rewrite it as a higher-dimensional first-order equation in the usual way. A sufficient condition for stability is that the magnitude of each eigenvalue of the Jacobian of g is less than 1. A sufficient condition for instability is that the magnitude of one of these eigenvalues is greater than 1.

Remark 3.1 The conditions we derive for stability or instability of steady states are in terms of δ, S, and the expectations rule. These conditions are the easiest to state and interpret when $\delta = 0$, and can then be extended (by continuity) to δ in a neighborhood of 0. Thus, each of the results in this section holds only for δ in some neighborhood of 0.[6] For conciseness, we use the notation "for $\delta \approx 0, \ldots$" or "if $\delta \approx 0$ then $\ldots$" to mean "there is $\bar{\delta} > 0$ such that if $\delta \in (0, \bar{\delta})$ then $\ldots$". If the ellipsis "$\ldots$" includes an expression such as "$f(\delta) \approx k$" then, for any $\epsilon > 0$, $\bar{\delta}$ can be chosen so that $|f(\delta) - k| < \epsilon$ if $\delta \in (0, \bar{\delta})$.

[5] An unstable steady state can also satisfy stability(b), but not stability(a). Chatterji and Chattopadhyay (1998) provide an example of an economic model in which a steady state is both unstable and *globally stable* (for all initial conditions, the equilibrium path converges to the steady state).

[6] This is not a mere technical simplification; for example, Bullard (1994) shows that dynamics in the least squares learning model of Marcet and Sargent (1989), which we consider in Section 7, becomes quite complicated for larger values of δ.

4 Rational Expectations

A price path $\{p_t\}_{t=0}^{\infty}$ is said to be a *rational expectations equilibrium (REE)* if and only if it is an equilibrium for the history-independent expectations rule $\pi_{t+1}^e = \pi_{t+1}$. The equilibrium condition $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$ can then be written

$$S(\pi_{t+1}) = S(\pi_t)/\pi_t + \delta \; . \tag{4.1}$$

On a suitable domain for π_t, we can rewrite (4.1) as $\pi_{t+1} = \Pi(\pi_t)$, where

$$\Pi(\pi) := S^{-1}(S(\pi)/\pi + \delta). \tag{4.2}$$

An inflation path $\{\pi_{t+1}\}_{t=0}^{\infty}$ is then a REE inflation path if and only if $S(\pi_1) > \delta$ and $\pi_{t+1} = \Pi(\pi_t)$ for $t \geq 1$. This claim is stated and proved precisely in Lettau and Van Zandt (2000, Section 4), where the domain of Π is also defined.

Our reduced form under RE is thus the difference equation $\pi_{t+1} = \Pi(\pi_t)$. Although autonomous, it does not make sense under RE to interpret stability as robustness to small perturbations, as after such a shock there is no rule that determines what the expectations should be. Instead, if we parameterize the set of equilibria by the first period's inflation factor (which is an endogenous variable rather than an exogenous historically-determined initial condition), then stability means that, for each neighborhood of the steady state, there is an open set of equilibria for which the path of inflation factors does not leave this neighborhood and converges to the steady state.

Proposition 4.1 *For all $\delta \in (0, \hat{\delta})$, $\pi^H(\delta)$ is stable and $\pi^L(\delta)$ is unstable with respect to RE dynamics.*

PROOF: We show that $\Pi'(\pi^L) > 1$ and $0 < \Pi'(\pi^H) < 1$ when $\delta = 0$, and hence (by continuity) when $\delta \approx 0$. Differentiate (4.1) to find $\Pi'(\cdot)$:

$$S'(\pi_{t+1})d\pi_{t+1} = \left(\frac{S'(\pi_t)}{\pi_t} - \frac{S(\pi_t)}{\pi_t^2}\right) d\pi_t$$
$$\Pi'(\pi_t) = \frac{1}{\pi_t} - \frac{S(\pi_t)}{S'(\pi_t)}\frac{1}{\pi_t^2} \; .$$

Then $\Pi'(1) = 1 - S(1)/S'(1) > 1$ and $\Pi'(\pi^a) = 1/\pi^a < 1$. □

Figure 4.1 shows Π for $S(\pi) = 1 - \pi/2$ and $\delta = 0.04$. Because $\pi^L(\delta)$ and $\pi^H(\delta)$ are the unique steady states in this affine example, there are no equilibrium paths with $\pi_t < \pi^L(\delta)$ and any equilibrium path with $\pi_t > \pi^L(\delta)$ converges monotonically to $\pi^H(\delta)$. Hence, most REE inflation paths converge to π^H.

5 Constant-gain adaptive expectations

5.1 Overview

In this section, we consider constant-gain expectations rules, in which inflation expectations π_{t+1}^e are recursively updated each period t by combining (e.g., averaging) the previous expected inflation factor π_t^e and an observed inflation factor π_t^i using a time-invariant rule. We say that information is *lagged* if $\pi_t^i = \pi_{t-1}$ and that it is *current* if $\pi_t^i = \pi_t$.

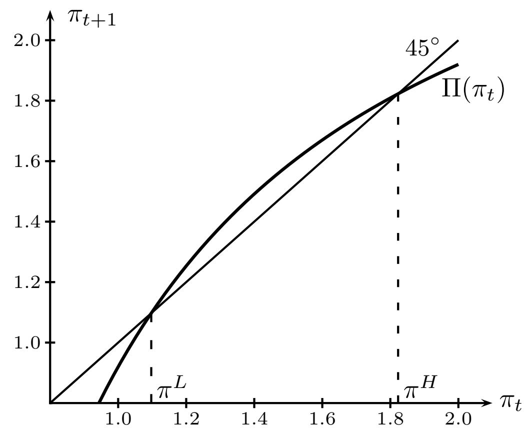

FIGURE 4.1. The dynamic equation of the RE equilibria for $S(\pi) = 1 - \pi/2$ and $\delta = 0.04$.

A principle example is the averaging rule

$$\pi^e_{t+1} \;=\; \alpha\pi^i_t + (1-\alpha)\pi^e_t \;,$$

where $\alpha \in (0,1]$. More generally, we consider rules of the form $\pi^e_{t+1} = \psi(\pi^i_t, \pi^e_t)$, where $\psi\colon \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is assumed to be continuously differentiable, to put positive weight on new information, and to leave expectations unmodified if the observed inflation equals the previously expected inflation. Denoting the first derivatives of ψ by $\psi_{\pi^i} := \partial\psi/\partial\pi^i$ and $\psi_{\pi^e} := \partial\psi/\partial\pi^e$, this assumption may be stated as follows.

Assumption 5.1

(1) For all $\pi \in \mathbb{R}_{++}$, $\psi(\pi,\pi) = \pi$;

(2) ψ is continuously differentiable;

(3) there is a $K \in (0,1)$ such that $1 - K < \psi_{\pi^i}(\pi,\pi) \leq 1$ for all $\pi \in \mathbb{R}_{++}$.

An implication of part (1) is that $\psi_{\pi^i}(\pi,\pi) + \psi_{\pi^e}(\pi,\pi) = 1$; hence, part (3) implies that $0 \leq \psi_{\pi^e}(\pi,\pi) < K$.

Remark 5.1 When $\pi^i_t = \pi_{t-1}$, π^e_1 and π^e_2 are not defined by ψ and are treated as *initial conditions* or parameters. When $\pi^i_t = \pi_t$, the initial condition is π^e_1. In each case, we are able to derive a reduced-form first- or second-order difference equation that involves only $\{\pi^e_t\}^\infty_{t=1}$ and that characterizes the possible paths of $\{\pi^e_t\}^\infty_{t=1}$. We study these reduced-form equations because, for $\hat\pi \in \mathbb{R}_{++}$, $(\hat\pi,\hat\pi)$ is a stable steady state for the system with state variables π_t and π^e_t if and only if $\hat\pi$ is a stable steady state of the reduced-form system with state variable π^e_t.[7]

[7] This is because $\pi_t \to \hat\pi$ if and only if $\pi^e_t \to \hat\pi$, which in turn holds because (i) the properties of ψ imply that if $\pi_t \to \hat\pi$ then $\pi^e_t \to \hat\pi$, and (ii) the properties of W imply that if $\pi^e_t \to \hat\pi$ then $\{\pi_t\}^\infty_{t=1}$ is convergent.

5.2 Lagged information

We begin with the case of lagged information: $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$. Given initial conditions π^e_1 and π^e_2 such that $S(\pi^e_1) > \delta$ and $S(\pi^e_2) > \delta$, $\{\pi^e_t\}^\infty_{t=1}$ and $\{\pi_t\}^\infty_{t=1}$ are equilibrium expected and realized inflation paths if and only if

$$\pi_{t-1} = W(\pi^e_t, \pi^e_{t-1}) \qquad \text{and} \qquad \pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$$

for $t \geq 2$. Combining these two equations, $\{\pi^e_t\}^\infty_{t=1}$ is an equilibrium expected inflation path if and only if $S(\pi^e_1) > \delta$, $S(\pi^e_2) > \delta$, and

$$\pi^e_{t+1} = \psi(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t) \tag{5.1}$$

for $t \geq 2$.

Proposition 5.1 *Assume $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$ for $t \geq 2$, where ψ satisfies Assumption 5.1 and π^e_1 and π^e_2 are initial conditions. Then π^H is unstable for $\delta \approx 0$. Furthermore, π^L is stable for $\delta \approx 0$ if*

$$-\frac{S(1)}{S'(1)} > \psi_{\pi^i}(1,1) \;, \tag{5.2}$$

whereas π^L is unstable for $\delta \approx 0$ if this inequality is reversed.

Proof: See Appendix A. See also Lettau and Van Zandt (2000, Section 8) for the simplest case, in which $\pi^e_{t+1} = \pi_{t-1}$. □

Thus, π^L is unstable when the supply function is sufficiently steep. A steeper S implies that agents decrease their savings more—and hence current inflation is higher—if they expect inflation to be high.

For some parameter values, the stability properties in Proposition 5.1 and Example 2 are the opposite of those under rational expectations dynamics. However, such a comparison has been made before; we are more interested in the comparison with adaptive expectations that use current information.

5.3 Current information

When the expectations rule uses lagged information, there is a single Walrasian equilibrium within each period. This has nothing to do with the assumption that S is downward sloping. Recall that π^e_{t+1} represents the expected terms of trade between period-t and period-$(t+1)$ consumption. When these terms are fixed, so is the real demand $S(\pi^e_{t+1})$ for money by households. The real supply by the government is always fixed at δ, and the nominal supply m_{t-1} going into the period is also fixed. The market-clearing price p_t is simply that which makes the nominal value of the net real demand for money, $p_t(S(\pi^e_{t+1}) - \delta)$, equal to the nominal supply, m_{t-1}.

If households instead use current-period information to update their inflation expectations, then a *higher price* for current consumption raises inflation expectations and hence makes current consumption seem *less dear* compared to tomorrow's consumption. Hence, demand for current consumption rises and *real* demand for money falls when p_t rises. The effect this has on the *nominal* demand for money is ambiguous, since the nominal value of a fixed quantity of real demand rises. There can be multiple prices at which the nominal demand and nominal supply of money are equal.

Specifically, suppose instead that $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$. Then the equilibrium condition $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$ can be written as

$$f(\pi_t, \pi^e_t) \ := \ W(\psi(\pi_t, \pi^e_t), \pi^e_t) - \pi_t \ = \ 0 \ .$$

Let $\varphi(\pi^e_t)$ be the set of equilibrium inflation factors, given π^e_t. Given the initial condition π^e_1, $\{\pi^e_t\}_{t=1}^{\infty}$ and $\{\pi_t\}_{t=1}^{\infty}$ are equilibrium expected and realized inflation paths if and only if $\pi_t \in \varphi(\pi^e_t)$ and $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ for $t \geq 1$. Combining these two conditions, we obtain a reduced-form condition $\pi^e_{t+1} \in \psi(\varphi(\pi^e_t), \pi^e_t)$ for the evolution of π^e_t, but it is not a conventional difference equation because $\pi^e_t \mapsto\!\!\!\!\to \psi(\varphi(\pi^e_t), \pi^e_t)$ is a correspondence rather than a function.

If we define an equilibrium selection F, where $F(\pi^e_t) \in \varphi(\pi^e_t)$, then we obtain a standard difference equation $\pi^e_{t+1} = \psi(F(\pi^e_t), \pi^e_t)$ and thus can define (in)stability in the usual way. If the equilibrium selection picks out the equilibrium point farthest from π^e_t, then stability means that there is a neighborhood of the steady state such that, for every initial condition in this neighborhood and *every* equilibrium path with this initial condition, the inflation factor converges to the steady state. This is rarely satisfied when when multiplicity is truly a problem. For example, if there are multiple equilibria at the steady state, then a path starting in the "steady state" can immediately jump away from it. Stability is more likely to be obtained if the equilibrium selection, in a neighborhood of each steady state, instead picks out an equilibrium that is closest to the steady state. Stability then means roughly that there is a neighborhood of the steady state such that, for every initial condition in this neighborhood, there is *some* equilibrium path with this initial condition that converges to the steady state.

As in Grandmont (1998)—who also allows agents to condition on current information—we adopt the latter approach by defining an equilibrium selection F that is obtained, in a neighborhood of each steady state, by applying the implicit function theorem to $f(\pi_t, \pi^e_t) = 0$. The *instability* results that so derived are robust in the sense that, if π is an unstable steady state for such a selection then it is also unstable—or perhaps not even a steady state—for other selections. On the other hand, one could take issue with a *stability*, given reasons to assume a different selection. This caveat is discussed further in Section 5.4.

Here are the results:[8]

Proposition 5.2 *Assume $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ for $t \geq 1$, where ψ satisfies Assumption 5.1 and π^e_1 is an initial condition. Then π^H is stable for $\delta \approx 0$. Furthermore, if*

$$-\frac{S(1)}{S'(1)} \ < \ \frac{2\psi_{\pi^i}(1,1)}{2 - \psi_{\pi^i}(1,1)} \tag{5.3}$$

then π^L is unstable for $\delta \approx 0$. If inequality (5.3) is reversed, then π^L is stable for $\delta \approx 0$.

Proof: See Appendix A. See also Lettau and Van Zandt (2000, Section 8) for the simplest case, in which $\pi^e_{t+1} = \pi_t$. □

These results indicate that stability under adaptive expectations is more like stability under rational expectations if agents are allowed to use current information in their expectations rule. The high steady state π^H is stable under rational expectations as well as when $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$, but it is unstable when $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$. The results for the low steady

[8] We first studied the basic ideas of this section via an example that is now in Lettau and Van Zandt (2000). This example was also studied independently and contemporaneously by Virasoro (1994).

state are similar but not as clear-cut. Under rational expectations it is unstable. If π^L is stable when information is current, it is also stable when information is lagged; however, if

$$\psi_{\pi^i}(1,1) \ < \ -\frac{S(1)}{S'(1)} \ < \ \frac{2\psi_{\pi^i}(1,1)}{2-\psi_{\pi^i}(1,1)}$$

then, for $\delta \approx 0$, π^L is unstable when information is current but is stable when information is lagged.

5.4 Further discussion equilibrium selection with current information

Including current information in the expectations rule introduces the technical complication of multiple within-period Walrasian equilibria. Does this not in itself justify the common practice of only including lagged information in the expectations rules?

That an assumption simplifies a model does not mean that it makes the model more consistent or a better description of world. The model we are studying is a reduced form of a temporary equilibrium model in which agents must know the current price at the time of market clearing, as this is how they choose nominal money balances given their real money demands. Hence, it makes sense that they also use this information to form expectations.

Perhaps one does not like Walrasian mechanisms, and would instead like to model within-period trade using a Shapley-Shubik (market-order) bidding mechanism, therefore taking price information out of the current period information set. If so, then this should be stated and modeled explicitly.

The mechanism by which current information affects stability is in any case intuitive. A price rise not only increases demand for money because the currency is devalued and hence households have to hold more money in order to store wealth, but it also reduces demand for money because it raises people's inflation expectations and makes them want to store less wealth.

That lagged information guarantees a unique temporary equilibrium mapping is a consequence of the fact that there is a single consumption good each period. Otherwise, there might still be multiple within-period Walrasian equilibria, and lagged versus current information would not buy additional simplicity. This is why the literature on temporary equilibrium with multiple goods typically assumes that agents use current information (e.g. see Grandmont (1998)).

Proposition 5.2 concludes that π^H is stable. However, Lettau and Van Zandt (2000, Section 6) show that implicit equilibrium selection on which this result is based is tatônnement unstable and that, when S is affine, there may be another equilibrium selection that is tatônnement stable and for which π^H is not a steady state.[9] If one requires tatônnement stability as a refinement, then π^H is eliminated, just as in the lagged information case. However, the reasons are completely different. The tatônnement stability argument says that π^H is not even a steady state because of a refinement on the static within-period Walrasian equilibria. If this is the justification for ruling out π^H rather than dynamic stability, then this argument should be made explicitly. Note that there is no indication that tatônnement stability with current information generally leads to the same refinement as dynamic stability with lagged information. This is not even true in the current model, as stability of π^L cannot be restored under current information by invoking this refinement.

[9]We are greatly indebted to Albert Marcet for bringing this fact to our attention. The views expressed here are those of the authors.

6 Expectational stability and slow updating

In this section, we reexamine recursive time-independent expectations rules when these put little weight on the last observation, and we consider time-dependent rules for which the weight on the last observation diminishes to zero.

6.1 Updating with constant but low weight on new information

Recall the expectations rules $\pi^e_{t+1} = \psi(\pi^i_t, \pi^e_t)$ studied in Section 5. Intuitively, if little weight is placed on the last observation (ψ_{π^i} is small), then stability should not depend on whether lagged or current information is used. For the low-inflation steady state π^L, this is easy to see from Propositions 5.1 and 5.2: when $\psi_{\pi^i} \approx 0$, the inequality in equation (5.2) holds and the inequality in equation (5.3) is reverse; hence π^L is stable whether information is lagged or current. However, it is not obvious that the high-inflation steady state is unstable whether information is lagged or current. Proposition 5.2 states that π^H is stable for $\delta \approx 0$ when information is current; the meaning of this result is that, for *fixed* ψ, there is a $\bar{\delta} > 0$ such that π^H is stable for $\delta < \bar{\delta}$. An inspection of the proof of Proposition 5.2 reveals it is also true that, for *fixed* $\delta > 0$, there is an $\bar{\alpha}$ such that π^H is unstable if $\psi_{\pi^i}(\pi^H, \pi^H) < \bar{\alpha}$.

We can also reach these conclusions by using the criterion of expectational stability, introduced by Evans (1985) and used extensively to characterize asymptotic stability in stochastic systems with decreasing-gain learning rules (see Evans and Honkapohja (2000) for an overview). In our model, a steady state is expectationally (un)stable if it is an (un)stable zero of the following differential equation:

$$\frac{d\pi^e}{d\tau} \; = \; W(\pi^e_\tau, \pi^e_\tau) - \pi^e_\tau \; . \tag{6.1}$$

That is, $\hat{\pi}$ is expectationally stable if

$$W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) < 1 \; ,$$

and it is expectationally unstable if this inequality is reversed.

Proposition 6.1 *For $\delta \approx 0$, π^L is expectationally stable and π^H is expectationally unstable.*

PROOF: According to Remark 2.2,

$$W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \; = \; -(\hat{\pi} - 1)S'(\hat{\pi})/S(\hat{\pi}) \; =: \; \Omega(\hat{\pi})$$

at a steady state $\hat{\pi}$. Since $\lim_{\delta \downarrow 0} S(\pi^L) = S(1) > 0$, we have $\lim_{\delta \downarrow 0} \Omega(\pi^L) = 0$ and hence π^L is expectationally stable for $\delta \approx 0$. However, since $\lim_{\delta \downarrow 0} S(\pi^H) = 0$, we have $\lim_{\delta \downarrow 0} \Omega(\hat{\pi}) = \infty$ and hence π^H is expectationally unstable for $\delta \approx 0$. □

One can think of the differential equation (6.1) as a fictitious continuous-time limit of our discrete-time model when the adjustment to the expected inflation factor is proportional to the length of the time period and to the gap between the expected and realized inflation factors. In this case,

$$\begin{aligned} \pi^e_{t+\Delta t} \; &= \; \pi^e_t + \Delta t \left(W(\pi^e_t, \pi^e_{t-\Delta t}) - \pi^e_t \right) \; , \\ \frac{\pi^e_{t+\Delta t} - \pi^e_t}{\Delta t} \; &= \; W(\pi^e_t, \pi^e_{t-\Delta t}) - \pi^e_t \; . \end{aligned}$$

(We are not deriving a true continuous-time limit of our model because we assume that the length of the time period does not affect δ or S.) Since the length of the time period only affects the speed of adjustment, this continuous-time equation should be an approximation to our discrete-time model when the rate of adjustment ψ_{π^i} is small. Thus, that π^L is stable (resp., π^H is unstable) when $\psi_{\pi^i} \approx 0$ should follow from the fact that π^L is expectationally stable (resp., π^H is expectationally unstable).

This is confirmed by deriving such a link for a more general class of models. Here we abuse notation slightly and let W stand for an arbitrary function. Otherwise, the dynamic system is as studied in Section 5, with state variables π_t and π_t^e.

Proposition 6.2 *Let $A \subset \mathbb{R}$ be open, let $W\colon A \times A \to \mathbb{R}$ be continuously differentiable, and let $\psi\colon A \times A \to A$ satisfy Assumption 5.1 (restated for the domain A). Consider the dynamic system with state variables π_t and π_t^e defined by (i) $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$ for $t \geq 1$ and (ii) $\pi_{t+1}^e = \psi(\pi_t^i, \pi_t^e)$. (In the case of* lagged *information, $\pi_t^i = \pi_{t-1}$, equation (ii) holds for $t \geq 2$, and π_1^e and π_2^e are initial conditions; in the case of* current *information, $\pi_t^i = \pi_t$, equation (ii) holds for $t \geq 1$, and π_1^e is an initial condition.) Assume that $\hat{\pi} \in A$ and $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$. If $\hat{\pi}$ is expectationally (un)stable then there is $\bar{\alpha}$ such that $\hat{\pi}$ is (un)stable if $\psi_{\pi^i}(\hat{\pi}, \hat{\pi}) < \bar{\alpha}$.*

Proof: See Appendix B. □

6.2 Diminishing gains

It is also intuitive that if $\hat{\pi}$ is (un)stable for ψ_{π^i} close to zero, then it should be (un)stable when the adjustment of expectations is time dependent and converges to zero, as in the expectations rule

$$\pi_{t+1}^e = \alpha_t \pi_t^i + (1 - \alpha_t)\pi_t^e ,$$

where π_t^i is equal either to π_{t-1} or π_t and where $\alpha_t \to 0$. The only caveat is that the sequence $\{\alpha_t\}$ should not converge too quickly, so that the system does not get stuck at a non-steady state.

This is the gist of Propositions 6.3 and 6.4 below. These results are similar to the use of expectational stability to characterize stability in *stochastic* systems with diminishing updating of expectations. However, that literature cannot be directly adapted to deterministic models. Evans and Honkapohja (1999) contains results for deterministic models that we adapt to our model when, in the proof of Proposition 6.3, $\pi_t^i = \pi_t$. When instead $\pi_t^i = \pi_{t-1}$, our model does not fit their framework. Therefore, we provide an independent proof of Proposition 6.4. We begin by specifying the parts of the model and the assumptions that are common to the two propositions.

Assumption 6.1 *Let $A \subset \mathbb{R}$ be open and let $W\colon A \times A \to \mathbb{R}$ be continuously differentiable. Consider the dynamic system with state variables π_t and π_t^e defined by (i) $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$ for $t \geq 1$ and (ii) $\pi_{t+1}^e = \alpha_t \pi_t^i + (1 - \alpha_t)\pi_t^e$. (In the case of* lagged *information, $\pi_t^i = \pi_{t-1}$, equation (ii) holds for $t \geq 2$, and π_1^e and π_2^e are initial conditions; in the case of* current *information, $\pi_t^i = \pi_t$, equation (ii) holds for $t \geq 1$, and π_1^e is an initial condition.) Assume that $0 < \alpha_t < 1$ for all t, $\alpha_t \to 0$, and $\sum_{t=1}^{\infty} \alpha_t = \infty$. Define a steady state to be $\hat{\pi} \in A$ such that $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$.*

Proposition 6.3 *Consider Assumption 6.1 with current information, and let $\hat{\pi}$ be a steady state. Assume $\alpha_t W_1(\hat{\pi},\hat{\pi}) \neq 1$ for all t. Then $\hat{\pi}$ is stable if it is expectationally stable. Assume also $W_2(\hat{\pi},\hat{\pi}) \neq -(1-\alpha_t)/\alpha_t$ for all t. Then $\hat{\pi}$ is unstable if it is expectationally unstable.*

PROOF: See Appendix B. □

Proposition 6.4 *Consider Assumption 6.1 with lagged information, and let $\hat{\pi}$ be a steady state. If $\{\alpha_t\}$ is weakly decreasing, then $\hat{\pi}$ is stable if it is expectationally stable. If $W_2(\hat{\pi},\hat{\pi}) < 0$, then $\hat{\pi}$ is unstable if it is expectationally unstable.*

PROOF: See Appendix B. □

7 OLS learning revisited

Marcet and Sargent (1989) study the dynamics of this model for the case of affine S, using an expectations rule in which π^e_{t+1} is the OLS estimate of $\bar{\pi}$ for the model

$$p_s = \bar{\pi} p_{s-1} + \epsilon_s \ , \tag{7.1}$$

using price data only up through period $t-1$. In our notation, we can write this expectations rule, which we call "$\text{OLS}_{p_{t-1}}$", as

$$\pi^e_{t+1} = \frac{\sum_{s=-1}^{t-1} p_s p_{s-1}}{\sum_{s=-1}^{t-1} p_{s-1}^2} = \frac{\sum_{s=-1}^{t-1} p_{s-1}^2 \pi_s}{\sum_{s=-1}^{t-1} p_{s-1}^2} \tag{$\text{OLS}_{p_{t-1}}$}$$

for $t \geq 0$, where p_{-2} and p_{-1} are initial conditions. The authors show that π^L is stable and π^H is unstable.

In this section, we generalize the characterization of stability for this expectations rule to the more general supply function S used in this paper, and we also consider three variations of OLS expectations rules. The first is the OLS estimate for the same model in equation (7.1), but including data from period t. This forecasting rule, which we refer to as "OLS_{p_t}", can be written as

$$\pi^e_{t+1} = \frac{\sum_{s=0}^{t} p_s p_{s-1}}{\sum_{s=0}^{t} p_{s-1}^2} = \frac{\sum_{s=0}^{t} p_{s-1}^2 \pi_s}{\sum_{s=0}^{t} p_{s-1}^2} \tag{OLS_{p_t}}$$

for $t \geq 0$, where p_{-1} is an initial condition.

An agent might instead use the OLS estimate of $\bar{\pi}$ for the model

$$\pi_s = \bar{\pi} + \epsilon_s \ , \tag{7.2}$$

using price data up through period $t-1$ in one case and period t in the other. We refer to these rules as "$\text{OLS}_{\pi_{t-1}}$" and "OLS_{π_t}", respectively. Of course, these OLS estimates are just the means of the inflation factors in the data sets,

$$\pi^e_{t+1} = \frac{1}{t+1} \sum_{s=-1}^{t-1} \pi_s \ , \tag{$\text{OLS}_{\pi_{t-1}}$}$$

$$\pi^e_{t+1} = \frac{1}{t+1} \sum_{s=0}^{t} \pi_s \ , \tag{OLS_{π_t}}$$

where π_{-1} and/or π_0 are initial conditions.

Proposition 7.1 *Each of the following stability properties holds for $\delta \approx 0$:*

Rule	π^L	π^H
$OLS_{p_{t-1}}$	*stable*	*unstable*
OLS_{p_t}	*stable*	*stable*
$OLS_{\pi_{t-1}}$	*stable*	*unstable*
OLS_{π_t}	*stable*	*unstable*

(Instability of π^H for $OLS_{\pi_{t-1}}$ also assumes $W_1(\pi^H, \pi^H) \neq t$ for all t.)

Proof: See Appendix C. □

In the rest of this section, we provide some intuition for these results; the details are in Appendix C.

Each of these rules can be written in the form $\pi^e_{t+1} = \alpha_t \pi_t + (1-\alpha_t)\pi^e_t$ or $\pi^e_{t+1} = \alpha_t \pi_{t-1} + (1-\alpha_t)\pi^e_t$, as follows:

$$(\mathrm{OLS}_{p_{t-1}}) \qquad \pi^e_{t+1} = \frac{p^2_{t-2}}{\sum_{s=-1}^{t-1} p^2_{s-1}} \pi_{t-1} + \frac{\sum_{s=-1}^{t-2} p^2_{s-1}}{\sum_{s=-1}^{t-1} p^2_{s-1}} \pi^e_t \ ,$$

$$(\mathrm{OLS}_{p_t}) \qquad \pi^e_{t+1} = \frac{p^2_{t-1}}{\sum_{s=0}^{t} p^2_{s-1}} \pi_t + \frac{\sum_{s=0}^{t-1} p^2_{s-1}}{\sum_{s=0}^{t} p^2_{s-1}} \pi^e_t \ ,$$

$$(\mathrm{OLS}_{\pi_{t-1}}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \pi_{t-1} + \frac{t}{t+1} \pi^e_t \ ,$$

$$(\mathrm{OLS}_{\pi_t}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \pi_t + \frac{t}{t+1} \pi^e_t \ .$$

Consider first $\mathrm{OLS}_{\pi_{t-1}}$ and OLS_{π_t}, for which the α_t are history-independent, sum to ∞, and converge to 0 (this case was studied in Section 6). According to Propositions 6.3 and 6.4, a steady state $\hat{\pi}$ is asymptotically (un)stable if it is expectationally (un)stable. According to Proposition 6.1, π^L is expectationally stable and π^H is not.

Now consider $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t}. Since the α_t are history dependent, we cannot directly apply the results of the preceding sections. Still, there is an interesting relationship between those results and the stability properties of $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t}. Observe that equation (7.1), when divided through by p_{s-1}, yields

$$(7.3) \qquad \pi_s = \bar{\pi} + \epsilon/p_{s-1} \ .$$

Given the OLS assumption that the disturbances $\{\epsilon_s\}_{s=2}^{\infty}$ are i.i.d., the difference between models (7.3) and (7.2) is that the former views the variance of the disturbances to the inflation rates as inversely proportional to the square of the previous period's price level. For this reason, if prices are rising, then $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t} put more weight on recent than on older observations of the inflation factor. If the inflation factor is above and bounded away from 1, then α_t is bounded away from 0. In particular, if $\pi_t \to \hat{\pi} \geq 1$ then $\alpha_t \to 1 - \hat{\pi}^{-2} =: \alpha_{\hat{\pi}}$.[10]

Consider a steady state $\pi \in \{\pi^L, \pi^H\}$. As long as $\hat{\pi} > 1$, so that $\alpha_{\hat{\pi}} > 0$, intuitively the stability of the steady state should be the same as for the constant-gain expectations rule

$$\pi^e_{t+1} = \alpha_{\hat{\pi}} \pi_{t-1} + (1-\alpha_{\hat{\pi}})\pi^e_t \qquad (\text{for } \mathrm{OLS}_{p_{t-1}})$$
$$\pi^e_{t+1} = \alpha_{\hat{\pi}} \pi_t + (1-\alpha_{\hat{\pi}})\pi^e_t \qquad (\text{for } \mathrm{OLS}_{p_t}) \ .$$

[10] A proof of this formula for the limit is in Marcet and Sargent (1989); see also Lettau and Van Zandt (2000, Section 7).

Since $\pi^L < \pi^H$, it follows that $\alpha_{\pi^L} < \alpha_{\pi^H}$; hence, the implicit assumption about the disturbances implies that $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t} place greater weight on recent information around the steady state π^H than around the steady state π^L. For $\delta \approx 0$, $\pi^H = \pi^a$ and $\alpha_{\pi^H} \approx 1 - (\pi^a)^{-2} > 0$. Propositions 5.1 and 5.2 therefore suggest that π^H is unstable for $\mathrm{OLS}_{p_{t-1}}$ and stable for OLS_{p_t}, as confirmed in our proof. In contrast, as $\delta \downarrow 0$, both $\pi^L \downarrow 1$ and $\alpha_{\pi^L} \downarrow 0$. When information is lagged, Propositions 5.1 and 6.1 both suggest that π^L is stable (also confirmed in our proof). However, these results do not immediately indicate whether π^L is stable when information is current (OLS_{p_t}): according to Proposition 5.2, for fixed α, π^L is stable if δ is close enough to 0; according to Proposition 6.1, for fixed δ, π^L is stable if α is close enough to 0. Our proof shows that the effect of low α dominates and hence π^L is stable.

8 Continuous-time models

The distinction between using current and one-period-old information cannot arise in a continuous-time model. This raises the question of whether the lagged- and current-information versions of our constant-gain model converge to the same continuous-time limit, and what this limiting model would look like. Rather than provide a proper answer to this question by deriving the limiting model, we compare our discrete-time model with a "continuous-time analog" that has already been studied in the literature by Evans and Yarrow (1981), Sargent and Wallace (1987), and Bruno and Fischer (1990). This model introduces a constant real debt financed by seigniorage to the continuous-time inflation model of Cagan (1956).

The model, which is summarized in Blanchard and Fischer (1989, Section 4.7), has been studied most commonly with exponential demand for real money balances, as in $S(\pi^e) = a\, e^{-c\pi^e}$, where π^e is the expected instantaneous inflation rate. In this case, there is a low and a high-inflation steady state, one and only one of which is stable. The low (resp. high) one is stable when the rate of expectations adjustment is low (resp. high). However, the exponential S does not satisfy the assumptions in the current paper, as it never reaches zero. Our analysis of the high-inflation steady state relies very much on the fact that there is a finite autarkic inflation factor.

In Lettau and Van Zandt (2000, Section 10), we explicitly characterize the existence and stability of low and high steady states in the continuous-time model under Assumption 2.1 on S (restated appropriately for the interpretation that π is an instantaneous inflation rate). The following proposition paraphrases the results of that section:

Proposition 8.1 *In a continuous-time analog of our model, there are low and high steady states π^L and π^H near 1 and π^a, respectively, for $\delta \approx 0$. π^L is unstable (resp., stable) if the weight put on recent information is large (resp., small) relative to $-S(\pi^L)/S'(\pi^L)$. π^H is stable.*

The comparison of our discrete-time model and the continuous-time analog is inconclusive for the low steady state, because the properties stated in Proposition 8.1 that determine (in)stability of π^L hold qualitatively in the discrete-time model whether agents use current or lagged information. However, stability of π^H unambiguously matches our constant-gain model with *current* information.

9 What do we mean by current information?

Whether we view agents as using "current" information depends on what variables we consider the agents to be updating and predicting. For example, in the expectations rules we have studied, agents use the current price to predict the next-period price. They calculate π^e_t from past and/or current information, then predict p_{t+1} to be $\pi^e_{t+1}p_t$.

Suppose instead that agents use a rule $\pi^e_{t+1} = a + b\pi_t$, where a and b are parameters that are calculated from past and/or current data. The agent uses current information about inflation to generate inflation expectations, even if only old data is used to update a and b. One may expect that stability will have the qualitative properties we derived for current information. This conclusion is not trivial because the expectations rule is overspecified with respect to steady state inflation factors, meaning that multiple values of the parameters can be consistent with a steady state (e.g., $a = \hat{\pi}$ and $b = 0$ or $a = 0$ and $b = 1$).

The sensitivity of expectational stability to the over specification is considered in Evans (1985, 1989). It is studied by Duffy (1994) for this macroeconomic model but without government deficits. With respect to the parameters a and b, all rational expectations equilibrium paths are steady states. Duffy finds that all are expectationally stable except the no-inflation steady state, which is the only one that does not converge to autarky. Duffy shows that the steady state values of b are greater than zero (except for the one corresponding to no inflation); this is consistent with our finding that putting weight on current inflation information makes the high-inflation steady state stable.

10 Conclusion

Both active researchers in and observers of the literature on stability under adaptive expectations in macroeconomic models are aware that changes in expectations rules affect the stability of steady states. Such nonrobustness is a fact of life when rationality and fulfilled expectations, whose specification is typically derived from deductive principles, are replaced by realistic models of boundedly rational behavior, the choice of which is essentially an empirical question. The indeterminacy that arises in models of rational expectations (or, e.g., of equilibria in games) is replaced by indeterminacy about the proper specification of expectations (or, e.g., of reputation or learning in games). Yet the development of such models helps us to understand how various kinds of human behavior lead to different outcomes.

Thus, the main results of this paper (outlined in the Introduction) are not intended to uncover a smoking gun of nonrobust models. Rather, the exercise provides concrete examples of nonrobustness in order to help us understand what factors affect stability. In particular, we show that the assumption that agents use lagged rather than current information should not be made casually and should not be justified solely by the simplification that such an assumption allows.

We also found this exercise useful for understanding the existing literature because we were able to experiment with a variety of expectations rules—similar to ones that have been used in this literature—in the context of a single simple macroeconomic model. We hope that some readers also benefit in this way.

A Proofs for constant-gain recursive expectations

Proof of Proposition 5.1: As explained prior to the statement of Proposition 5.1, it suffices to study the stability of steady states of the following difference equation:

$$\pi^e_{t+1} \;=\; \psi(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t) \;=:\; g(\pi^e_t, \pi^e_{t-1}) \; . \tag{A.1}$$

Let $g_{\pi^e_t}$ and $g_{\pi^e_{t-1}}$ be the partial derivates of g. When $g_{\pi^e_t}$ and $g_{\pi^e_{t-1}}$ are evaluated at a steady state $\hat{\pi}$, the characteristic equation for the difference equation linearized around $\hat{\pi}$ is

$$x^2 - g_{\pi^e_t} x - g_{\pi^e_{t-1}} \;=\; 0 \; . \tag{A.2}$$

Note that $\hat{\pi}$ is stable if both roots of this equation lie in the unit circle and is unstable if either root lies outside the unit circle.

Evaluated at any (π^e_t, π^e_{t-1}), the derivatives of g are

$$\begin{aligned} g_{\pi^e_t} \;&=\; \psi_{\pi^i} W_1 + \psi_{\pi^e} \;>\; 0 \; , \\ g_{\pi^e_{t-1}} \;&=\; \psi_{\pi^i} W_2 \;<\; 0 \; , \end{aligned}$$

where ψ_{π^i} and ψ_{π^e} are evaluated at $(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t)$.

Consider first the steady state π^H, at which all derivatives are now evaluated. Note that a root of the characteristic equation (A.2) lies outside the unit circle if $g_{\pi^e_t} > 2$; since $\psi_{\pi^e} \geq 0$, a sufficient condition is that $\psi_{\pi^i} W_1 > 2$. By Assumption 5.1, ψ_{π^i} is bounded away from zero; by Remark 2.2, $\lim_{\delta \downarrow 0} W_1(\pi^H, \pi^H) = \infty$. Hence, for $\delta \approx 0$, $\psi_{\pi^i} W_1 > 2$ and so π^H is unstable.

Consider now the steady state π^L. Let $A(\delta) := \psi_{\pi^i} W_1$, $B(\delta) := \psi_{\pi^e}$, and $C(\delta) := -\psi_{\pi^i} W_2$, with each derivative evaluated at π^L (hence the dependence on δ). Then the characteristic equation can be written as

$$x^2 - (A(\delta) + B(\delta))x + C(\delta) \;=\; 0 \; . \tag{A.3}$$

Let $A := \lim_{\delta \downarrow 0} A(\delta)$, and define B and C similarly. The roots depend continuously on the coefficients of the quadratic equation, so if the roots of $x^2 - (A+B)x + C$ are both inside the unit circle (resp., if one root lies outside the unit circle) then the same holds for equation (A.3) when $\delta \approx 0$.

Observe that $A = C = -\psi_{\pi^i}(1,1)S'(1)/S(1)$, and hence the relevant equation is $x^2 - (A+B)x + A$. In Lettau and Van Zandt (2000, Section 5.2) we show that, because $A > 0$ and $0 \leq B < 1$, the roots of this quadratic equation lie inside the unit circle if $A < 1$ and at least one of the roots lies outside the unit circle if $A > 1$. Note that $A < 1$ is just the condition $S(1) > -S'(1)\psi_{\pi^i}(1,1)$ stated in the proposition. □

Proof of Proposition 5.2: The proof begins with the discussion of equilibrium selections that precedes Proposition 5.2. Recall that the period-t equilibrium condition is

$$f(\pi_t, \pi^e_t) \;:=\; W(\psi(\pi_t, \pi^e_t), \pi^e_t) - \pi_t \;=\; 0 \; . \tag{A.4}$$

We let F be an equilibrium selection that, in a neighborhood of the steady states π^L and π^H, selects the equilibrium closest to the steady state (or to the previous inflation expectations). Specifically, F is defined in a neighborhood of each of these steady states by application of the implicit function theorem. The dynamic system thus becomes $\pi_t = F(\pi^e_t)$ and $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ for $t \geq 2$. Combining these, we obtain a single equation $\pi^e_{t+1} = \psi(F(\pi^e_t), \pi^e_t) =:$

$g(\pi_t^e)$ governing $\{\pi_t^e\}_{t=1}^{\infty}$. As explained in Remark 5.1, it suffices to characterize the stability of π^L and π^H as steady states of this reduced-form system.

Let $\hat{\pi}$ be a steady state and let f_π and f_{π^e} denote the partial derivatives of f. In what follows, partial derivatives are evaluated at $\pi_t = \pi_t^e = \hat{\pi}$, and their arguments are omitted for clarity. We thus have

$$\begin{aligned} f_\pi &= W_1\psi_{\pi^i} - 1 \ , \\ f_{\pi^e} &= W_1\psi_{\pi^e} + W_2 \ . \end{aligned}$$

As long as $f_\pi \neq 0$, there is a neighborhood of $\hat{\pi}$ on which F coincides with a function obtained by applying the implicit function theorem to $f(\pi_t, \pi_t^e) = 0$ at $\pi_t = \pi_t^e = \pi$. It follows that F is differentiable at $\hat{\pi}$ and that $F'(\hat{\pi}) = -f_{\pi^e}/f_\pi$. Hence, g is differentiable at $\hat{\pi}$ and

$$g'(\hat{\pi}) \ = \ F'(\hat{\pi})\psi_{\pi^i} + \psi_{\pi^e} \ = \ -\frac{W_1\psi_{\pi^e} + W_2}{W_1\psi_{\pi^i} - 1}\psi_{\pi^i} + \psi_{\pi^e} \ = \ -\frac{W_2\psi_{\pi^i} + \psi_{\pi^e}}{W_1\psi_{\pi^i} - 1} \ .$$

Into the right-hand side we substitute the expressions (for W_1 and W_2) found in Remark 2.2, and so obtain

$$g'(\hat{\pi}) \ = \ \frac{\hat{\pi}S'(\hat{\pi})\psi_{\pi^i} + S(\hat{\pi})\psi_{\pi^e}}{\hat{\pi}^2 S'(\hat{\pi})\psi_{\pi^i} + S(\hat{\pi})} \ . \tag{A.5}$$

Thus, $\hat{\pi}$ is a stable steady state of g if $|g'(\hat{\pi})| < 1$ and is unstable if $|g'(\hat{\pi})| > 1$. If $f_\pi = 0$ and $f_{\pi^e} \neq 0$, then for π_t^e close to π, there are no solutions to $f(\pi, \pi) = 0$ as close as π_t^e to π; hence π is unstable.

Consider the steady state π^L for $\delta \approx 0$. Then $\pi^L \approx 1$ and

$$g'(\pi^L) \ \approx \ \frac{S'(1)\psi_{\pi^i} + S(1)\psi_{\pi^e}}{S'(1)\psi_{\pi^i} + S(1)} \ =: \ \frac{A+B}{A+C} \ ,$$

where $A := S'(1)\psi_{\pi^i}$, $B := S(1)\psi_{\pi^e}$, and $C := S(1)$. By assumption, $\psi_{\pi^e} < 1$ and hence $B < C$. One can therefore show (see Lettau and Van Zandt (2000, Section 5.3)) that $|(A+B)/(A+C)| > 1$ if and only if $-A - B > A + C$, that is,

$$-2S'(1)\psi_{\pi^i} \ > \ S(1)(1 + \psi_{\pi^e}) \ . \tag{A.6}$$

Thus, inequality (A.6) is a sufficient condition for instability of π^L when $\delta = 0$ and, by continuity of the derivatives, for $\delta \approx 0$. Substituting $\psi_{\pi^e} = 1 - \psi_{\pi^i}$ and rearranging yields the inequality in equation (5.3) of the proposition. Similarly, if this inequality (A.6) is reversed, then π^L is stable for $\delta \approx 0$ as long as $f_\pi(1,1) \neq 0$, that is, $S(1) + S'(1)\psi_{\pi^i}(1,1) \neq 0$. This latter condition is implied by the reversal of inequality (A.6).

Now consider the steady state π^H. Then $\pi^H \approx \pi^a$ and $S(\pi^H) \approx 0$ and so $g'(\pi^H) \approx 1/\pi^a < 1$. Hence, π^H is stable for $\delta \approx 0$. □

B Proofs for expectational stability

Proof of Proposition 6.2: Remark 5.1 applies, so one can study reduced form systems involving only π_t^e.

Consider first the case of $\pi_{t+1}^e = \psi(\pi_{t-1}, \pi_t^e)$. As in the proof of Proposition 5.1, we have that

(1) π_t^e is governed by the difference equation $\pi_{t+1}^e = g(\pi_t^e, \pi_{t-1}^e)$, where $g(\pi_t^e, \pi_{t-1}^e) = \psi(W(\pi_t^e, \pi_{t-1}^e), \pi_t^e)$;

(2) the partial derivatives of g are $g_{\pi_t^e} = \psi_{\pi^i} W_1 + \psi_{\pi^e}$ and $g_{\pi_{t-1}^e} = \psi_{\pi^i} W_2$; and

(3) the characteristic equation for the difference equation linearized around the steady state $\hat{\pi}$ is $x^2 - g_{\pi_t^e} x - g_{\pi_{t-1}^e} = 0$, which can be written (using that, for any π, $\psi_{\pi^i}(\pi, \pi) + \psi_{\pi^e}(\pi, \pi) = 1$) as

$$\chi(x; \psi_{\pi^i}) \ := \ x^2 - (\psi_{\pi^i} W_1 + 1 - \psi_{\pi^i}) x - \psi_{\pi^i} W_2 \ = \ 0 \ .$$

When $\psi_{\pi^i} = 0$ this characteristic equation is $x^2 - x = 0$ and the roots are 0 and 1. The roots depend continuously on ψ_{π^i} and so, for $\psi_{\pi^i} \approx 0$, the two roots are ≈ 0 and ≈ 1. We must check whether the root close to 1 is less than 1 (then $\hat{\pi}$ is stable) or greater than 1 (then $\hat{\pi}$ is unstable). To do so, we consider the root $\hat{x}(\psi_{\pi^i})$ given by the implicit function theorem around $\psi_{\pi^i} = 0$ and $x = 1$. Since

$$\begin{aligned} \frac{\partial \chi}{\partial x}(x, \psi_{\pi^i}) \ &= \ 2x - \psi_{\pi^i} W_1 - 1 + \psi_{\pi^i} \ , \\ \frac{\partial \chi}{\partial \psi_{\pi^i}}(x, \psi_{\pi^i}) \ &= \ -W_1 x + x - W_2 \ , \end{aligned}$$

we have

$$\hat{x}'(0) \ = \ -\frac{\frac{\partial \chi}{\partial x}(1,0)}{\frac{\partial \chi}{\partial x}(1,0)} \ = \ W_1 + W_2 - 1 \ .$$

If $\hat{\pi}$ is expectationally stable, then $\hat{x}'(0) < 0$; hence, and for $\psi_{\pi^i} \approx 0$, $\hat{x}(\psi_{\pi^i}) < 1$ and $\hat{\pi}$ is stable. If, instead, $\hat{\pi}$ is expectationally unstable, then $\hat{x}'(0) > 0$; hence, and for $\psi_{\pi^i} \approx 0$, $\hat{x}(\psi_{\pi^i}) > 1$ and $\hat{\pi}$ is unstable.

Consider next the case $\pi_{t+1}^e = \psi(\pi_t, \pi_t^e)$. Here we have an equilibrium selection problem as discussed in Section 5.3, and we follow the approach outlined there. Our proof initially parallels that of Proposition 5.1.

The period-t equilibrium condition is

$$f(\pi_t, \pi_t^e) \ = \ W(\psi(\pi_t, \pi_t^e), \pi_t^e) - \pi_t \ = \ 0 \ .$$

For a steady state $\hat{\pi}$, $f_\pi(\hat{\pi}, \hat{\pi}) = W_1 \psi_{\pi^i} - 1$; hence, for $\psi_{\pi^i} \approx 0$, $f_\pi(\hat{\pi}, \hat{\pi}) \neq 0$. Thus, by the implicit function theorem, there is an equilibrium selection $F(\pi_t^e)$ such that $\hat{\pi} = F(\hat{\pi})$, F is differentiable at $\hat{\pi}$, and

$$F'(\hat{\pi}) \ = \ -\frac{f_{\pi^e}(\hat{\pi}, \hat{\pi})}{f_\pi(\hat{\pi}, \hat{\pi})} \ = \ -\frac{W_1 \psi_{\pi^e} + W_2}{W_1 \psi_{\pi^i} - 1} \ .$$

The dynamic system governing π_t^e in a neighborhood of $\hat{\pi}$ is $\pi_{t+1}^e = \psi(F(\pi_t^e), \pi_t^e) =: g(\pi_t^e)$. Then

$$\begin{aligned} g'(\hat{\pi}) \ = \ \psi_{\pi^i} F' + \psi_{\pi^e} \ &= \ \frac{-\psi_{\pi^i} W_1 \psi_{\pi^e} - W_2 \psi_{\pi^i} + \psi_{\pi^e} W_1 \psi_{\pi^i} - \psi_{\pi^e}}{W_1 \psi_{\pi^i} - 1} \ , \\ &= \ \frac{-W_2 \psi_{\pi^i} - \psi_{\pi^e}}{W_1 \psi_{\pi^i} - 1} \ = \ \frac{1 - \psi_{\pi^i} + W_2 \psi_{\pi^i}}{1 - \psi_{\pi^i} W_1} \end{aligned}$$

and $g'(\hat{\pi}) > 0$ for $\psi_{\pi^i} \approx 0$. We have also $g'(\hat{\pi}) < 1$ (hence $\hat{\pi}$ is stable) if

$$\begin{aligned} 1 - \psi_{\pi^i} + W_2 \psi_{\pi^i} \ &< \ 1 - \psi_{\pi^i} W_1 \\ 0 \ &< \ \psi_{\pi^i}(1 - W_1 - W_2) \ . \end{aligned}$$

This holds if $\hat{\pi}$ is expectationally stable and hence $1 - W_1 - W_2 > 0$. Similarly, if $\hat{\pi}$ is expectationally unstable, then $g'(\hat{\pi}) > 1$ and $\hat{\pi}$ is unstable. □

Proof of Proposition 6.3: Evans and Honkapohja (1999) study a (multidimensional) system of the form

$$\pi^e_{t+1} \;=\; \alpha_t F(\pi^e_t, \alpha_t) + (1-\alpha_t)\pi^e_t \;,$$

where the sequence $\{\alpha_t\}$ satisfies the assumptions of Proposition 6.3 and F satisfies certain assumptions to be described shortly. Our system can be written in this form when $\pi^i_t = \pi_t$ and when F is an equilibrium selection—that is, $F(\pi^e;\alpha)$ is a solution π to

$$f(\pi, \pi^e; \alpha) \;:=\; W(\alpha\pi + (1-\alpha)\pi^e, \pi^e) - \pi \;=\; 0$$

for any $\pi^e \in A$ and $\alpha \in [0,1)$. In the rest of this proof, we explain how to apply their results.

Observe that f is continuously differentiable, even for negative α, as long as $\alpha\pi + (1-\alpha)\pi^e \in A$. Hence, since A is open, for any steady state $\hat{\pi}$ there is a neighborhood of $(\hat{\pi}, \hat{\pi}, 0)$ in $A \times A \times \mathbb{R}$ on which f is continuously differentiable, and

$$\begin{aligned} f_\pi(\hat{\pi}, \hat{\pi}; 0) &= -1 \;, \\ f_{\pi^e}(\hat{\pi}, \hat{\pi}; 0) &= W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \;, \\ f_\alpha(\hat{\pi}, \hat{\pi}; 0) &= 0 \;. \end{aligned}$$

By the implicit function theorem, we can choose an equilibrium selection F that is continuously differentiable in a neighborhood U of $(\hat{\pi}, 0)$, with

$$\begin{aligned} F_{\pi^e}(\hat{\pi}; 0) &= W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \;, \\ F_\alpha(\hat{\pi}; 0) &= 0 \;. \end{aligned}$$

Furthermore, we can choose F so that $F(\hat{\pi}, \alpha) = \hat{\pi}$ for α such that $(\hat{\pi}, \alpha) \in U$.

There may be finitely many periods t such that $(\hat{\pi}, \alpha_t)$ is not in U. For such t, we note that

$$\begin{aligned} f_\pi(\hat{\pi}, \hat{\pi}, \alpha_t) &= \alpha_t W_1(\hat{\pi}, \hat{\pi}) - 1 \;, \\ f_{\pi^e}(\hat{\pi}, \hat{\pi}; \alpha_t) &= (1-\alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \;. \end{aligned}$$

By assumption, $f_\pi(\hat{\pi}, \hat{\pi}, \alpha_t) \neq 0$. Hence, we can invoke the implicit function theorem for each of these periods to choose F so that (a) $F(\hat{\pi}, \alpha_t) = \hat{\pi}$, (b) F is continuously differentiable in a neighborhood of $(\hat{\pi}, \alpha_t)$, and (c)

$$F_{\pi^e}(\hat{\pi}, \alpha_t) \;=\; \frac{(1-\alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})}{1 - \alpha_t W_1(\hat{\pi}, \hat{\pi})} \;.$$

Evans and Honkapohja (1999) assume that (a) F is continuously differentiable in a neighborhood of $(\hat{\pi}, 0)$ and (b) for all t, $F(\hat{\pi}, \alpha_t) = \hat{\pi}$ and F is continuous in a neighborhood of $(\hat{\pi}, \alpha_t)$. We have shown that these conditions are satisfied.

Their Proposition 1 states that $\hat{\pi}$ is stable(b) (see Definition 3.1) if $F_{\pi^e}(\hat{\pi}; 0) < 1$; since $F_{\pi^e}(\hat{\pi}; 0) = W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})$, this condition is equivalent to expectational stability. An inspection of their proof indicates that they have also shown that $\hat{\pi}$ is stable, rather than merely stable(b).

Their Proposition 2 states that $\hat{\pi}$ is unstable if $F_{\pi^e}(\hat{\pi}, 0) > 1$ (i.e., if $\hat{\pi}$ is expectationally unstable) and if $F_{\pi^e}(\hat{\pi}, \alpha_t) \neq -(1-\alpha_t)/\alpha_t$ for all t. The latter condition is

$$\begin{aligned} \frac{(1-\alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})}{1 - \alpha_t W_1(\hat{\pi}, \hat{\pi})} &\neq -\frac{1-\alpha_t}{\alpha_t} \;, \\ \alpha_t(1-\alpha_t) W_1(\hat{\pi}, \hat{\pi}) + \alpha_t W_2(\hat{\pi}, \hat{\pi}) &\neq -(1-\alpha_t) + \alpha_t(1-\alpha_t) W_1(\hat{\pi}, \hat{\pi}) \;, \\ W_2(\hat{\pi}, \hat{\pi}) &\neq -\frac{1-\alpha_t}{\alpha_t} \;, \end{aligned}$$

which we assumed for the instability result. □

The following lemma is used in the proof of Proposition 6.4, and is proved in Lettau and Van Zandt (2000, Section 9).

Lemma B.1 *Suppose* $\{\alpha_t\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{t\to\infty}\alpha_t = 0$ *and* $\sum_{t=1}^{\infty}\alpha_t = \infty$. *Let* $K \in \mathbb{R}$ *and let* $y_s := \prod_{t=1}^{s}(1+\alpha_t K)$. *Then* $\lim_{s\to\infty} y_s = 0$ *if* $-1 < K < 0$ *and* $\lim_{s\to\infty} y_s = \infty$ *if* $K > 0$.

Proof of Proposition 6.4: Consider now the case of lagged information, $\pi^e_{t+1} = \alpha_t\pi_{t-1} + (1-\alpha_t)\pi^e_t$. Then $\{\pi^e_t\}$ follows the difference equation

$$\pi^e_{t+1} \;=\; \alpha_t W(\pi^e_t, \pi^e_{t-1}) + (1-\alpha_t)\pi^e_t \;. \tag{B.1}$$

In two ways this is simpler than the case of current information: (a) we need not derive an equilibrium selection and its differentiability properties, and (b) the function W (which replaces F in the proof of Proposition 6.3) does not depend on α_t. However, because W depends on both π^e_t and π^e_{t-1}, this system does not fit the class studied by Evans and Honkapohja (1999).

Specifically, suppose we write equation (B.1) as a two-dimensional first-order equation

$$\begin{pmatrix}\pi^e_{t+1}\\ \pi^e_t\end{pmatrix} \;=\; G_t\begin{pmatrix}\pi^e_t\\ \pi^e_{t-1}\end{pmatrix} \;:=\; \begin{pmatrix}\alpha_t W(\pi^e_t,\pi^e_{t-1}) + (1-\alpha_t)\pi^e_t\\ \pi^e_t\end{pmatrix}.$$

Let M_t be the Jacobian of G_t evaluated at the steady state $(\hat\pi,\hat\pi)$:

$$M_t \;:=\; \begin{pmatrix}1-\alpha_t+\alpha_t W_1 & \alpha_t W_2\\ 1 & 0\end{pmatrix}, \tag{B.2}$$

where W_1 and W_2 are evaluated at $(\hat\pi,\hat\pi)$. In the proofs of Evans and Honkapohja (1999), it is important that the time-variant term α_t can be separated in the form $M_t = I + \alpha_t J$, where I is the identity matrix and J is a time-invariant matrix. This is not possible here, and so we provide our own proof.

As usual, the main arguments of the proof concern the linear approximation, and then additional arguments show that the residual does not alter the conclusions. Denote the residual of the linearization of W around $(\hat\pi,\hat\pi)$ by r. Then

$$\pi^e_{t+1} - \hat\pi \;=\; (1-\alpha_t+\alpha_t W_1)(\pi^e_t - \hat\pi) + \alpha_t W_2(\pi^e_{t-1} - \hat\pi) + \alpha_t r(\pi^e_t, \pi^e_{t-1}) \;. \tag{B.3}$$

The residual $r(\pi^e_t,\pi^e_{t-1})$ satisfies the following Lipschitz condition: for all $k > 0$, there is $\epsilon > 0$ such that $|r(\pi^e_t,\pi^e_{t-1})| \le k(|\pi^e_t| + |\pi^e_{t-1}|)$ if $|\pi^e_t| < \epsilon$ and $|\pi^e_{t-1}| < \epsilon$. We will choose k below; ϵ is then selected accordingly and U_ϵ denotes the ϵ-ball around 0.

Stability We write the difference equation (B.1) as a two-dimensional first-order difference equation, linearized and with a substitution of variables so that the steady state is $(0,0)$:

$$\theta_{t+1} \;=\; M_t\theta_t + R_t \;,$$

where M_t is the Jacobian of G evaluated at $(\hat\pi,\hat\pi)$,

$$\theta_t \;:=\; \begin{pmatrix}\pi^e_t - \hat\pi\\ \pi^e_{t-1} - \hat\pi\end{pmatrix} \qquad \text{and} \qquad R_t \;=:\; \begin{pmatrix}\alpha_t r(\pi^e_t,\pi^e_{t-1})\\ 0\end{pmatrix}.$$

Let $M(\alpha)$ be the matrix in equation (B.2), substituting α for α_t, so that we can write $M_t = M(\alpha_t)$ and thereby emphasize that M_t depends on t only through α_t. The first property of $M(\alpha)$ that we need is the following, which will be proved below:

Lemma B.2 *There is $\bar{\alpha}$ such that the eigenvectors of $M(\alpha)$ are linearly independent and depend continuously on α for $\alpha \in [0, \bar{\alpha}]$.*

For each t, let S_t be the matrix whose columns are the eigenvectors of M_t and let Λ_t be the diagonal matrix whose diagonal entries are the eigenvalues of M_t. Let τ be such that $\alpha_t \leq \bar{\alpha}$ for $t \geq \tau$. Then for $t \geq \tau$, it follows from Lemma B.2 that M_t can be diagonalized as $M_t = S_t \, \Lambda_t \, S_t^{-1}$ and hence our difference equation can be written $\theta_{t+1} = S_t \, \Lambda_t \, S_t^{-1} \, \theta_t + R_t$. Multiply this equation by S_{t+1}^{-1} to obtain

$$S_{t+1}^{-1} \theta_{t+1} \; = \; S_{t+1}^{-1} \, S_t \, \Lambda_t \, S_t^{-1} \theta_t + S_{t+1}^{-1} \, R_t \; .$$

Define $\zeta_t := S_t^{-1} \theta_t$ and $\Gamma_t := S_{t+1}^{-1} \, S_t \, \Lambda_t$. Then $\zeta_{t+1} = \Gamma_t \zeta_t + S_{t+1}^{-1} \, R_t$. If $\|\cdot\|$ denotes a norm on $\mathbb{R}^2$ and a matching linear operator norm on matrices in $\mathbb{R}^{2 \times 2}$, then

$$\|\zeta_{t+1}\| \; = \; \|\Gamma_t\| \cdot \|\zeta_t\| + \|S_{t+1}^{-1}\| \cdot \|R_t\| \; .$$

Since $\{S_t\}$ converges to a non-singular matrix, $\{\theta_t\}$ converges to $(0,0)$ if and only if $\{\zeta_t\}$ does.

Ignoring momentarily the residual, we have $\|\zeta_{t+1}\| \leq \left(\prod_{s=\tau}^{t} \|\Gamma_t\| \right) \|\zeta_\tau\|$, and convergence follows if we can show that $\lim_{t \to \infty} \prod_{s=\tau}^{t} \|\Gamma_t\| = 0$. Expectational stability implies that the eigenvalues of M_t are less then 1 in absolute value, and hence the norm of Λ_t is less than 1. If this were a time-invariant system, then $S_{t+1}^{-1} \, S_t$ would be exactly equal to the identity; hence $\|\Gamma_t\| = \|\Lambda_t\| < 1$ and convergence is obtained. In this time-variant system, the terms $S_{t+1}^{-1} \, S_t$ do not drop out. However, because $\{\alpha_t\}$ converges, S_t is approximately equal to S_{t+1} and hence $S_{t+1}^{-1} \, S_t$ is approximately equal to the identity. On the other hand, as $t \to \infty$, one of the eigenvalues of M_t converges to 1 and hence $\|\Lambda_t\|$ and $\|\Gamma_t\|$ converge to 1. Convergence of θ_t thus depends on how quickly $\|\Gamma_t\|$ converges to 1, which in turn depends delicately on the interplay between the sequences $\{S_{t+1}^{-1} \, S_t\}$ and $\{\Lambda_t\}$.

We verify through brute calculation that the deviations of $S_{t+1}^{-1} \, S_t$ from the identity do not swamp the convergence due to the eigenvalues of M_t. For this purpose, we need $\|\cdot\|$ to be the L_1 vector norm on $\mathbb{R}^2$ and the associated matrix norm on $\mathbb{R}^{2 \times 2}$. That is, for $\|x\| = |x_1| + |x_2|$ for $x \in \mathbb{R}^2$, and $\|A\| = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$. We shall prove the following lemma:

Lemma B.3 *Suppose $\hat{\pi}$ is expectationally stable. Then there are $\rho > 0$ and $\bar{\alpha} > 0$ such that if $\alpha_{t+1} \leq \alpha_t \leq \bar{\alpha}$ then $\|\Gamma_t\| \leq 1 - \rho \alpha_t$.*

Redefine τ so that $t \geq \tau \Rightarrow \alpha_t \leq \bar{\alpha}$ for the $\bar{\alpha}$ in both Lemmas B.2 and B.3. It follows from Lemmas B.3 and B.1 that $\lim_{t \to \infty} \prod_{s=\tau}^{t} \|\Gamma_t\| = 0$.

We need to be sure this result is not disrupted by the residual. Let k and U_ϵ be as described in the Lipschitz condition for r. Then $\|R_t\| < \alpha_t k \, \|\theta_t\|$ if $\pi_t^e, \pi_{t-1}^e \in U_\epsilon$. Since $\theta_t = S_t \, \zeta_t$, $\|\theta_t\| \leq \|S_t\| \cdot \|\zeta_t\|$. Hence,

$$\begin{aligned} \|R_t\| \; &\leq \; \alpha_t k \, \|S_t\| \cdot \|\zeta_t\| \\ \|S_{t+1}^{-1}\| \cdot \|R_t\| \; &\leq \; \alpha_t k \, \|S_{t+1}^{-1}\| \cdot \|S_t\| \cdot \|\zeta_t\| \; . \end{aligned}$$

From Lemma B.2, both S_t and S_t^{-1} converge to nonsingular matrices, so $\|S_t\|$ and $\|S_t^{-1}\|$ are bounded. Define $K = \sup_t k \, \|S_{t+1}^{-1}\| \cdot \|S_t\|$. Then

$$\|S_{t+1}^{-1}\| \cdot \|R_t\| \; \leq \; \alpha_t K \|\zeta_t\| \; .$$

Choose k small enough that $K < \rho$. Suppose $t \geq \tau$ and $\pi^e_t, \pi^e_{t-1} \in U_\epsilon$. Then

$$\|\zeta_{t+1}\| \;\leq\; (1 - \alpha_t(\rho - K))\|\zeta_t\| \;.$$

Note that therefore $\|\zeta_{t+1}\| < \|\zeta_t\|$. Iterating this inequality yields

$$\|\zeta_{\tau+s}\| \;\leq\; \left(\prod_{t=\tau}^{\tau+s-1} (1 - \alpha_t(\rho - K))\right) \|\zeta_\tau\| \;.$$

Since $\rho - K > 0$, Lemma B.1 shows that $\lim_{s\to\infty} \zeta_{\tau+s} = (0,0)$, and hence $\lim_{s\to\infty} \theta_{\tau+s} = (0,0)$.

To conclude, we need to account for what might happen in the first τ periods. Fix any neighborhood $U \subset U_\epsilon$ of $\hat{\pi}$. We show that there is a neighborhood of $\hat{\pi}$ such that for initial conditions in this neighborhood, $\pi^e_t, \pi^e_{t+1} \in U$ for $t = 1, \ldots, \tau$. It then follows from the above that $\pi^e_t, \pi^e_{t+1} \in U$ for $t \geq \tau$ and that $\pi^e_t \to \hat{\pi}$. (Furthermore, $\|\psi_{\tau+s}\|$ decreases monotonically and so stability(a) is satisfied.) Hence, $\hat{\pi}$ is a stable steady state.

This final step follows in the usual way from the local continuity of G_t. Specifically, let $U_\tau := U_\epsilon$. For $t \in 1, \ldots, \tau - 1$, given U_{t+1}, let $U_t \subset U_\epsilon$ be a neighborhood of $\hat{\pi}$ such that $G(U_t \times U_t) \subset U_{t+1} \times U_{t+1}$; such a neighborhood exists because G_t is continuous in a neighborhood of $(\hat{\pi}, \hat{\pi})$ and $G_t(\hat{\pi}, \hat{\pi}) = (\hat{\pi}, \hat{\pi})$. If $\pi^e_1, \pi^e_2 \in U_1$ then $\pi^e_t, \pi^e_{t+1} \in U_t$ for $t = 1, \ldots, \tau$.

To conclude the proof for stability (and before proceeding to the proof for instability), we provide the proof of the two lemmas.

Proof of Lemmas B.2 and B.3: We used Mathematica to compute the eigenvectors and eigenvalues of $M(\alpha)$ (see Lettau and Van Zandt (2000, Appendix B)). We can diagonalize $M(\alpha) = S(\alpha)\,\Lambda(\alpha)\,S(\alpha)^{-1}$, where $\Lambda(\alpha)$ is the diagonal matrix whose diagonal entries are the eigenvalues of $M(\alpha)$ and $S(\alpha)$ is the matrix whose columns are the eigenvectors of $M(\alpha)$. These matrices are

$$\Lambda(\alpha) \;=\; \begin{pmatrix} h(\alpha) & 0 \\ 0 & g(\alpha) \end{pmatrix} \quad , \quad S(\alpha) \;=\; \begin{pmatrix} h(\alpha) & g(\alpha) \\ 1 & 1 \end{pmatrix} \quad , \quad S(\alpha)^{-1} \;=\; \begin{pmatrix} -\frac{1}{f(\alpha)} & \frac{g(\alpha)}{f(\alpha)} \\ \frac{1}{f(\alpha)} & -\frac{h(\alpha)}{f(\alpha)} \end{pmatrix} \;,$$

where

$$\begin{aligned} f(\alpha) \;&=\; \sqrt{(1 - \alpha + \alpha W_1)^2 + 4\alpha W_2} \;, \\ g(\alpha) \;&=\; \big((1 - \alpha + \alpha W_1) + f(\alpha)\big)/2 \;, \\ h(\alpha) \;&=\; \big((1 - \alpha + \alpha W_1) - f(\alpha)\big)/2 \;. \end{aligned}$$

Note in particular that $S(\alpha)$ depends continuously on α, and that $S(\alpha)$ is non-singular (hence $S(\alpha)^{-1}$ is well-defined) when $f(\alpha) \neq 0$, which holds for α in a neighborhood of 0. This completes the proof of Lemma B.2.

We can show (see Lettau and Van Zandt (2000, Appendix B)) that

$$\Gamma_t \;=\; \begin{pmatrix} h(\alpha_t)\frac{-h(\alpha_t)+g(\alpha_{t+1})}{f(\alpha_{t+1})} & g(\alpha_t)\frac{-g(\alpha_t)+g(\alpha_{t+1})}{f(\alpha_{t+1})} \\ h(\alpha_t)\frac{h(\alpha_t)-h(\alpha_{t+1})}{f(\alpha_{t+1})} & g(\alpha_t)\frac{g(\alpha_t)-h(\alpha_{t+1})}{f(\alpha_{t+1})} \end{pmatrix} \;.$$

Examine $\|\Gamma_t\|$ for small α_t (say, for $t \geq \tau$). First, observe that

$$\begin{aligned} g'(\alpha) \;&=\; \Big(-1 + W_1 + (1/2)f(\alpha)^{-1/2}\big(2(-1 + \alpha - \alpha W_1)(1 - W_1) + 4W_2\big)\Big)/2 \;, \\ g'(0) \;&=\; \Big(-1 + W_1 + (1/2)\big(-2(1 - W_1) + 4W_2\big)\Big) \;=\; W_1 + W_2 - 1 \;<\; 0 \;. \end{aligned}$$

Since $g'(0) < 0$ and since $\{\alpha_t\}$ is decreasing, for α_t small enough (t large enough), $g(\alpha_{t+1}) \geq g(\alpha_t)$ and hence the top-right term of Γ_t is positive.

Second, note that

$$\lim_{\alpha\to 0} f(\alpha) \ = \ 1 \quad , \quad \lim_{\alpha\to 0} g(\alpha) \ = \ 1 \quad , \quad \lim_{\alpha\to 0} h(\alpha) = 0 \quad .$$

Therefore, for t large enough, (a) the top-left and bottom-left terms in Γ_t are close to 0, (b) the top-right term is positive and close to zero, and (c) the bottom-right term is close to 1. Hence, $\|\Gamma_t\|$ is the sum of the terms (which are positive) in the right column:

$$\begin{aligned} \|\Gamma_t\| \ &= \ g(\alpha_t)\frac{-g(\alpha_t) + g(\alpha_{t+1})}{f(\alpha_{t+1})} + g(\alpha_t)\frac{g(\alpha_t) - h(\alpha_{t+1})}{f(\alpha_{t+1})} \\ &= \ g(\alpha_t)\frac{g(\alpha_{t+1}) - h(\alpha_{t+1})}{f(\alpha_{t+1})} \ = \ g(\alpha_t) \ . \end{aligned}$$

Pick ρ such that $0 < \rho < -g'(0)$. Then, for t large enough, $\|\Gamma_t\| \leq 1 - \rho\alpha_t$. This completes the proof of Lemma B.3. □

Instability: Abusing but economizing on notation, we normalize the steady state to zero by interpreting π_t^e as $\pi_t^e - \hat{\pi}$. Then equation (B.3) becomes

$$\pi_{t+1}^e \ = \ (1 - \alpha_t + \alpha_t W_1)\pi_t^e + \alpha_t W_2 \pi_{t-1}^e + \alpha_t r(\pi_t^e, \pi_{t-1}^e) \ .$$

Choose $k > 0$ such that

$$W_1 + W_2 - 2k \ > \ 1 \qquad \text{and} \qquad W_2 + k \ < \ 0$$

(possible since we assume $W_1 + W_2 > 1$ and $W_2 < 0$). Let U_ϵ then be as described in the Lipschitz condition for r. If $\pi_t^e, \pi_{t-1}^e \in U_\epsilon$ then

$$|\pi_{t+1}^e| \ \geq \ \big(1 + \alpha_t(W_1 - 1 - k)\big)|\pi_t^e| - \big(\alpha_t|W_2| + \alpha_t k\big)|\pi_{t-1}^e| \ .$$

Since $W_2 < 0$,

$$|\pi_{t+1}^e| \ \geq \ \big(1 + \alpha_t(W_1 - 1 - k)\big)|\pi_t^e| + \alpha_t(W_2 - k)|\pi_{t-1}^e|$$

Suppose $|\pi_t^e| \geq |\pi_{t-1}^e|$. Since $W_2 - k < 0$, we can replace $|\pi_{t-1}^e|$ by $|\pi_t^e|$, obtaining

$$|\pi_{t+1}^e| \ \geq \ \big(1 + \alpha_t(W_1 + W_2 - 1 - 2k)\big)|\pi_t^e| \ = \ (1 + \alpha_t K)|\pi_t^e| \ , \tag{B.4}$$

where $K := W_1 + W_2 - 1 - 2k > 0$. Hence $|\pi_{t+1}^e| > |\pi_t^e|$; by induction, it follows that inequality (B.4) holds until $\{\pi_t^e\}$ leaves U_ϵ.

We now show that if $\pi_1^e, \pi_2^e \in U_\epsilon$ and if $|\pi_2^e| > |\pi_1^e| > 0$ then the sequence $\{\pi_t^e\}$ leaves U_ϵ. Therefore, 0 is unstable. Suppose π_1^e and π_2^e satisfy the stated conditions. We have shown that the sequence $\{|\pi_t^e|\}$ satisfies $|\pi_{t+1}^e| \geq (1 + \alpha_t)|\pi_t^e|$ until it leaves U_ϵ. If it does not leave U_ϵ, we can iterate this inequality to obtain

$$|\pi_t^e| \ \geq \ |\pi_2^e| \prod_{s=2}^{t-1}(1 + \alpha_s K) \ .$$

Then Lemma B.1 shows that $\lim_{t\to\infty}|\pi_s^e| = \infty$; hence $\{\pi_t^e\}$ must leave U_ϵ. □

C Proofs for OLS learning

Proof of Proposition 7.1: For $\mathrm{OLS}_{\pi_{t-1}}$ and OLS_{π_t}, see the paragraph (following the proposition) in which these results are derived as corollaries to Propositions 6.3 and 6.4.

$\mathrm{OLS}_{p_{t-1}}$: Marcet and Sargent (1989) prove these stability results for the case of affine S in their Proposition 3. Since the stability properties are obtained, in any case, by studying a linear approximation of a difference equation, the extension to nonlinear S is trivial (we omit the details). Note that the condition $k \leq 1$ in Marcet and Sargent (1989, Proposition 3) holds for $\delta \approx 0$.

OLS_{p_t}: We presume that the reader has read the part of Section 7 that follows Proposition 7.1. In particular, recall that (i) we can write $\pi^e_{t+1} = \alpha_t \pi_t + (1-\alpha_t)\pi^e_t$, where $\alpha_t := p^2_{t-1}/\sum_{s=0}^t p^2_{s-1}$, and (ii) if $\pi_t \to \hat{\pi}$, then $\alpha_t \to \alpha_{\hat{\pi}} := 1 - \hat{\pi}^{-1}$.

We can thus write the period-t equilibrium condition $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$ as

$$f(\pi_t, \pi^e_t; \alpha_t) \;:=\; W(\alpha_t \pi_t + (1-\alpha_t)\pi^e_t, \pi^e_t) - \pi_t \;=\; 0 \;.$$

In period t, α_t is a fixed parameter in this equation, so we can write the set of solutions as $\varphi(\pi^e_t; \alpha_t)$. As explained in Section 5.3, there may be multiple solutions; we therefore choose an equilibrium selection $F(\pi^e_t; \alpha_t)$ (on the domain for which φ has non-empty values) such that for a steady state $\hat{\pi} \in \{\pi^L, \pi^H\}$ and $\hat{\alpha} := 1 - \hat{\pi}^{-2}$, there is a neighborhood of $(\hat{\pi}, \hat{\alpha})$ on which $F(\pi^e_t; \alpha_t)$ is the element of $\varphi(\pi^e_t; \alpha_t)$ that is closest to $\hat{\pi}$. As long as $f_\pi(\hat{\pi}, \hat{\pi}; \hat{\alpha}) \neq 0$, the implicit function theorem implies that F is continuously differentiable in a neighborhood of $(\hat{\pi}, \hat{\alpha})$. Observe that $f_\pi(\hat{\pi}, \hat{\pi}; \hat{\alpha}) = \hat{\alpha} W_1 - 1$, which is not equal to 0 for $\delta \approx 0$ as follows: $W_1(\pi^L, \pi^L) \approx W_1(1,1) < \infty$ and $\alpha_{\pi^L} \approx 1 - (1)^{-2} = 0$, so $f_\pi(\pi^L, \pi^L; \alpha_{\pi^L}) < 0$; whereas $W_1(\pi^H, \pi^H) \approx \infty$ and $\alpha_{\pi^H} \approx 1 - (\pi^a)^{-2} > 0$, so $f_\pi(\pi^H, \pi^H; \alpha_{\pi^H}) > 0$.

Mimicking the proof of Proposition 3 in Marcet and Sargent (1989), we write the evolution of $\{\pi^e_t, \alpha_t\}_{t=1}^\infty$ as

$$\begin{aligned} \pi^e_{t+1} &= \alpha_t F(\pi^e_t; \alpha_t) + (1-\alpha_t)\pi^e_t \;, \\ \alpha_{t+1} &= \left(1 + \alpha_t^{-1} F(\pi^e_t; \alpha_t)^{-2}\right)^{-1} \;. \end{aligned}$$

The first equation is the OLS_{p_t} expectations rule, with π_t replaced by $F(\pi^e_t; \alpha_t)$. The second equation is obtained from

$$\begin{aligned} \alpha_{t+1} \;=\; \frac{p_t^2}{\sum_{s=0}^{t+1} p^2_{s-1}} \;&=\; \left(\frac{p_t^2}{p_t^2} + \frac{\sum_{s=0}^t p^2_{s-1}}{p_t^2}\right)^{-1} \\ &=\; \left(1 + \alpha_t^{-1}\pi_t^{-2}\right)^{-1} \;=\; \left(1 + \alpha_t^{-1} F(\pi^e_t; \alpha_t)^{-2}\right)^{-1} \;. \end{aligned}$$

We check stability of this difference equation at a steady $(\hat{\pi}, \hat{\alpha})$, where $\hat{\pi} \in \left\{\pi^L, \pi^H\right\}$ and $\hat{\alpha} = 1 - \hat{\pi}^{-2}$. Since $\hat{\pi}$ is a steady state, $F(\hat{\pi}; \alpha_t) = \hat{\pi}$ for any α_t and so $\partial F(\hat{\pi}; \hat{\alpha})/\partial \alpha_t = 0$. It follows that

$$\left.\frac{\partial \pi^e_{t+1}}{\partial \alpha_t}\right|_{\pi^e_t = \hat{\pi}, \alpha_t = \hat{\alpha}} \;=\; \hat{\alpha}\frac{\partial F}{\partial \alpha_t}(\hat{\pi}, \hat{\alpha}) + F(\hat{\pi}, \hat{\alpha}) - \hat{\pi} \;=\; 0 \;.$$

Hence, the eigenvalues of the linearization of these difference equations around a steady state are $\partial \pi^e_{t+1}/\partial \pi^e_t$ and $\partial \alpha_{t+1}/\partial \alpha_t$.

Since $\partial F(\hat{\pi};\hat{\alpha})/\partial \alpha_t = 0$,

$$\left.\frac{\partial \alpha_{t+1}}{\partial \alpha_t}\right|_{\pi_t^e=\hat{\pi},\alpha_t=\hat{\alpha}} = \hat{\alpha}^{-2}\hat{\pi}^{-2}\left(1+\hat{\alpha}^{-1}\hat{\pi}^{-2}\right)^{-2}$$

$$= \left(\hat{\alpha}\hat{\pi}+\hat{\pi}^{-1}\right)^{-2} = \left((1-\hat{\pi}^{-2})\hat{\pi}+\hat{\pi}^{-1}\right)^{-2} = \hat{\pi}^{-2} .$$

Thus, for any steady state $\hat{\pi} > 1$, we have $|\partial \alpha_{t+1}/\alpha_t| < 1$.

Stability therefore hinges on the magnitude of $\partial \pi_{t+1}^e/\partial \pi_t^e$. At a steady state $(\hat{\pi},\hat{\alpha})$, $\partial F(\hat{\pi};\hat{\alpha})/\partial \pi_t^e$ is equal to $-f_{\pi^e}(\hat{\pi},\hat{\pi};\hat{\alpha})/f_\pi(\hat{\pi},\hat{\pi};\hat{\alpha})$. Then $\partial \pi_{t+1}^e/\partial \pi_t^e$ (evaluated at $\pi_t^e = \hat{\pi}$ and $\alpha_t = \hat{\alpha}$) is just $g'(\hat{\pi})$ from the proof of Proposition 5.2 (equation (A.5)) with $\psi_{\pi_t} = \hat{\alpha}$ and $\psi_{\pi_t^e} = 1-\hat{\alpha}$:

$$\left.\frac{\partial \pi_{t+1}^e}{\partial \pi_t^e}\right|_{\pi_t^e=\hat{\pi},\alpha_t=\hat{\alpha}} = \frac{\hat{\pi}S'(\hat{\pi})\hat{\alpha}+S(\hat{\pi})(1-\hat{\alpha})}{\hat{\pi}^2 S'(\hat{\pi})\hat{\alpha}+S(\hat{\pi})} . \tag{C.1}$$

Consider first the steady state (π^L, α_{π^L}). For $\delta \approx 0$, the numerator and denominator of equation (C.1) are both positive, since $S(\pi^L) > 0$ and $\alpha_{\pi^L} \approx 0$. Thus, $\left|\partial \pi_{t+1}^e/\partial \pi_t^e\right| < 1$ if and only if

$$\begin{aligned} \pi^L S'(\pi^L)\alpha_{\pi^L} + S(\pi^L)(1-\alpha_{\pi^L}) &\overset{?}{<} (\pi^L)^2 S'(\pi^L)\alpha_{\pi^L} + S(\pi^L) , \\ \pi^L S'(\pi^L)\alpha_{\pi^L} &\overset{?}{<} (\pi^L)^2 S'(\pi^L)\alpha_{\pi^L} + S(\pi^L)\alpha_{\pi^L} , \\ -(\pi^L-1)\pi^L S'(\pi^L) &\overset{?}{<} S(\pi^L) , \end{aligned} \tag{C.2}$$

which holds for $\delta \approx 0$ because $\pi^L \approx 1$ and $S(1) > 0$.

Now consider the steady state (π^H, α_{π^H}). For $\delta \approx 0$, the numerator and denominator of the RHS of equation (C.1) are both *negative* because $S(\pi^H) \approx 0$ and $\alpha_{\pi^H} \approx 1-(\pi^a)^2 > 0$. Hence, the condition for stability is the reverse of the inequality in equation (C.2). That is, $-(\pi^H-1)\pi^H S'(\pi^H) > S(\pi^H)$, which also holds because $S(\pi^H) \approx 0$ and $\pi^H \approx \pi^a > 1$. Therefore, π^H is also stable. □

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