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Business Models for Off-Grid Energy Access at the Bottom of the Pyramid

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A large proportion of the world's population has no access to electricity and so relies on noxious kerosene for their lighting needs. Solar-based solutions require a large upfront investment and are often unaffordable in this market owing to consumers' tight liquidity constraints. As an alternative, there are business models relying on rechargeable light bulbs that are sold at a subsidized price (which renders them affordable) and require regular micropayments for recharges (which eases liquidity constraints). These bulbs provide a cheaper and healthier light source than kerosene, yet their adoption is lower than expected and some consumers continue to use kerosene. We propose a stylized consumer behavior model to explain these observations. In addition to monetary cost incurred while using a particular light source, our model accounts for the inconvenience cost (due to repeated travel to the purchase center) and blackout cost (due to liquidity constraints) associated with that source. Although kerosene lighting is more expensive than bulbs, consumers who face either high inconvenience costs or high blackout costs prefer kerosene to bulbs because the former's flexibility, with regard to quantity, helps reduce whichever cost is dominating. At the firm level, there is an optimal bulb capacity that trades off demand for rechargeable bulbs against recharge revenue from consumers; furthermore, firm can reverse the preferences of kerosene consumers by increasing its product's flexibility (e.g., by allowing partial recharges). Although strategies - such as price discounts and mobile micropayments - based on alleviating liquidity constraints are not in themselves sufficient to ensure higher adoption rates, increased bulb use becomes more likely when they are combined with strategies to reduce consumers' inconvenience.

Keywords: Bottom of the Pyramid; Electricity; Sustainability; Business Models

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1. Introduction

It is a disturbing fact that nearly a fifth of humankind still does not have access to electricity (IEA 2015). More than 95% of this population inhabits countries in Sub-Saharan Africa and developing Asia. Not surprisingly, countries with low electrification rates are those in which most citizens live on less than $2 (US) per day (IEA 2015). This part of the world's population is often referred to as the *bottom of the pyramid* (BoP).

Grid-based models of electricity supply, which are those most often used to serve the top of the pyramid, have been unsuccessful in these countries because they require substantial capital investment. About 80% of the world's off-grid population resides in rural areas (IEA 2015). Because households in villages are often scattered, it may neither be technically feasible nor economical to extend grid electricity to these regions (IFC 2010). Even in the electrified parts of these countries, poor households either are not connected to the grid (because of high connection fees) or are supplied with low-quality light with frequent power outages (IFC 2010). Hence there is a huge market for off-grid energy in these countries.

The importance of access to energy at the BoP cannot be overemphasized. Energy is needed for efficient lighting, heating, cooking, transportation, communication, sanitation, and healthcare services, all of which contribute to increasing productivity, improving health, alleviating poverty, and promoting economic growth (IEA 2015). The focus of this paper is on the lighting needs of the poor, and we analyze the viability of novel off-grid lighting sources at the BoP.

Poor households spend an estimated $18 billion each year on low-quality lighting solutions – this represents an appreciable fraction (up to 10%) of their monthly expenditures. Most of this expenditure goes toward kerosene which remains to be the predominant source of off-grid lighting (IFC 2012a).[1] In addition to its high costs, kerosene poses great fire and health hazards. This highly flammable liquid releases millions of tons of carbon dioxide annually and is responsible for 2 million deaths each year – more than the number due to malaria. Home solar systems (i.e., solar panels sold with modular products such as lamps and plugs) offer a clean source of light; yet each system costs more than $100, placing solar energy well beyond the reach of people at the BoP (IFC 2010). In addition to home systems, firms such as d.light[2] and Onergy[3] sell cheaper, portable solar bulbs (with self-sufficient solar panels) costing from $10 to $50. However, these are also unaffordable to the poor because of their extremely low income levels and near total lack of access to efficient mechanisms for saving and borrowing (IFC 2010).

Prahalad (2006) argues that effectively serving such a liquidity-constrained market will require innovative and fundamentally different business models at the BoP. For example, fast-moving consumer goods (FMCG) companies such as Unilever and P&G have adapted some of their products for the BoP market by repackaging goods in smaller volumes to make them more affordable (Prahalad 2006). The equivalent of this model for delivering light is rechargeable bulb technology.[4] Instead of selling rechargeable bulbs to consumers at full price, firms can either rent them or sell them at a subsidized price. Continued use of such bulbs requires that they be recharged at a (usually village-level) recharge center for a small recharging fee. The revenue stream from

[1] Many households at the BoP do not use kerosene for their cooking or heating needs because it is too expensive for these purposes. Instead, they use firewood and charcoal for cooking and heating on inefficient stoves and fireplaces (IFC 2012a).

[2] http://www.dlight.com/

[3] http://onergy.in/

[4] Light sources powered by *replaceable* batteries (e.g., torches) can also provide light to consumers in small portions, but they usually cost consumers more than do rechargeable versions. Moreover, as argued by Bensch et al. (2015), the improper disposal of replaced batteries in rural areas degrades the environment and threatens the health of those living there. Although our model and insights can be extended to these sources, they are not discussed in this paper.

repeated recharges makes it possible for the firm to subsidize the upfront price by financing it through ongoing payments. For example, consumers in Laos can rent bulbs from Sunlabob[5], which assesses a fee of $0.12 for recharging; each recharge lasts for 10 hours.[6] In Bangladesh, Shidhulai[7] retrofits hurricane lamps with light-emitting diode (LED) bulbs that are likewise rejuvenated at a solar recharge center. Shidhulai's lamps are sold for $3 to consumers who bring their own kerosene lamps to be retrofitted; otherwise the charge is $4. Each lamp recharge lasts for 8 hours and costs $0.07. It takes 1–2 hours to recharge a bulb with these solar-based technologies. In addition to solar-based recharging, Nuru Energy[8] (based in Rwanda) uses a stationary bicycle, the so-called PowerCycle generator, that can fully recharge five bulbs with 20 minutes of pedaling (Carrick and Santos 2013). Nuru sells its bulbs for $1.50 and charges $0.20 for a single recharge, which lasts for 18 hours.

Because rechargeable bulbs are sold at a low (subsidized) price, upfront technology cost is unlikely to be a barrier to adoption. At the same time, extremely affordable micropayments for recharges help consumers overcome their liquidity constraints. These bulbs provide clean and smoke-free light. Selling hours of light instead of the equipment that produces it resembles the pay-per-use service provided by a grid connection, so this business model brings to consumers the advantages of electric lighting. Most firms that market rechargeable bulbs price their recharges so that the technology costs consumers much less than using kerosene. For instance, 100 ml of kerosene costs $0.20 in Rwanda and produces close to six hours of light. Thus, with Nuru's recharge price, bulbs are 3 times cheaper than kerosene.[9]

These features of rechargeable light bulbs lead one to suppose that most of the BoP population would prefer them to other, costlier sources of light and hence that these bulbs would displace kerosene. However, this is not always the case; kerosene is still prevalent in regions where rechargeable bulbs are available. For instance, the number of purchases of Nuru bulbs is high (about 67%, according to a pilot study in Rwanda; Beuggert 2014) thanks to subsidized prices, but the true adoption rate (i.e., the actual usage) is lower than anticipated – as reflected in the low recharge frequencies (only 1.2–1.6 times per month, on average; Beuggert 2014). Some villagers continue to use kerosene even though it costs significantly more (Beuggert 2014), which seems to be an economically irrational preference.

[5] http://www.sunlabob.com/

[6] The price and bulb capacity details for the examples here are taken from http://energymap-scu.org/. Most such bulbs are designed to prevent being recharged by any other technology.

[7] http://www.shidhulai.org/afftechnology.html

[8] http://nuruenergy.com/

[9] Here we are quoting the price of kerosene in Rwanda's urban areas. But since prices in rural areas are generally higher (IFC 2012b), the cost saving we report is a conservative estimate.

In this paper, we explore the potential drivers of such preferences and propose strategies to alter them. Understanding the drivers of consumer preferences is of utmost importance because it can lead to better business models and product designs, which in turn should result in higher adoption rates and less use of kerosene. These outcomes would benefit firms, consumers, and the environment.

We approach this topic on two fronts, describing consumer behavior and then offering prescriptions for the firm. In the first part of the paper (Sections 3 and 4) we build a stylized model to characterize consumer preference for kerosene and bulbs. This model accounts for three aspects that are key to light consumption under poverty: (i) *procurement* dynamics, which are driven by the necessity to purchase light (in the form of kerosene or bulb recharges); (ii) *repeated* purchases, which entail inconvenience for consumers who must periodically replenish their light supply; and (iii) *liquidity* constraints, which can lead to periods during which the consumer has no access to light. These features lead to three types of consumer costs – respectively, the monetary cost, the inconvenience cost, and the blackout cost. The overall cost incurred by a consumer is the sum of these three costs. We assume that the consumer prefers the light source associated with the lowest long-run average overall cost. We find that not all of these BoP consumers prefer bulbs despite the lower monetary cost of recharging as compared with purchasing kerosene. In our model, consumers whose inconvenience costs are high relative to their blackout costs are referred to as *inconvenience-averse* consumers; analogously, those with high blackout costs relative to inconvenience costs we call *blackout-averse* consumers. Both these consumer types prefer kerosene to bulbs because of its flexible nature: kerosene can be procured in any quantity whereas bulbs have a fixed recharge capacity. That flexibility allows inconvenience-averse (resp. blackout-averse) consumers to purchase in large (resp. small) quantities, helping them save on their long-run inconvenience (resp. blackout) cost.

In the second, prescriptive part of the paper (Section 5) we discuss several strategies to increase both the adoption rate of bulbs and the revenue of firms operating those bulbs. These strategies build on the first part's structural analysis of consumer costs and preferences. We find that, at a given recharge price, increasing bulb capacity increases the demand for bulbs in the market; however, it also reduces the firm's revenue per consumer because higher-capacity bulbs require less frequent recharging. This trade-off implies there is an optimal bulb capacity, which in turn suggests that more capacity is not always better. Because some market segments prefer the flexibility of kerosene, as just described, the firm could gain consumers from those segments by making its bulb technology more flexible. That could be achieved by scaling up bulb capacity while allowing consumers to recharge them only partially. Blackouts are reduced by strategies that alleviate consumers' liquidity constraints; such strategies include price discounts and mobile micropayments.

Yet the result may not always be higher adoption rates, since these strategies lead to increased consumption and thus to more consumer inconvenience in the long run; thus their overall effect on consumer costs, and hence on bulb adoption, is ambiguous. That being said, liquidity-oriented strategies are likely to improve adoption when they are *combined* with strategies designed to reduce inconvenience by, for example, offering door-to-door recharge service or increasing the number of recharge centers.

Our paper makes three main contributions. First, we present a novel model of the off-grid consumption of light in impoverished regions. It is a simple model that nonetheless accounts for several important operational features: liquidity constraints, consumer inconvenience, and technology flexibility. Hence this model could serve as a template for future research in contexts involving similar trade-offs. Second, our analysis offers a plausible explanation for why people may prefer a technology, such as kerosene, that is both inferior to and more expensive than the alternative (here, rechargeable bulbs). Although that preference could be plausibly explained in other ways – for example, by habit formation or as a lack of trust in new technology – we are not aware of any research that incorporates, as we do here, the flexibility and convenience of kerosene. Finally, we operationalize the preference structure of consumers toward the end of evaluating strategies that firms could implement to increase the adoption of rechargeable off-grid technologies.

2. Literature Review

Our paper is positioned at the intersection of two streams of literature: sustainable operations and the economics of poverty.

The challenges of devising sustainable business models are discussed extensively in Kleindorfer et al. (2005), Drake and Spinler (2013), Girotra and Netessine (2013), and Plambeck (2013). A number of studies have analyzed the effect of operational decisions and business models on the environment in a variety of contexts, including supply chains (Cachon 2014, Plambeck and Taylor 2015, Vedantam et al. 2015), choice of revenue models (Agrawal et al. 2012, Belavina et al. 2015), product designs (Plambeck and Wang 2009, Raz et al. 2013), and fleet operations (Chocteau et al. 2011, Kleindorfer et al. 2012). Our paper is directly related to the growing body of research on sustainable energy-related business models, a literature that spans several areas: the adoption of electric vehicles (Avci et al. 2015, Lim et al. 2015), the effect of energy policies on supply and demand in electricity markets (Daniels and Lobel 2014, Sunar and Birge 2014), the pricing of renewable energy sources (Kok et al. 2015, Alizamir et al. 2016), and the adoption of "green" technologies (Lobel and Perakis 2011, Krass et al. 2013, Cohen et al. 2014, Aflaki and Netessine 2015). Much as in these papers, the focus of our research is on understanding how the firm's decisions can affect adoption of a clean energy technology. Yet our paper differs from those just cited because the context we investigate is the bottom of the pyramid – a market with unique consumer characteristics and adoption challenges.

With regard to understanding the drivers of technology adoption in the context of poverty, our paper is also related to the behavioral economics literature. Some examples from this research stream are Cohen and Dupas (2010), who study the adoption of insecticide-treated bed nets, and Dupas (2014), who studies the effect of short-term subsidies on the long-term adoption of those bed nets. Also, Ashraf et al. (2010) explore how price affects the use of water purification solutions, Devoto et al. (2012) examine what drives the adoption of piped water connections, and Hanna et al. (2012) seek to explain the low adoption rates of eco-friendly cooking stoves; Dercon and Christiaensen (2011) identify factors that discourage the application of fertilizers, and Duflo et al. (2011) propose nudges to increase the use of fertilizers. In addition to this rich vein of research papers, several books have been published on the topic. Collins et al. (2009) explain the saving and expenditure patterns of the world's poor, Karlan and Appel (2011) investigate why seemingly beneficial policies often do not have a positive effect on the lives of impoverished populations, and Banerjee and Duflo (2011) critically analyze the decision making of poor people in several contexts. Mullainathan and Shafir (2013) use the notion of scarcity to frame both the behavior of busy people (who lack time) and the behavior of poor people (who lack money).

3. Consumer Model

In this section, we begin by listing the features that are arguably the most important for consumer preferences in the context of light consumption under poverty; we then build models which account for these features.

First, consumers in our model are typically villagers, living in poor countries, whose main occupation is subsistence farming. Most of these consumers live on an extremely low income (less than $2 per day) and have no access to efficient saving and borrowing mechanisms (IEA 2015). Moreover, on a daily basis they juggle dealing with a variety of needs, both expected and unexpected; some of those needs – such as food and health – receive higher priority than light. These characteristics severely constrain the cash available for purchasing light. Given such liquidity constraints, consumers who want to purchase light may be unable to afford doing so. In such cases they must wait for sufficient money to accrue, during which time they do not have light and so suffer from *blackouts*. Thus liquidity constraints (and the resulting blackouts) are an important factor affecting impoverished consumers' preferences for light consumption.

Second, the sources of light considered in this paper require repeated purchases. In particular, consumers must either recharge their bulbs at a recharging center or buy kerosene from a nearby store. Few villages have more than one or two kerosene stores, and there is usually only just a single recharging center. The villages in East African countries, for example, are spread over hills and typically have neither efficient public transportation nor even well-laid roads for walking. So for some consumers, purchasing light involves the physically demanding task of hill climbing, and a round trip can take longer than two hours (IFC 2010). Since many villagers work as daily labor on farms owned by others, the time required to purchase light is a significant inconvenience – another crucial factor in the context of repeated light purchases.

The third key feature of this market is that, as a source of light, kerosene differs markedly from bulbs. The light from bulbs is white, smoke-free, and focused; in contrast, kerosene produces a yellow, smoky, and diffuse light. More importantly, kerosene is a liquid – and thus a *flexible resource* – that can be bought in almost any quantity, whereas bulbs have a fixed capacity. The former aspects characterize the quality of light whereas the latter characterizes the quantity of light purchased, which reflects the amount of money spent to purchase that light. These inherent differences between the two light sources are likely to play an important role in determining consumer preferences. In the lives of poor populations, the constraints due to lack of funds are more binding than those due to quality considerations; for this reason, our consumer model accounts only for quantity flexibility. Note that incorporating quality preferences, although easy enough to do, would yield self-evidently trivial results.

If we define a *cycle* as the duration between two purchases, then each such cycle involves the purchase of light, the consumption of that light, and making the next purchase – possibly after waiting until sufficient funds have accrued. When these factors are considered, we can identify three types of consumer costs as follows: the *monetary* cost of purchasing light, the *inconvenience* cost associated with the need to replenish the light supply, and the *blackout* cost due to the disutility incurred when needed light is unavailable because of liquidity constraints. The overall cost that the consumer incurs in a cycle is the sum of these three costs.

Since the purchase of light is cyclic and since the consumer incurs a finite cost in each cycle, we believe that the most suitable metric for evaluating a light source is the *long-run* cost incurred by the users of that source. Thus we assume that our consumer is a "long-run–average cost minimizer". In what follows, we devise a model that incorporates the features just described and thereby characterize long-run costs. We remark that the technology *acquisition* cost (i.e., the purchase price of a rechargeable bulb or a kerosene lamp) is not considered here because that cost is unlikely to hinder adoption: households typically own a kerosene lamp (IFC 2010), and the subsidized upfront price of bulbs makes their purchase affordable to most (Beuggert 2014). Furthermore, since our analysis concerns costs incurred in the long-run, these one-time initial costs get washed out in such setting and hence have a trivial impact.

3.1. Consumption of Light: Bulbs

Let P be the recharge price and let Q be the rechargeable bulb's capacity. After each recharge, we assume that the light is consumed at a constant rate of λ hours per day. This assumption is reasonable because the need for light is almost deterministic: about three to four hours in the late evening and night. Hence a bulb runs through its charge (i.e., becomes "discharged") in Q/λ days, as shown in Figure 1. We refer to this duration as the bulb's *consumption time*.

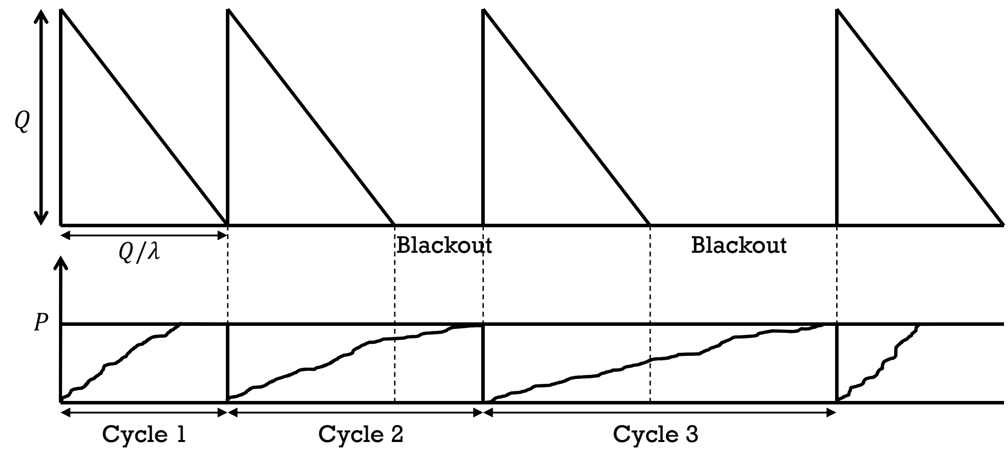

Figure 1 **Consumption cycles (upper graph) and income cycles (lower graph). The consumer has sufficient money when the bulb is discharged in cycle 1 but not in cycles 2 and 3. Hence the consumer does not experience any blackouts in cycle 1 but does experience blackouts, of uncertain duration, in cycles 2 and 3.**

When the bulb is discharged, if the consumer can afford the next recharge (as in cycle 1 of Figure 1) then recharging proceeds without delay. (The consequences of accounting for delays in purchases is discussed in Section 4.) However, liquidity constraints may be such that the consumer cannot afford recharging when the bulb is discharged (as in cycles 2 and 3 of Figure 1). In that case, the consumer must wait for sufficient money to accrue before recharging and therefore experiences blackouts during the interim.

3.1.1. Modeling Liquidity Constraints and Blackouts. The time spent waiting for the next recharge is tied directly to the consumer's cash in hand that can be spent on light, so characterizing this duration requires that we model her income process. Because earned income is used for a variety of needs (food, health, education, etc.), some of which may be of higher priority than the need for light, we assume that the consumer uses "mental accounting" (Thaler 1985) to manage her overall income where she has separate accounts for each of her important needs. There is substantial evidence from the literature on psychology and experimental economics that people have a strong tendency to budget portions of their overall income into separate accounts and then to track their expenses against those budgets (see e.g. Henderson and Peterson 1992, Heath and

Soll 1996, Thaler 1999). For the purposes of our paper, the only such account of interest is the one dedicated to acquiring light; the income entering this account is that part of the consumer's total income that can be spent on light.

For the purpose of illustration, we could also think of mental account for light as a *savings box* used by the consumer to save money for purchasing light. After every purchase, we assume that the money in this box is reset to zero and the process begins anew (as shown in Figure 1); therefore, at no time will the consumer have saved more than P dollars for light. Heath and Soll (1996) use several experiments to show that people set such maximum limits/budgets across categories of expenses in advance of actual consumption. Budgeting of this type helps consumers with self-control issues (Shefrin and Thaler 1988) and simplifies the computational costs associated with complex allocation mechanisms (Simon 1947). Heath and Soll also argue that people usually adhere to these pre-set budgets and resist transferring money across accounts. Furthermore, Thaler (1999) argues that "the tighter the budget, the more explicit are the budgeting rules, both in households and organizations. Families living near the poverty level use strict, explicit budgets" (p. 193). We therefore expect these mental accounts to be quite inflexible in the current context.[10]

The simplest income model presumes that the consumer adds μ dollars every day to this mental account/savings box. Under that deterministic model, P/μ days are required for the consumer to accrue enough money for a recharge. We call this the *hitting time*, i.e., the time required by a income process to "hit" the threshold P. If the hitting time P/μ is shorter than the consumption time Q/λ, then the consumer always has enough money to replenish a discharged bulb and so does not experience any blackouts. But when the hitting time is longer than the consumption time, the consumer experiences blackouts that last $P/\mu - Q/\lambda$ days.

Since the consumer's needs and also her overall income are uncertain, it is unlikely that her disposable income for light actually grows at a constant rate. In other words: instead of adding μ dollars every day to the savings box, the consumer adds an uncertain amount that could be less or greater than μ depending on her daily needs and income. As evident from the deterministic model, we require only the hitting time to characterize blackout duration. The traditional approach to modeling hitting time is to first model income as a standard stochastic process and then characterize the associated hitting time. For example, if income were modeled as a Brownian motion then its hitting time is inverse Gaussian distributed (Johnson et al. 1995). However, Brownian motion can take negative values and so using it to model income leads to several unrealistic results.[11] Alternately, we could model income as a process that takes only positive values such as reflected

[10] As we shall see shortly, although the mental accounting model assumed here simplifies the analysis, it does not drive any of our main results.

[11] The analysis based on Brownian motion is available from the authors upon request.

Brownian motion, Gamma process, etc. But these processes have intractable hitting times which complicates the analysis.

Since we need only the hitting time of the income process to model consumer purchase behavior, instead of modeling income explicitly, we offer a simple model of the hitting time. Recall that the hitting time is P/μ days in the deterministic case but that, when income is uncertain, it could take either a longer or a shorter time than P/μ days to hit the threshold P. We represent this randomness using a positive random variable $\tilde{\varepsilon}$ and model the hitting time as

$$\tilde{\tau}(P) = \frac{P}{\mu}\,\tilde{\varepsilon}, \tag{1}$$

where P/μ is the deterministic hitting time and $\tilde{\varepsilon}$ is the randomness associated with that hitting time. Since $\tilde{\varepsilon}$ could be any positive random variable (with finite expectation), we believe that our hitting time model is quite general. Although it is not necessary, we could normalize the mean of $\tilde{\varepsilon}$ to 1 to be consistent with the deterministic model. In this setup, the variance of $\tilde{\varepsilon}$ captures the magnitude of uncertainty in a consumer's life. Since $\tilde{\varepsilon}$ is always positive, it follows that $\tilde{\tau}(P)$ is also positive and that $\tilde{\tau}(0) = 0$ almost surely – that is, no time is needed to hit a threshold when it is the same as the starting point. Throughout this paper we denote the probability density function (PDF) of $\tilde{\varepsilon}$ by f, and its cumulative distribution function (CDF) by F.

Under this hitting-time model, the duration of blackouts can be written simply as $[\tilde{\tau}(P) - Q/\lambda]^+$; here $[z]^+$ denotes the positive part of z.

3.1.2. Modeling Long-Run Consumer Costs. Next we construct the long-run average consumer cost C associated with using bulbs. Recall from Figure 1 that consumption cycles are aligned with income cycles and that both renew after every purchase. Thus we have perfect renewals and so can use Wald's theory (Ross 1996) to compute the long-run average cost, which is simply the ratio of expected cost during a cycle to that cycle's expected length. We now characterize each of these two components. Our hitting-time model implies that the expected blackout duration L is given by

$$L = \mathbb{E}[P\tilde{\varepsilon}/\mu - Q/\lambda]^+. \tag{2}$$

The expected cycle length Ψ is then the sum of consumption time and expected blackout duration:

$$\Psi = Q/\lambda + L. \tag{3}$$

As discussed previously, the consumer's expected cost in a cycle comprises three components: expected monetary, inconvenience, and blackout costs. The monetary cost in a cycle is simply the amount P paid by a consumer for a recharge. Although there are various forms of inconvenience incurred by consumers seeking to recharge their bulbs, we collapse all of them into a single parameter I; this *inconvenience level* is the "dollar equivalent" of the inconvenience that the consumer incurs.

To model blackout cost, we assume that if the consumer experiences z consecutive days of blackouts then the corresponding blackout cost is $\beta b(z)$. Here b is the *blackout function*, which is increasing, twice differentiable, and strictly convex with $b(0) = 0$; the term β captures the consumer's *sensitivity* to blackouts and serves also to convert experienced blackouts to dollars. The strict convexity of b means, for example, that experiencing six consecutive days of blackouts is costlier to a consumer than experiencing three consecutive days of blackouts on two different occasions. To establish boundary conditions for some of our results, we further assume that $\lim_{z\to\infty} b(z) = \lim_{z\to\infty} b'(z) = \lim_{z\to\infty} zb'(z) - b(z) = \infty$. These conditions reflect the increasing discomfort of extended blackouts and so ensure that consumers prefer some light source to having no light at all. Thus the expected blackout cost is given by βB, where

$$B = \mathbb{E}b[P\tilde{\varepsilon}/\mu - Q/\lambda]^+. \tag{4}$$

The following lemma lists some intuitive properties of L, B, and Ψ that will be useful in proving the results to follow.

Lemma 1. [Shapes of expected blackout duration, blackout cost, and cycle length]

(i) *L and B are decreasing in bulb capacity Q, increasing in recharge price P, and decreasing in the saving rate μ.*

(ii) *Ψ is increasing in both Q and P but is decreasing in μ.*

When we combine all the costs discussed so far, the final expression for long-run average consumer cost is written as

$$\begin{aligned} C &= \frac{\text{Expected sum of inconvenience, monetary and blackout costs}}{\text{Expected cycle length}} \\ &= \frac{I + P + \beta B}{\Psi}. \end{aligned} \tag{5}$$

We can summarize this development as follows. The assumed consumption and income processes lead to perfect renewals, which enable an elegant characterization of the consumer's long-run cost of using bulbs. As evident from (5), long-run costs are both simple and tractable, which makes them useful for the tasks that are more complex – namely, characterizing consumer preferences and optimizing firm-level decisions. Moreover, unlike complex dynamic consumption models, making decisions based on long-run costs does not require that the consumers have strong cognitive abilities. Under our model, consumers need only estimate two simple numbers to compute the long-run cost of using a light source: the (average) length of time between two purchases of light; and the (average) cost incurred *between* two purchases.

3.2. Consumption of Light: Kerosene

Let p_k be the unit price of kerosene. To facilitate comparisons, we base our discussion on *hours of light*. For example, if 100 ml of kerosene costs \$0.20 and lasts about six hours, then $p_k = 0.033$ dollars/hour. Unlike bulbs, kerosene offers consumers the flexibility of choosing the consumption quantity Q_k (hours of light). To keep the model simple, we assume that a consumer purchases the same, but *optimally chosen* quantity Q_k in every cycle (e.g., that a standard bottle is used for each purchase). This assumption is not entirely realistic, however, since the consumer could vary her purchase quantity across cycles depending on a variety of constraints. Yet we restrict our analysis to this simple model for two reasons. First, if we can rationalize the seemingly irrational preference for kerosene under this model, then a more complex model (that incorporates kerosene quantity dynamics) can also rationalize this preference; thus the insights derived here would continue to hold (at least as a special case) in a more complex setting. Second, we are interested in the effects of flexibility in choosing a quantity and this simple model allows us to separate that factor from the effects of quantity dynamics. It is trivial that the ability to vary the purchased quantity across cycles drives consumers' preference toward kerosene, so we do not include this feature in our model.

Given these assumptions, we can readily use the framework just developed to characterize the long-run consumer cost of using kerosene as follows:

$$C_k = \frac{I + p_k Q_k + \beta B_k}{\Psi_k}, \tag{6}$$

where

$$\Psi_k = Q_k/\lambda + \mathbb{E}[p_k Q_k \tilde{\varepsilon}/\mu - Q_k/\lambda]^+ \equiv Q_k \hat{\Psi}_k \quad \text{and} \quad B_k = \mathbb{E} b[p_k Q_k \tilde{\varepsilon}/\mu - Q_k/\lambda]^+. \tag{7}$$

Because the consumer has the flexibility of choosing Q_k, we assume that she chooses the quantity that minimizes her long-run cost. Let that cost-minimizing quantity be $Q_k^* = \arg\min_{Q_k} C_k$ and let C_k^* be the corresponding optimal cost. The following proposition establishes the existence and uniqueness of Q_k^*; it also characterizes the effects of inconvenience level and blackout sensitivity on Q_k^*, which will be useful when we analyze consumer preferences. Other than the constraint that Q_k be positive, the analysis here assumes no bounds on Q_k. We discuss the consequences of relaxing this possibly unrealistic assumption in Section 4.

Proposition 1. [Optimal kerosene purchase quantity]

(i) *C_k is U-shaped in Q_k with a unique minimum Q_k^*.*

(ii) *The optimal kerosene quantity Q_k^* is increasing in I but decreasing in β.*

Let us now interpret these results. As the purchase quantity Q_k increases, the consumer experiences a lower inconvenience cost yet a higher blackout cost in the long run. At the same time, the long-run monetary cost remains unaffected because $p_k Q_k/\Psi_k = p_k/\hat{\Psi}_k$ is independent of Q_k (by (7)). Thus the optimal purchase quantity Q_k^* is obtained by balancing the marginal *decrease* in long-run inconvenience cost against the marginal *increase* in long-run blackout cost.

We show that it is optimal for a consumer to purchase in large quantities if she faces high inconvenience level (relative to her blackout sensitivity). This description would apply, for example, to a consumer who resides far from the kerosene store and does not have a strong need for light. By purchasing in large quantities at each visit, and thereby making fewer visits in the long run, this consumer could save on her inconvenience cost and reduce her overall cost. Conversely, a consumer who is highly sensitive to blackouts (relative to her inconvenience level) will find it optimal to purchase in small quantities. This scenario would be embodied by a consumer who lives close to the kerosene store and has children who must do their homework at night (or has cattle that must be fed at night). For such a consumer, more frequent purchases of small quantities reduces blackouts in the long run while lowering the overall cost. In the next section we shall see that this trade-off – between inconvenience cost and blackout cost – plays a crucial role in determining consumer preferences.

4. Preference for Bulbs vs. Kerosene

Here we compare the consumer's cost C of using bulbs with her optimal cost C_k^* of using kerosene. Before comparing these two costs, we discuss our assumptions on the constituent parameters. First, we assume that the consumption rate λ and the income characteristics (μ and F) do not differ across the two light sources. Second, we assume that the consumer's sensitivity to blackouts (β) and the blackout function (b) are also the same for both sources; thus, blackouts due to discharged light bulbs are no more or less painful than blackouts due to an unreplenished kerosene supply.

It may be that the inconvenience level I associated with using bulbs differs from that associated with using kerosene (call it I_k). For example, if we consider the physical inconvenience of traveling to the purchase center, then currently I_k tends (on average) to be less than I because a village usually has more kerosene sellers than recharging centers. So all else held equal, reducing the inconvenience level of bulbs reduces C and increases the preference for bulbs. Hence we ignore these trivial differences and instead assume the same value of I for both bulbs and kerosene. We shall establish that, even when the inconvenience level and blackout cost structure are the same across these sources of light, consumers may still prefer one over the other (for the reasons we explore next). This approach enables us to derive prescriptions for increasing the preference for rechargeable light bulbs that go beyond merely suggesting that I be reduced.

For a given consumer, let

$$\Delta = C - C_k^* = \frac{I + P + \beta B}{\Psi} - \frac{I + p_k Q_k^* + \beta B_k^*}{\Psi_k^*} \tag{8}$$

be the difference between the long-run costs of bulbs and kerosene. Then the consumer prefers bulbs (resp., kerosene) if $\Delta \le 0$ (resp., if $\Delta > 0$).[12] Our next result characterizes the shape of Δ as a function of the inconvenience level I.

Lemma 2. *The difference Δ is convex in I, and it has an interior minimum in $(0, \infty)$.*

Figure 2 shows various shapes that Δ could take as a function of I. If the minimum of Δ lies above the horizontal axis (as in Figure 2(a)), then Δ is completely positive and so there is no preference region for bulbs. In contrast, if the minimum lies below the horizontal axis (as in panels (b) and (c) of that figure), then depending on the value of $\lim_{I\to 0}\Delta$, Δ crosses that axis either once or twice. (Note that $\lim_{I\to\infty}\Delta = \infty$.) It is easy to see that

$$\lim_{I\to 0}\Delta \le 0 \iff \beta \le \frac{\Psi}{B}\left\{\frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi}\right\} \equiv \hat{\beta}. \tag{9}$$

Our next proposition formally establishes the necessary and sufficient conditions for Δ to take the various shapes plotted in Figure 2.

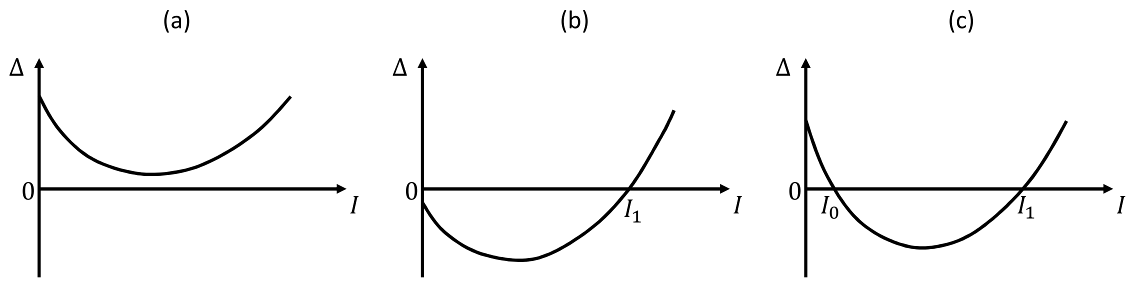

Figure 2 **Shapes of Δ. (a) $p_k < P/Q$: Δ is always positive. (b) $p_k \ge P/Q$ and $\beta \le \hat{\beta}$: Δ crosses the horizontal axis only once. (c) $p_k \ge P/Q$ and $\beta > \hat{\beta}$: Δ crosses the horizontal axis twice.**

Proposition 2. [Consumer preference structure]

(i) *If $p_k < P/Q$, then the consumer prefers kerosene to bulbs.*

[12] The analysis here compares the cost of using bulbs with the cost of using kerosene. Yet consumers could use more than one source of light, and some households have a primary source of light as well as a less preferred secondary source (e.g., candles, torches, firewood) on which they rely in the absence of their primary source. Hence "blackout" could be interpreted as a period during which the consumer uses her secondary source. Our model can accommodate such behavior by changing the value of β, in which case the preceding analysis would address the consumer's choice of bulbs versus kerosene as her *primary* source of light. However, this model does not address the scenario in which kerosene (resp. bulbs) is a secondary source of light when bulbs (resp. kerosene) are the primary source. Instead, the model reflects our objective to analyze the cost structure when a consumer *abandons* kerosene in adopting rechargeable bulbs.

(ii) *If* $p_k \geq P/Q$ *and* $\beta \leq \hat{\beta}$ *then – for given* μ, F, λ, P, Q, p_k, b, *and* β *– there exists a threshold* I_1 *such that the consumer prefers bulbs (resp., kerosene) if the associated inconvenience level* $I \leq I_1$ *(resp.,* $I > I_1$*).*

(iii) *If* $p_k \geq P/Q$ *and* $\beta > \hat{\beta}$ *then – for given* μ, F, λ, P, Q, p_k, b, *and* β *– there exist two thresholds,* I_0 *and* I_1*, such that the consumer prefers bulbs if* $I_0 \leq I \leq I_1$ *and prefers kerosene otherwise.*

For ease of exposition, hereafter we simply assume that $I_0 = 0$ when $\beta \leq \hat{\beta}$; then we can say that the consumer prefers bulbs to kerosene if $I_0 \leq I \leq I_1$ irrespective of the value of her β. Proposition 2 can now be interpreted as follows. First consider the case when $p_k < P/Q$; that is, suppose the unit price of kerosene is lower than that of bulbs. Then, as shown in Figure 2(a), Δ is always positive. Because kerosene is a flexible resource and also cheaper, in this case the consumer is always better-off using kerosene. The implication is that to be operable in the market, bulb recharges should be priced *below* kerosene. So in countries like India and Sri Lanka, where kerosene is heavily subsidized (IFC 2010), adoption of bulbs is expected to be low. Yet where kerosene is costlier than bulbs, as in most African countries, the preference structure is more complex. Before explaining it in detail, we state one more result that enables us to visualize preferences in the β–I plane.

Lemma 3. *Both* I_0 *and* I_1 *are increasing in* β*.*

Figure 3 plots the zeros of Δ (which exist only when $p_k \geq P/Q$) as a function of blackout sensitivity. The graph reveals that there are two regions of preference for kerosene ($\Delta > 0$) and one for bulbs ($\Delta \leq 0$). Region (I) consists of consumers with high levels of inconvenience relative to their blackout sensitivity; we call them "inconvenience-averse" consumers. Because of their high inconvenience levels, these consumers prefer to purchase less frequently and in large quantities to reduce their inconvenience cost. Since kerosene offers them the flexibility of purchasing in large quantities but bulbs do not, they prefer kerosene to bulbs. In contrast, region (III) consists of consumers with high blackout sensitivity relative to their inconvenience levels. Such "blackout-averse" consumers prefer to purchase more frequently but in small quantities to reduce their blackout cost. As in region (I), these consumers prefer kerosene because it gives them the quantity flexibility that bulbs do not. In region (II), consumers have moderate levels of inconvenience and are moderately sensitive to blackouts. They prefer neither large quantities nor small quantities; the capacity offered by bulbs is optimal for them and hence they prefer bulbs to kerosene.

In summary, if kerosene is cheaper than bulbs (i.e., $p_k < P/Q$) then the entire market prefers kerosene; otherwise the market is split into three types of consumers, of which two types (the inconvenience-averse and the blackout-averse) prefer kerosene and one prefers bulbs. We have shown that the key drivers of this preference structure are (i) consumers' liquidity constraints (which lead to undesired blackouts), (ii) the need to balance blackout cost with inconvenience cost

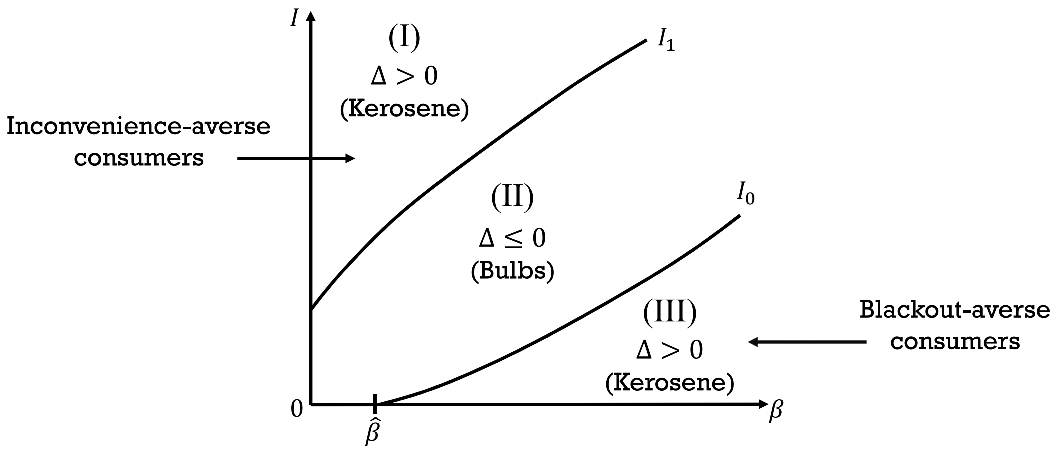

Figure 3 **Preference for bulbs versus kerosene when $p_k \geq P/Q$ (when $p_k < P/Q$, all consumers prefer kerosene).**

associated with repeated purchases, and (iii) the flexibility of kerosene, which is preferred by both inconvenience-averse and blackout-averse consumers. We emphasize that these preferences result not from dissimilar inconvenience levels across the sources but rather from the need to *balance* the inconvenience cost (with the same inconvenience levels across the sources) against the blackout cost. So it should be clear that, if the inconvenience level of consumers with bulbs is lower than that with kerosene, then the lower overall cost of bulbs will make them preferable.

Thus the preference for kerosene, which on the surface may seem economically irrational, makes more sense when we account for the operational underpinnings of consumer behavior. This interplay among liquidity, inconvenience, and flexibility in explaining seemingly irrational consumer preferences has also been noted in other contexts. Karlan and Appel (2011) observe that "people settle for second-best because first-best is inconvenient. They borrow from moneylenders at high rates because microfinance banks have inflexible repayment schedules. They save their money in non-interest-bearing clubs because the clubs offer deposit collection at subscribers' businesses. They send their children to more expensive private schools because private school tuition can be paid in installments. And they treat their broken bones with herbal salve because they don't have to endure a week in the waiting room—and give up a week's earnings in the process—to do it" (p. 227).

Given the preference structure just described, the crucial question faced by a firm is how to increase adoption of its bulbs. We answer this question in Section 5, but we preface that firm-level analysis with the three remarks to follow.

Remark 1. Our model assumes that consumers manage their income via mental accounting and save only for their next purchase. However, if the consumer does save money for *future* light purchases then she is less liquidity constrained than previously supposed. Our model cannot capture such behavior because we would then lose the renewal structure central to our analysis. Yet we can show that our results persist in the extreme case of a consumer who is a good enough saver that she faces no liquidity constraints and therefore no blackouts. That scenario can be expressed in our modeling framework as

$$\Delta = \frac{I+P}{Q/\lambda} - \frac{I_k + p_k Q_k}{Q_k/\lambda} = \lambda\left(\frac{I}{Q} - \frac{I_k}{Q_k}\right) - \lambda\left(p_k - \frac{P}{Q}\right).$$

If we assume (as in Figure 3) that the second term is positive, then Δ could be positive if the inconvenience level of kerosene I_k is significantly lower than I *or* if the kerosene purchase quantity Q_k is significantly larger than Q. Thus inconvenience and flexibility drive preferences for kerosene even when the consumer is not liquidity constrained.

Remark 2. We have assumed that consumers do not delay purchasing light once sufficient money has accrued (see Figure 1). That assumption seems reasonable enough, but here we discuss how the preference structure is affected by relaxing it. For this purpose we consider two plausible scenarios. First, the consumer could delay her purchase for a *random duration* because of inertia, lethargy, or other pre-scheduled tasks. In this case, blackout duration increases by a random amount in the expressions for C and C_k (an increase that affects both cycle length and blackout costs). One can easily verify that, with this modification, Δ continues to be U-shaped in I and that its minimum is less than or greater than zero according as whether $P/Q \le p_k$ or $P/Q > p_k$. It follows that accounting for such random delays in purchases does not affect the preference structure.

Now consider the scenario in which the consumer delays her purchase by some duration that is *strategically chosen* to minimize her overall cost. This setting is analytically complex under a stochastic income model, so our discussion here is based on a deterministic income model with $b(z) = z^2$ and $p_k/\mu > 1/\lambda$.[13] Consumers who use bulbs optimize the cycle length T, whereas those who use kerosene optimize the combination of cycle length T_k and purchase quantity Q_k. The difference between the corresponding optimal costs is then given by

$$\Delta = \min_{T \ge P/\mu} \frac{I + P + \beta([T - Q/\lambda]^+)^2}{T} - \min_{\substack{Q_k \ge 0, T_k \ge 0,\\ T_k \ge p_k Q_k/\mu, T_k \le Q_k M}} \frac{I + p_k Q_k + \beta([T_k - Q_k/\lambda]^+)^2}{T_k}. \tag{10}$$

The constraints $T \ge P/\mu$ and $T_k \ge p_k Q_k/\mu$ ensure that cycle lengths are greater than the time it takes to accrue sufficient money for the purchase. The constraint $T_k \le Q_k M$ (for M an arbitrarily chosen large number) ensures that, if $Q_k = 0$, then T_k is also equal to zero. Lemma A.5 (in the Appendix A) shows that the Δ in (10) is U-shaped in I. The minimum value of Δ is less than (resp., greater than) zero if $P/Q \le p_k(1+\zeta)$ (resp., if $P/Q > p_k(1+\zeta)$), where $\zeta \ge 0$. Because in

[13] The assumption $p_k/\mu > 1/\lambda$ implies that the consumption time Q_k/λ with kerosene is always less than the hitting time $p_k Q_k/\mu$; that is, the kerosene user will always encounter blackouts. This assumption is reasonable given that kerosene is a costly source of light.

this model the consumer is behaving strategically with respect to bulb usage, the preference region for bulbs is larger than in the previous model (as shown also by the higher threshold, $p_k(1+\zeta)$ versus p_k). Hence, our base model is actually the more conservative one in defining the preference region for bulbs.

We believe that the first scenario is more realistic than the second: if consumers have sufficient money, then they simply purchase light at the earliest they can without being too strategic about this decision. In both scenarios, Δ is U-shaped in I and crosses the horizontal axis at most twice – creating two regions of preference for kerosene as before. Since both of these models yield results that differ little from those predicted by our base model, they are not explored further.

Remark 3. The analysis in this section assumes that consumers can purchase kerosene without any quantity restrictions. However, there could be both lower and upper bounds on the purchase quantity due to seller-side measuring and holding constraints. We can easily show that (a) if the lower bound is too small then the preference regions are unaffected whereas (b) if this bound is too large then blackout-averse consumers switch to using bulbs (because kerosene is no longer a flexible resource). A similar intuition applies with regard to the upper bound. Therefore, constraints on kerosene quantity sold do *not* negatively affect the preference region for bulbs.

5. Strategies for Increasing Bulb Adoption and Firm Revenue

As discussed in Section 4, consumers' preference for kerosene is driven by the interplay among liquidity, inconvenience, and flexibility. In this section we discuss several strategies that target each of these three aspects toward the end of mitigating that preference.

5.1. Effects of Preference Structure on Bulb Design

The current design of bulbs is mainly a function of ergonomic factors (size, shape, usability, etc.) and technological considerations (e.g., manufacturing costs, brightness constraints, bulb life). In view of consumer preference structure, however, the firm would do well to consider the additional factors discussed next.

5.1.1. Setting the Bulb Capacity Optimally. In the long run, bulb capacity Q is an important decision variable for the firm. To investigate how changing a rechargeable bulb's capacity can affect adoption rates and also firm revenue, we first translate the preference structure delineated in Section 4 into a market-level demand for bulbs. Thus we augment this model by assuming that consumers are heterogeneous in their levels of inconvenience, as characterized by the density function g and distribution function G. We then define the demand for bulbs as the probability that a randomly chosen consumer in the market prefers bulbs to kerosene. According to Proposition 2, this probability is zero if $P/Q > p_k$. In that case, it would make sense for the firm to abandon the market. Hence we shall assume in this section that $P/Q \le p_k$, thereby imposing a natural lower bound on Q; formally, $Q \ge \underline{Q} = P/p_k$. Then demand is equal to the probability mass lying between the two zeros of Δ and is given by

$$D(Q) = \Pr\big(I_0(Q) \le I \le I_1(Q)\big) = G\big(I_1(Q)\big) - G\big(I_0(Q)\big). \tag{11}$$

Then we have the following comparative statics with respect to Q.

Lemma 4. *When bulb capacity Q increases:*

(i) *the consumer's long-run inconvenience cost, monetary cost, and blackout cost decrease;*

(ii) *I_0 decreases and I_1 increases; and*

(iii) *demand D increases.*

As bulb capacity increases, the consumer visits the recharge center less frequently and so reduces her long-run inconvenience cost. Because the recharge price is unchanged, she also pays less in the long run; this reduces her monetary cost. Moreover, the resulting longer consumption time allows her to accrue more money before the bulb is discharged, which reduces her blackout cost in the long run. Thus, increasing bulb capacity reduces the overall long-run cost of consumers.

Because increased bulb capacity reduces both long-run inconvenience and blackout costs, it attracts some consumers from both the blackout-averse market and the inconvenience-averse market (as shown by the decrease in I_0 and the increase in I_1, respectively); hence the demand for bulbs increases. The long-run revenue derived by the firm from a single consumer who uses bulbs, or the *revenue per consumer*, is equal to the consumer's long-run monetary cost $P/\Psi(Q)$. We can now write the long-run market-level revenue, or the *aggregate revenue*, as

$$R(Q) = \frac{P}{\Psi(Q)} D(Q). \tag{12}$$

Lemma 4 establishes that increasing capacity has two opposing effects on aggregate revenue: it boosts demand but reduces revenue per consumer. This trade-off leads us to characterize the optimal bulb capacity as described in our next proposition. Owing to the analytical complexity of dealing with a generic $b(z)$, in the rest of Section 5.1 we restrict the analysis to $b(z) = z^2$ because doing so yields closed-form expressions for the zeros of Δ (see Lemma A.1). Numerical analysis suggests that these findings continue to hold for powers greater than 2.

Proposition 3. *Suppose that* (a) $b(z) = z^2$, (b) $g(z)/(1-G(z))$ *is increasing in* z, *and* (c) $zg(z)/G(z)$ *is decreasing in* z. *Then* $R(Q)$ *is unimodal in the interval* $\big[\underline{Q}, \overline{Q}\big]$, *where the upper bound* $\overline{Q}$ *uniquely solves the equation* $\frac{\partial I_1/\partial Q}{I_1} = 2\frac{\partial \Psi/\partial Q}{\Psi}$.

Condition (b) is same as increasing hazard rate, which is satisfied by a wide class of distributions (Barlow and Proschan 1965). We show in Lemma A.3 that the distributions commonly used to model positive random variables – such as gamma, log-normal, chi-squared, chi, Weibull, exponential, power-law, and uniform distributions – satisfy condition (c). Hence Proposition 3 holds for several of these distributions. We can prove the unimodality of aggregate revenue only in the interval $[\underline{Q}, \overline{Q}]$ because the behavior of $R(Q)$ beyond $\overline{Q}$ is unclear and characterizing it is analytically cumbersome. Although the upper bound on capacity may seem restrictive, extensive numerical analysis indicates that $\overline{Q}$ is large enough to be not restrictive in practice.

Regardless of the shape that $R(Q)$ takes, there remains a trade-off between demand and revenue per consumer. Although offering more capacity always benefits consumers and increases demand, doing so reduces the firm's revenue derived from consumers in the long run; it follows that an intermediate bulb capacity is optimal. Hence firms should design bulbs that reflect this trade-off. Next we discuss a strategy that increases demand *without* affecting revenue per consumer.

5.1.2. Making Bulbs Flexible. Section 4 revealed that kerosene's flexibility plays an important role in driving consumer preference in its direction. Therefore, making rechargeable bulbs flexible should effectively reduce kerosene usage. Recall that some consumers prefer kerosene because bulb capacity – and the corresponding recharge price – are too high; these blackout-averse consumers would prefer to pay a smaller amount for a lower quantity of light. The firm could attract such consumers by allowing for *partial* recharges. This strategy could be implemented if the firm provided indicators on the bulbs so that consumers and recharging centers could track a bulb's current charge level. Suppose the firm allowed bulbs to be recharged only halfway; then some consumers inclined to purchase lower quantities of light would likely prefer these "half-bulb" recharges even as those who prefer full bulb recharges maintain that preference. So by leveraging consumer heterogeneity in the market, the firm can increase the demand for bulbs by making them flexible. We next establish this result formally.

We start by introducing the concept of *scaling* a bulb which later helps us understand bulb flexibility. If a bulb with recharge price P and capacity Q is scaled by x, then the new recharge price and capacity are (respectively) xP and xQ. If $x < 1$ (resp., $x > 1$) then the bulb is scaled down (resp., up). Let $\Psi^x = xQ/\lambda + \mathbb{E}[xP\tilde{\varepsilon}/\mu - xQ/\lambda]^+$ and $B^x = \mathbb{E}b[xP\tilde{\varepsilon}/\mu - xQ/\lambda]^+$ denote (respectively) the expected cycle length and blackout costs associated with a scaled bulb. First, we note that scale does not affect revenue per consumer:

$$\frac{xP}{\Psi^x} = \frac{xP}{xQ/\lambda + \mathbb{E}[xP\tilde{\varepsilon}/\mu - xQ/\lambda]^+} = \frac{P}{Q/\lambda + \mathbb{E}[P\tilde{\varepsilon}/\mu - Q/\lambda]^+}.$$

This follows because consumers pay a low recharge price more frequently when x is small but pay a high price less frequently when x is large. In our model, the amount paid in the long run under both scenarios is the same. It follows that any strategy that relies solely on scale will affect demand but not revenue per consumer.

Much as in Section 4 (cf. (8)), let

$$\Delta^x = C^x - C_k^* = \frac{I + xP + \beta B^x}{\Psi^x} - \frac{I + p_k Q_k^* + \beta B_k^*}{\Psi_k^*}$$

be the difference between the long-run cost with a scaled bulb (C^x) and the optimal long-run cost with kerosene (C_k^*). Also, let the zeros of Δ^x be I_0^x and I_1^x. We forgo detailing the necessary conditions for these zeros to exist because they are similar to the conditions prescribed in Section 4. The following result establishes the movement of these zeros and characterizes the shape of demand with respect to scale.

Proposition 4. *If* (a) $b(z) = z^2$, (b) $g'(z)/g(z)$ *is decreasing in* z, *and* (c) $zg'(z)/g(z)$ *is decreasing in* z, *then the following statements hold*:

(i) *both* I_0^x *and* I_1^x *increase with* x; *and*

(ii) *demand is unimodal in* x.

Although this result is for $b(z) = z^2$, numerical analysis suggests that it continues to hold for powers greater than 2. Conditions (b) and (c) are not that restrictive because (b) is equivalent to the log-concavity of I's PDF, which is satisfied by many commonly used distributions (see Bagnoli and Bergstrom 2005); several of these distributions also satisfy condition (c) (per Lemma A.3).

Let us now interpret this result. On the one hand, if bulbs are scaled down (i.e., when x is reduced) then consumers use smaller bulbs and pay less for recharging. Since blackout-averse consumers prefer smaller quantities, it follows that the firm gains some of these consumers (i.e., I_0^x moves leftward) but also loses some consumers who prefer bulbs with a greater capacity (i.e., also I_1^x moves leftward). Thus, the blackout-averse market shrinks while the inconvenience-averse market expands. On the other hand, if bulbs are scaled up then the inconvenience-averse market shrinks while the blackout-averse market expands. Hence the firm cannot always increase its demand simply by changing the scale factor. In fact, Proposition 4 shows that demand first increases and then decreases with scale.

Now suppose that the firm allows bulbs to be recharged halfway. This scenario is equivalent to offering consumers a choice between two bulbs: one with $x = 1$ (no scaling) or one with $x = 0.5$. Depending on the consumer's trade-off between the inconvenience cost and the blackout cost, she will prefer one of these bulbs and will end up incurring the minimum of $C^{0.5}$ and C^1. Then

$$\Delta = \min\{C^{0.5}, C^1\} - C_k^* = \min\{C^{0.5} - C_k^*, C^1 - C_k^*\} = \min\{\Delta^{0.5}, \Delta^1\}. \tag{13}$$

Figure 4(a) plots the shapes of $\Delta^{0.5}$, Δ^1 and Δ. As argued previously, consumers who prefer a full bulb recharge to kerosene (i.e., those between I_0^1 and I_1^1) will continue to prefer bulbs. But now the firm gains, in addition, a part of the blackout-averse market (between $I_0^{0.5}$ and I_0^1); the result is an expansion of the preference region for bulbs (spanning from $I_0^{0.5}$ to I_1^1). It is interesting that, unlike the case of when *only* scaled-down bulbs are available, the inconvenience-averse market remains unaffected by scaling.

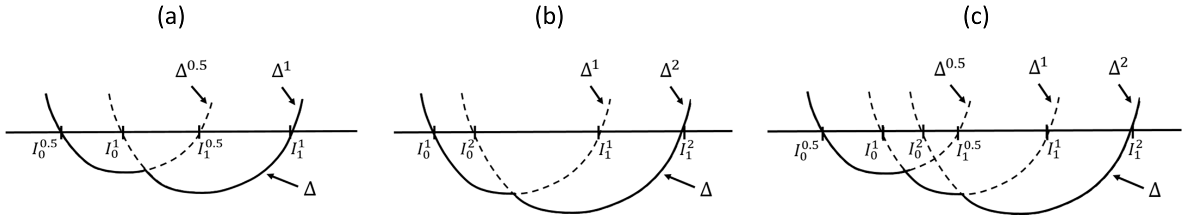

Figure 4 Preference structure with flexible bulbs. (a) Option set $\{0.5, 1\}$: firm gains from blackout-averse market. (b) Option set $\{1, 2\}$: firm gains from inconvenience-averse market. (c) Option set $\{0.5, 1, 2\}$: firm gains from both markets.

Recall that scale does not affect the firm's revenue per consumer. Those who prefer $x = 0.5$ (resp., $x = 1$) pay smaller (resp., larger) amounts but recharge more often (resp., less often), and the firm benefits equally in the long run from both of these revenue streams. Thus partial recharging – unlike increasing capacity, as in Section 5.1.1 – is a strategy that increases demand but does not affect revenue per consumer; the outcome is higher aggregate revenue. We can easily show that this result holds for any $x < 1$.

We have seen that providing the option to choose between $x = 0.5$ and $x = 1$ shrinks the blackout-averse market without affecting the inconvenience-averse market. Figure 4(b) shows that, conversely, allowing consumers to choose between $x = 1$ and $x = 2$ shrinks the inconvenience-averse market without affecting the blackout-averse market. Matters improve still further (from the firm's standpoint) if consumers can choose among all three of these scaling factors: as can be seen in panel (c) of the figure, both the inconvenience-averse and blackout-averse markets shrink. So by scaling up and offering multiple partial recharge levels, the firm can offer a light source that mimics the flexibility of kerosene and so results in higher demand.[14]

It is important to note that there are practical limits to the extent of scaling bulbs and to the number of partial recharge levels. First of all, larger bulbs cost more to manufacture, increasing their upfront price and hence resulting in fewer purchases; larger bulbs may also prove inconvenient for consumers' applications and to carry them for recharges. Second, offering a variety of partial recharge levels could negatively impact the bulb's battery life while also requiring recharge centers to upgrade their technology to track the charge level in the bulb. Yet if implemented in moderation, this strategy could lead to considerable improvements in adoption rates and aggregate revenue.

[14] We demonstrate this effect using $x = 0.5$ and $x = 2$, but clearly it will persist for other scale values.

5.2. Alleviating Liquidity Constraints

Recall from Section 4 that not only flexibility concerns but also liquidity constraints and inconvenience levels play an important role in determining consumer preferences. Here we investigate the strategies for (and effects of) alleviating liquidity constraints. In Section 3 we described how a consumer's liquidity constraints determine the time it takes for her to accrue enough money to purchase light (i.e., the hitting time), which in turn determines the blackout duration. Recall as well that the hitting time $\tilde{\tau} = (P/\mu)\tilde{\varepsilon}$ has two components: P/μ, which captures the *affordability* of recharges; and $\tilde{\varepsilon}$, which characterizes the *uncertainty* in a consumer's life. We next discuss strategies aimed at each of these components and their effect on the adoption of bulbs.

5.2.1. Affecting Affordability. Recharges become more affordable when either the recharge price P falls or the saving rate μ rises. Strategies involving the recharge price (e.g., price discounts, subsidies) reduce the price paid by consumers for recharges. However, firms serving the BoP have virtually no flexibility with regard to prices. Because of the context (i.e., impoverished regions) in which they operate, there is nearly always an upper bound on what can be charged. At the same time, there is a lower bound stemming from the annual cash flow required to repay venture capitalists or service bank debt. Our discussions with Nuru's CEO revealed that consumers tend to view bulb recharges as analogous to mobile phone recharges and so "anchor" on the latter's fees.[15] Furthermore, to the extent that bulb recharges are costlier than mobile phone recharges, consumers have an incentive to "hack" bulbs so they can be recharged at a mobile phone recharge center. Hence the firm's full bulb recharge price cannot differ much from the recharge price of a mobile phone. The effect of all these constraints is that the firm is forced to operate within a very narrow price range.

That being said, the saving rate μ could be *artificially* increased by strategies such as cash drops and allowing consumers to recharge on credit. As argued next, the saving rate could also be *naturally* increased by providing consumers with a "safe box" to save money for light. It is well established that poor people have difficulty saving as much as they would like. Their *scarcity mindset* (Mullainathan and Shafir 2013) makes efficient money management difficult, which in turn leads to more scarcity in their lives – the so-called scarcity trap. Yet there is evidence from a field experiment in Kenya (Dupas and Robinson 2013) that a simple box with a lock and key results in consumers substantially increasing their investment in preventative health and hence reducing their vulnerability to health shocks. These authors argue that this safe box "can act as a commitment device: once money was put into the box, it was labeled as health savings, which made

[15] Conversation with Sameer Hajee (CEO of Nuru) in February 2014. People at the BoP can recharge their mobile phones only by visiting a dedicated recharge center.

it less fungible and therefore less susceptible to friends' requests and daily spending" (p. 1163). We can extend these arguments to the context of light and argue that providing consumers with a safe box for light might increase their saving rate.

By Lemma 1(i), either a decrease in P or an increase in μ will reduce blackouts experienced by the consumer. Note, however, that reducing blackouts is not the same as reducing a consumer's *sensitivity* to blackouts (β). Although reducing the latter would lower consumer cost, the intrinsic nature of sensitivity makes it difficult to alter. Yet even reducing blackouts per se need not reduce the consumer's overall cost, a claim we formalize as follows.

Proposition 5. [Shapes of long-run cost components]

(i) *The long-run inconvenience cost I/Ψ is decreasing in P and increasing in μ.*

(ii) *The long-run monetary cost P/Ψ is increasing in P and decreasing in μ.*

(iii) *The long-run blackout cost $\beta B/\Psi$ is increasing in P and decreasing in μ.*

This proposition shows that all the components of consumer cost are *not* monotonic – in the same direction – with respect to P and μ. So even though one might expect the long-run cost C simply to increase with P and decrease with μ, this need not always be the case. As price increases, the consumer naturally uses less light. If our overall cost consisted of *only* the consumption, then the cost would (as expected) increase with P. However, inconvenience is another factor in our cost model. If consumption declines then so does the consumer's inconvenience. Thus, a price increase hurts the consumer in terms of reduced consumption but also benefits her in terms of reduced inconvenience. Similarly, the consumer's light consumption increases with her saving rate; when only consumption is considered, her cost decreases with μ. Yet because she is consuming more, the consumer must pay more money and also incurs higher inconvenience. In short: because our model incorporates several cost dimensions, changes in P and μ could result in the consumer benefiting in terms of some dimensions but not others. Hence it is hardly surprising that we observe non-monotonicities in C.[16]

Remark 4. *Generally speaking, long-run consumer cost C is non-monotonic in P and μ. For example, if the PDF of $\tilde{\varepsilon}$ is log-concave and if $b(z) = z^m$ for $m > 1$ an integer, then the following statements hold.*

(i) *C is U-shaped in the saving rate μ.*

[16] The non-monotonicity of cost arises because consumers in our base model do not strategically delay their purchases. We can use the model from Remark 2 to verify that the consumer cost *with strategic delays* is monotonically increasing in P and decreasing in μ. As pointed out previously, our base model takes a more conservative approach in defining preferences for bulbs than does the model with strategic delays. It follows that the prescription derived from the former is also more conservative than that derived from the latter. Therefore, if consumers purchase strategically then the firm can increase adoption simply by alleviating liquidity constraints.

(ii) *C is N-shaped in the recharge price P if $I > \hat{I}$ and is otherwise increasing; here the threshold $\hat{I}$ is a function of μ, F, λ, Q, b, and β.*

Proof is given in the Appendix A.

In other words, increasing affordability by reducing P and/or boosting μ may not always lower the consumer's overall cost. The main reason for these ambiguous effects is that increased affordability benefits consumers in terms of consumption but affects them negatively in terms of inconvenience (since increased consumption in the long run also increases inconvenience). As it could also lead to significantly higher costs, it could induce some consumers to prefer using kerosene. These dynamics explain why the strategies discussed above need not lead to greater adoption of rechargeable bulbs. Given that more inconvenience in the long run is the culprit here, those strategies are more likely to increase the adoption rate if they are implemented in conjunction with inconvenience-reducing strategies; these are discussed in Section 5.3.

5.2.2. Affecting Income Uncertainty. Here we argue that micropayments using mobile technology, when implemented effectively, could reduce consumers' income uncertainty. With the increased market penetration of this technology in Sub-Saharan Africa and in developing Asia – penetration levels in 2014 were, respectively, 39% and 45% (GSMA 2015) – mobile phone–based money transfer services have become fairly prominent in these regions. In Kenya, for instance, the solar-based lighting solutions marketed by M-Kopa[17] depend on monitoring consumer payments via the M-Pesa mobile payment service. In our context, this approach would translate into the firm asking a consumer not to pay P all at once but instead to pay a smaller amount p over n periods (such that $P = np$). We remark that this approach will not reduce inconvenience for the consumer, who must still travel to the recharging center to get the bulb recharged. Yet as we show next, it does reduce her income uncertainty and therefore reduces the expected blackout duration.

The logistics of such an arrangement could be as follows. The consumer purchases a bulb that is fully charged. While using this bulb, she makes micropayments for the next recharge through her mobile phone. Once the n required payments have been made, the consumer's bulb can be recharged at the center at no additional cost. This process then repeats after the bulb is recharged.

Figure 5 plots the hitting times without micropayments (panel (a)) and with micropayments where $n = 3$ (panel (b)). As we can see in Figure 5(b), a consumer's income needs to hit a lower threshold p yet must do so n times per cycle. In this case the hitting time is given by $\tilde{\tau}_n = p\tilde{\varepsilon}_1/\mu + \cdots + p\tilde{\varepsilon}_n/\mu = P\tilde{e}_n/\mu$, where $\tilde{e}_n = \sum_i \tilde{\varepsilon}_i/n$. The corresponding expected blackout duration

[17] http://www.m-kopa.com/

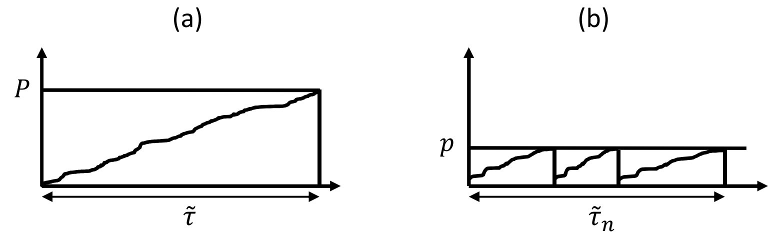

Figure 5 Mobile micropayments and hitting time in a cycle. (a) $\tilde{\tau}$ is the hitting time under our base model's payment scheme. (b) $\tilde{\tau}_n$ is the hitting time under a mobile micropayment scheme with $n=3$ and $P=3p$.

and blackout cost are given by $L_n = \mathbb{E}[P\tilde{e}_n/\mu - Q/\lambda]^+$ and $B_n = \mathbb{E}b[P\tilde{e}_n/\mu - Q/\lambda]^+$, respectively. Then our next lemma follows once we note that $\tilde{e}_n$ is a mean-preserving spread of $\tilde{e}_{n+1}$.[18]

Lemma 5. *With mobile micropayments, both the expected blackout duration L_n and the expected blackout cost B_n are decreasing in n.*

The effect on blackouts of increasing n is identical to that of increasing μ. Numerical analyses suggest that, much as in Remark 4, the consumer's overall cost is non-monotonic in n; hence mobile micropayments may not always result in higher adoption rates. Once again, however, if this approach is combined with inconvenience-reducing strategies then the adoption of bulbs is likely to increase.

5.3. Reducing Inconvenience

Although we did not account for any difference in inconvenience levels between bulbs and kerosene when modeling the structure of consumer preferences, it is clear that reducing the inconvenience associated with using bulbs would decrease consumer cost and increase the adoption of bulbs. In this section we suggest several strategies for reducing consumer inconvenience and also discuss their limitations.

The firm could significantly reduce inconvenience by implementing a door-to-door bulb recharge service. For example, employees at the recharging center could collect discharged bulbs from consumers at their respective residences and then deliver the recharged bulbs later that day. Yet not only would implementing such a service be costly, it is not clear how employees could be incentivized to do so. An alternative approach would be for the firm to increase the number of recharging centers. If these centers are optimally located, then the average consumer would not have to travel so far for a recharge; this inconvenience-reducing strategy could have the further benefit of increasing competition among centers, which might motivate employees to improve the quality of their service. Of course, such an approach would increase the firm's fixed costs associated with establishing multiple recharge centers (hiring and training employees, setting up the equipment, etc.).

[18] It is easy to see that the variance of $\tilde{e}_n$ is greater than that of $\tilde{e}_{n+1}$. Moreover, we can write $\tilde{e}_n = \tilde{e}_{n+1} + \tilde{z}$ for $\tilde{z} = [(\tilde{\varepsilon}_1 - \tilde{\varepsilon}_2) + 2(\tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) + \cdots + n(\tilde{\varepsilon}_n - \tilde{\varepsilon}_{n+1})]/((n+1)n)$, where $\mathbb{E}[\tilde{z}] = 0$. Then, by the definition in Rothschild and Stiglitz (1970), $\tilde{e}_n$ is a mean-preserving spread of $\tilde{e}_{n+1}$.

Recharging bulbs now takes about 1–2 hours with a solar-based system or 20–30 minutes with a mechanical system. Firms could reduce consumer waiting time to zero by adopting the business model of "battery switching" stations for electric cars (Avci et al. 2015). Under this model, the consumer would not purchase a bulb but instead would subscribe to a service that provides access to one whenever needed. It would then be the recharge center's responsibility to maintain an inventory of recharged bulbs and to exchange (for a modest fee) consumers' discharged bulbs with recharged ones. The firm could also set up multiple such centers in each village, which would reduce consumer travel distances in addition to their waiting times. However, the firm should be cautious when implementing this business model. If the firm has already sold bulbs to consumers, then the "endowment" effect (Thaler 1980) could render the service ineffective; consumers might be unwilling to exchange their own bulbs with some random other bulbs. Hence this subscription approach would likely be far more successful if implemented in villages that constitute a new market.

Firm could also encourage (or incentivize) consumers to "pool" their bulb recharges. Thus, for example, a single household among five to ten like households could take the responsibility of collecting and transporting all their bulbs to be recharged; members of the group would share this burden by taking turns. So instead of reducing each household's inconvenience, this strategy would reduce the group's collective inconvenience. One can expect, though, that the strategy would involve coordination problems among households with different usage patterns.

The technology used to recharge bulbs can often be used also to recharge mobile phones. It follows that the firm could incentivize consumers to combine their bulb recharging with mobile phone recharging, thereby reducing inconvenience in the long run. Once again, however, coordinating difficulties will likely arise to the extent that mobile recharging frequencies differ from those for bulb recharging. We believe that the efficacy of these various strategies is best understood by experimenting with them in the field. One could also build formal models to investigate which strategy should perform better and under what circumstances. That task is beyond the scope of this paper, so we leave it for future research.

6. Conclusions

Understanding why poor people prefer one technology over another is crucial in designing effective policies and implementing suitable business models. Many technologies, although perceived at the outset to be beneficial to the poor, are not easily adopted; examples include insecticide-treated bed nets (Cohen and Dupas 2010), water purification solutions (Ashraf et al. 2010), fertilizers (Duflo et al. 2011), roundabout water pumps (Sodhi and Tang 2011), and improved stoves (Hanna et al. 2012). Several books have been written to demystify seemingly irrational preferences of the poor, leading to subsequently designed programs that made a profound difference in their lives (Banerjee and Duflo 2011, Karlan and Appel 2011, Mullainathan and Shafir 2013). We believe that our research contributes to this impactful stream of literature by explaining preferences for light sources at the bottom of the pyramid and designing strategies to increase adoption of clean alternatives.

We build a novel consumer behavior model that accounts for the liquidity constraints faced by poor consumers and for the inconvenience they incur from blackouts and repeated purchases. This model allows us to characterize consumer preferences for each technology by assuming that they choose the light source which leads to lower long-run cost. Our analysis reveals that rechargeable bulbs are a viable market alternative only if they are offered at a lower marginal price than kerosene. So in countries such as India and Sri Lanka, where kerosene is heavily subsidized (IFC 2010), the adoption of rechargeable bulb technology is expected to be low. In fact, Nuru Energy's recharge revenue from India is but a fifth of its revenue from African markets (Carrick and Santos 2013). Yet in African countries, where using kerosene is significantly more expensive than bulbs, we find that preferences for the two technologies are mixed. Our model indicates that consumers will continue to prefer kerosene if they are strongly averse either to blackouts or to recharge inconvenience. The reason is that kerosene offers consumers *flexibility* with regard to quantity, which helps them reduce long-run blackout and inconvenience costs.

Of course, there are some other plausible explanations for the observed consumer preference for kerosene: lack of information about bulbs or low appreciation for their benefits, a mistrust of new technologies, and habit formation. Yet our model suggests that consumers might prefer kerosense even when behavioral factors (e.g., ignorance, trust, habits) do *not* play a role.

We propose several ways in which a firm offering clean rechargeable technologies could gain consumers from market segments that prefer kerosene. At any given recharge price, an increase in bulb capacity allows the firm to capture consumers from both inconvenience-averse and blackout-averse segments of the market. However, with larger bulbs the recharge frequency declines and so the firm's revenue per consumer decreases. So even though increasing bulb capacity always benefits consumers, beyond a certain threshold such increases are detrimental to the firm's long-run revenue; therefore, an intermediate bulb capacity is optimal. The firm can preclude declines in revenue per consumer by varying the *scale*, rather than the *capacity*, of its rechargeable bulbs. Then the firm gains consumers from one market segment, but it also loses consumers from the other. Yet by enabling *partial* recharges (which is equivalent in theory to providing multiple scale options), the firm could gain more blackout-averse consumers without losing any inconvenience-averse consumers. Moreover, the combination of scaling bulbs up and partial recharging allows the firm to gain from both market segments without any decline in its revenue per consumer.

In essence, this strategy increases adoption rates by leveraging the heterogeneity in consumer preferences across the market.

Strategies that reduce consumer inconvenience also reduce the overall long-run cost of bulbs relative to kerosene, leading to increased adoption. The same cannot be said, however, of strategies that aim to alleviate liquidity constraints: providing consumers with a "safe box" to save money for light (increases the saving rate), introducing mobile micropayments (helps reduce income uncertainty), offering price discounts or subsidies, and so forth. Strategies of this type reduce blackouts, but they result in higher consumption that in turn leads to increased long-run inconvenience costs. It follows that the effect of such strategies on *overall* consumer cost, and hence on adoption, is ambiguous. However, adoption levels are likely to increase when these strategies are implemented in conjunction with strategies aimed at reducing consumer inconvenience. Thus our results highlight the crucial role that inconvenience plays in determining consumer preferences and hence the need to mitigate that inconvenience if increased bulb adoption is to occur.

Our research could be extended in several directions. On the modeling side, we have assumed that consumers use either bulbs or kerosene – whichever costs less in the long run. In practice, however, a single household might well use both light sources. Thus one could explicitly model *portfolios* of light sources and investigate the preference structure over feasible portfolios. On the empirical side, the efficacy of our proposed strategies for increasing the adoption of bulbs could be tested in the field using randomized controlled trials. Such efforts are currently undertaken by, for example, International Growth Center[19] and ENERGIA[20] with Nuru Energy and Innovations for Poverty Actions[21] in Rwanda. These experiments could serve also as a source of fine-grained data on the characteristics (light usage, expenditures, etc.) of poor households, which would help us better understand their consumption and saving patterns.

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[19] http://www.theigc.org/wp-content/uploads/2015/08/Energy-projects.pdf

[20] http://energia.org/2015/03/gender-research-area-5/

[21] http://www.poverty-action.org/

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Appendix A: Proofs

Proof of Lemma 1. Let $\rho = Q\mu/(\lambda P)$.

(i) The results for L follow by noting that the corresponding partial derivatives have the desired sign:

$$\frac{\partial L}{\partial P} = \frac{1}{\mu}\int_{\rho}^{\infty} \varepsilon\, dF(\varepsilon) \geq 0, \quad \frac{\partial L}{\partial Q} = -\frac{1}{\lambda}(1 - F(\rho)) \leq 0, \quad \frac{\partial L}{\partial \mu} = -\frac{P}{\mu^2}\int_{\rho}^{\infty} \varepsilon\, dF(\varepsilon) \leq 0. \tag{14}$$

Similarly, the results for B follow because b is increasing:

$$\begin{aligned} \frac{\partial B}{\partial P} &= \frac{1}{\mu}\int_{\rho}^{\infty} b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\varepsilon\, dF(\varepsilon) \geq 0, \\ \frac{\partial B}{\partial Q} &= -\frac{1}{\lambda}\int_{\rho}^{\infty} b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right) dF(\varepsilon) \leq 0, \\ \frac{\partial B}{\partial \mu} &= -\frac{P}{\mu^2}\int_{\rho}^{\infty} b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\varepsilon\, dF(\varepsilon) \leq 0. \end{aligned} \tag{15}$$

(ii) Recall that $\Psi = Q/\lambda + L$. Hence Ψ is increasing in Q because

$$\frac{\partial \Psi}{\partial Q} = \frac{F(\rho)}{\lambda} \geq 0. \tag{16}$$

The results with respect to (w.r.t.) P and μ follow immediately from part (i). $\square$

Proof of Proposition 1. Let $\rho_k = \mu/(\lambda p_k)$.

(i) Let $\Psi_k = Q_k/\lambda + \mathbb{E}[p_k Q_k \tilde{\varepsilon}/\mu - Q_k/\lambda]^+ = Q_k(1/\lambda + \mathbb{E}[p_k\tilde{\varepsilon}/\mu - 1/\lambda]^+) = Q_k\hat{\Psi}_k$, where $\hat{\Psi}_k$ is independent of Q_k. Then

$$C_k = \frac{I + p_k Q_k + \beta B_k}{\Psi_k} = \frac{p_k}{\hat{\Psi}_k} + \frac{I + \beta B_k}{Q_k \hat{\Psi}_k}.$$

The long-run monetary cost is independent of Q_k. Note that

$$\frac{B_k}{Q_k} = \frac{1}{Q_k}\int_{\rho_k}^{\infty} b\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) dF(\varepsilon) \quad \text{and} \quad \frac{\partial (B_k/Q_k)}{\partial Q_k} = \frac{H_k}{Q_k^2};$$

here

$$H_k = \int_{\rho_k}^{\infty} \left[\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) b'\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) - b\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right)\right] dF(\varepsilon).$$

The integrand (in brackets) is of the form $zb'(z) - b(z)$, where $z \geq 0$; it is positive and increasing because it equals zero when $z = 0$ and its derivative is $zb''(z) \geq 0$ (as follows from the convexity of b). Therefore, H_k is also positive and increasing in Q_k. The derivative of C_k w.r.t. Q_k is then given by

$$\frac{\partial C_k}{\partial Q_k} = \frac{1}{Q_k^2 \hat{\Psi}_k}[-I + \beta H_k].$$

Since $\lim_{Q_k\to 0} H_k = 0$ and since $\lim_{Q_k\to\infty} H_k = \infty$ (by our assumption that $\lim_{z\to\infty} zb'(z) - b(z) = \infty$), it follows that the term in brackets crosses the x-axis exactly once from below, which yields the unique minimum Q_k^*.

(ii) Observe that

$$\frac{\partial^2 C_k}{\partial I \partial Q_k} = -\frac{1}{Q_k^2 \hat{\Psi}_k} \le 0 \quad \text{and} \quad \frac{\partial^2 C_k}{\partial \beta \partial Q_k} = \frac{H_k}{Q_k^2 \hat{\Psi}_k} \ge 0.$$

The desired result now follows from Topkis (1998). $\square$

Proof of Lemma 2. The derivative of Δ w.r.t. I is

$$\frac{\partial \Delta}{\partial I} = \frac{\partial C}{\partial I} - \frac{\partial C_k^*}{\partial I} = \frac{1}{\Psi} - \frac{1}{Q_k^* \hat{\Psi}_k}. \tag{17}$$

According to Proposition 1(ii), Q_k^* is increasing in I. Hence the derivative in (17) is also increasing in I and so Δ is convex in I. It is easy to see that $\lim_{I\to 0} Q_k^* = 0$ and $\lim_{I\to\infty} Q_k^* = \infty$, from which it follows that $\lim_{I\to 0} \partial\Delta/\partial I = -\infty$ and $\lim_{I\to\infty} \partial\Delta/\partial I = 1/\Psi > 0$. So here the derivative of Δ crosses the x-axis exactly once from below, which leads to a unique minimum. $\square$

Proof of Proposition 2. First note that the preference region for bulbs exists if and only if (iff) the minimum value of Δ is less than zero. We shall demonstrate that this minimum is negative iff $P/Q \le p_k$. Let Δ_m denote the minimum value of Δ, and let this minimum be achieved at I_m. Then

$$\Delta_m = \left(\frac{I}{\Psi} - \frac{I}{Q_k^* \hat{\Psi}_k}\right)\bigg|_{I_m} + \left(\frac{P}{\Psi} - \frac{p_k}{\hat{\Psi}_k}\right)\bigg|_{I_m} + \left(\frac{\beta B}{\Psi} - \frac{\beta B_k^*}{Q_k^* \hat{\Psi}_k}\right)\bigg|_{I_m}. \tag{18}$$

By (17), the first term in parentheses is equal to zero at I_m. The second term in (18) is independent of I_m. To show that it is negative iff $P/Q \le p_k$, we rewrite P/Ψ as $p_l/\hat{\Psi}_l$; here $p_l = P/Q$ is the unit price of bulbs and $\hat{\Psi}_l = \Psi/Q$. Now note that the function $p/(1/\lambda + \mathbb{E}[p\tilde{\varepsilon}/\mu - 1/\lambda]^+)$ is increasing in p because its derivative,

$$\frac{F(\mu/(\lambda p))/\lambda}{(1/\lambda + \mathbb{E}[p\tilde{\varepsilon}/\mu - 1/\lambda]^+)^2},$$

is positive. Therefore, if $p_l \le (>)\, p_k$ then $p_l/\hat{\Psi}_l \le (>)\, p_k/\hat{\Psi}_k$.

We now show that the third term of (18) is also negative iff $P/Q \le p_k$. By (17), the sign of this term depends only on the sign of $B - B_k^*$ evaluated at I_m, which is given by

$$\mathbb{E}b\left[\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right]^+ - \mathbb{E}b\left[\frac{p_k Q_k^* \tilde{\varepsilon}}{\mu} - \frac{Q_k^*}{\lambda}\right]^+\Bigg|_{I_m} = \mathbb{E}b\left[\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right]^+ - \mathbb{E}b\left[\frac{\Psi}{\hat{\Psi}_k}\left(\frac{p_k\tilde{\varepsilon}}{\mu} - \frac{1}{\lambda}\right)\right]^+.$$

Now we can see that

$$\left(\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right) - \frac{\Psi}{\hat{\Psi}_k}\left(\frac{p_k\tilde{\varepsilon}}{\mu} - \frac{1}{\lambda}\right) = \frac{\Psi\tilde{\varepsilon}}{\mu}\left(\frac{p_l}{\hat{\Psi}_l} - \frac{p_k}{\hat{\Psi}_k}\right) + \frac{Q}{\lambda\hat{\Psi}_k}(\hat{\Psi}_l - \hat{\Psi}_k)$$

is negative iff $p_l \le p_k$. The desired result follows by noting that the positive part, the monotonically increasing b, and the expectation preserve this relationship. $\square$

Proof of Lemma 3. Because the I_j ($j \in \{0,1\}$) are zeros of Δ, we can use the implicit function theorem to write their derivatives w.r.t. β as

$$\frac{\partial I_j}{\partial \beta} = -\frac{\partial \Delta / \partial \beta}{\partial \Delta / \partial I}\bigg|_{I_j}.$$

By definition, $\partial \Delta / \partial I$ is negative (resp. positive) at I_0 (resp. I_1). Using the technique employed for proving Lemma 2 and Proposition 2, we can show that, for any given I, there are two zeros of Δ w.r.t. β; we label them β_0 and β_1 (with $\beta_0 < \beta_1$). At any zero of Δ, by definition we have

$$I\left(\frac{1}{\Psi} - \frac{1}{\Psi_k^*}\right) + \beta\left(\frac{B}{\Psi} - \frac{B_k^*}{\Psi_k^*}\right) = \frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi} \implies I\frac{\partial \Delta}{\partial I} + \beta\frac{\partial \Delta}{\partial \beta} > 0. \tag{19}$$

Consider a point on the curve $I_0(\beta)$. By definition, $\partial \Delta / \partial I < 0$. We can use (19) to show that $\partial \Delta / \partial \beta > 0$ at this zero; hence $I_0(\beta)$ is increasing in β. Now consider a point on the curve $\beta_0(I)$. By definition, $\partial \Delta / \partial \beta < 0$, and again by (19) we have $\partial \Delta / \partial I > 0$ at this zero. Hence the $\beta_0(I)$ curve is increasing and so its inverse, the $I_1(\beta)$ curve, must also be increasing. $\square$

Proof of Lemma 4.

(i) Long-run inconvenience and monetary costs decrease with Q because, by Lemma 1(ii), cycle length increases with Q. Since B is decreasing in Q, Ψ is increasing in Q, and both B and Ψ are positive, it follows that their ratio (and hence the long-run blackout cost) is decreasing in Q.

(ii) Because I_j ($j \in \{0,1\}$) are zeros of Δ, we can use the implicit function theorem and write

$$\frac{\partial I_j}{\partial Q} = -\frac{\partial \Delta / \partial Q}{\partial \Delta / \partial I}\bigg|_{I_j}.$$

The result then follows by noting that Δ decreases with Q (since, by part (i), C decreases with Q) and that Δ is downward sloping (resp., upward sloping) at I_0 (resp., at I_1).

(iii) The derivative of demand w.r.t. Q is

$$\frac{\partial D}{\partial Q} = g(I_1)\frac{\partial I_1}{\partial Q} - g(I_0)\frac{\partial I_0}{\partial Q}.$$

The result now follows easily from part (ii). $\square$

The four lemmas that follow will be used to prove Propositions 3 and 4. Lemmas A.1 and A.2 characterize the properties of I_0 and I_1 when $b(z) = z^2$; Lemmas A.3 and A.4 characterize the properties of the distribution of I.

Lemma A.1. *If $b(z) = z^2$, then the zeros of Δ are given by*

$$\sqrt{I_0} = \Psi\left(\frac{\sqrt{\beta \hat{B}_k}}{\hat{\Psi}_k} - \sqrt{\delta}\right) \quad \textit{and} \quad \sqrt{I_1} = \Psi\left(\frac{\sqrt{\beta \hat{B}_k}}{\hat{\Psi}_k} + \sqrt{\delta}\right), \tag{20}$$

where

$$\delta = \beta\left(\frac{\hat{B}_k}{\hat{\Psi}_k^2} - \frac{B}{\Psi^2}\right) + \frac{1}{\Psi}\left(\frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi}\right) \quad \textit{and} \quad \hat{B}_k = \int_{\frac{\mu}{\lambda p_k}}^{\infty} \left(\frac{p_k \varepsilon}{\mu} - \frac{1}{\lambda}\right)^2 dF(\varepsilon).$$

Proof of Lemma A.1. We first obtain the optimal kerosene cost in closed form by rewriting C_k as

$$C_k = \frac{p_k}{\hat{\Psi}_k} + \frac{1}{\hat{\Psi}_k}\left(\frac{I}{Q_k} + \beta Q_k \hat{B}_k\right).$$

The term in parentheses is a convex function of Q_k, so it follows that

$$Q_k^* = \sqrt{\frac{I}{\beta \hat{B}_k}} \quad \text{and} \quad C_k^* = \frac{p_k}{\hat{\Psi}_k} + \frac{2\sqrt{I\beta\hat{B}_k}}{\hat{\Psi}_k}. \tag{21}$$

Using (21), we can now rewrite (8) as

$$\Delta = \frac{I + P + \beta B}{\Psi} - \left(\frac{p_k}{\hat{\Psi}_k} + \frac{2\sqrt{I\beta\hat{B}_k}}{\hat{\Psi}_k}\right).$$

Then $\Delta = 0$ is a quadratic equation in $\sqrt{I}$. Its solutions are given by (20), and δ is the determinant of this equation. $\square$

Lemma A.2. *Let* $i_0 = \sqrt{I_0}$ *and* $i_1 = \sqrt{I_1}$*. If* $b(z) = z^2$*, then:*

(i) Ψ/I_1 *is decreasing in* Q;

(ii) Ψ/i_1 *is U-shaped in* Q;

(iii) $\frac{\partial i_1/\partial Q}{\partial \Psi/\partial Q}$ *is decreasing in* Q; *and*

(iv) $-\Psi\frac{\partial i_0/\partial Q}{\partial \Psi/\partial Q}$ *is decreasing in* Q.

Proof of Lemma A.2.
(i) From (20) we obtain

$$\frac{I_1}{\Psi} = \left(\sqrt{\Psi}\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} + \sqrt{\Psi\delta}\right)^2 \quad \text{for} \quad \Psi\delta = \Psi\beta\frac{\hat{B}_k}{\hat{\Psi}_k^2} - \frac{\beta B}{\Psi} + \left(\frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi}\right).$$

The result follows once we note that, by Lemma 4(i), $\Psi\delta$ is increasing in Q.
(ii) By (20), $i_1/\Psi = \sqrt{\beta\hat{B}_k}/\hat{\Psi}_k + \sqrt{\delta}$. Therefore, it is enough to show that δ is unimodal in Q. Its derivative w.r.t. Q is given by

$$\frac{\partial\delta}{\partial Q} = \frac{1}{\Psi^2}\frac{\partial\Psi}{\partial Q}\left(-\beta\frac{\partial B/\partial Q}{\partial\Psi/\partial Q} + \frac{2\beta B}{\Psi} + \frac{2P}{\Psi} - \frac{p_k}{\hat{\Psi}_k}\right).$$

From (15) and (16), we see that $\partial B/\partial Q$ is negative and increasing in Q whereas $\partial\Psi/\partial Q$ is positive and increasing in Q. Therefore, $-\frac{\partial B/\partial Q}{\partial\Psi/\partial Q}$ is decreasing in Q. Since B/Ψ and P/Ψ are also decreasing in Q (by Lemma 4), the term in parentheses decreases with Q. At $Q = P/p_k$ (which is the minimum possible capacity), this term takes a positive value because $P/\Psi = p_k/\hat{\Psi}_k$. Its limit as $Q \to \infty$ is $-p_k/\hat{\Psi}_k < 0$ and so it crosses the x-axis only once from above, which renders δ unimodal.
(iii) Using the expression for i_1 from (20), we have

$$\frac{\partial i_1/\partial Q}{\partial\Psi/\partial Q} = \frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} - \frac{\beta}{2\Psi\sqrt{\delta}}\frac{\partial B/\partial Q}{\partial\Psi/\partial Q} + \frac{1}{2\sqrt{\delta}}\left\{\frac{2\beta\hat{B}_k}{\hat{\Psi}_k^2} + \frac{p_k}{\hat{\Psi}_k\Psi}\right\}.$$

The first term on the right-hand side (RHS) is independent of Q whereas the second term is decreasing in Q because, as argued in part (ii), $-\frac{\partial B/\partial Q}{\partial \Psi/\partial Q}$ is positive and decreasing in Q while $\Psi\sqrt{\delta}$ is positive and increasing in Q. The third term is also decreasing in Q because its derivative w.r.t. Q is

$$\frac{1}{4\delta\sqrt{\delta}}\left\{-\frac{1}{\Psi^3}\frac{\partial \Psi}{\partial Q}\left(\frac{p_k}{\hat{\Psi}_k}\right)^2+\frac{\beta}{\Psi^2}\frac{\partial B}{\partial Q}\left(\frac{2\beta\hat{B}_k}{\hat{\Psi}_k^2}+\frac{p_k}{\hat{\Psi}_k\Psi}\right)-\frac{4\beta\hat{B}_k}{\hat{\Psi}_k^2}\frac{\partial \Psi}{\partial Q}\frac{(P+\beta B)}{\Psi^3}\right\}\leq 0.$$

(iv) Recall from (9) that i_0 exists only when $\beta>\hat{\beta}$. When this inequality holds, it is easy to verify that δ increases with Q. Using the expression for i_0 from (20), we obtain

$$-\Psi\frac{\partial i_0}{\partial Q}=\frac{1}{2\sqrt{\delta}}\left\{\Psi^2\frac{\partial \delta}{\partial Q}+2\delta\Psi\frac{\partial \Psi}{\partial Q}-\frac{2\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k}\frac{\partial \Psi}{\partial Q}\Psi\sqrt{\delta}\right\}=\frac{1}{2\sqrt{\delta}}\left\{\frac{p_k}{\hat{\Psi}_k}\frac{\partial \Psi}{\partial Q}-\beta\frac{\partial B}{\partial Q}+\frac{2\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k}\frac{\partial \Psi}{\partial Q}i_0\right\}.$$

The term in braces is positive and, when divided by $\partial\Psi/\partial Q$, is decreasing in Q; the reason is that i_0 and $-\frac{\partial B/\partial Q}{\partial \Psi/\partial Q}$ are both decreasing in Q. Thus $-\Psi\frac{\partial i_0/\partial Q}{\partial \Psi/\partial Q}$ is the ratio of positive decreasing function and positive increasing function and so is decreasing in Q. $\square$

Lemma A.3. *Let g and G be the PDF and the CDF, respectively, of a positive random variable Z such that $zg'(z)/g(z)$ and $zg(z)/G(z)$ are decreasing in z. Call this property* (P).

(i) *Random variable Y such that $Z=uY^v$ also satisfies property* (P) *for all $u>0$ and $v>0$.*

(ii) *The gamma, log-normal, Erlang, chi-squared, chi, Weibull, exponential, power-law, and uniform distributions satisfy property* (P).

Proof of Lemma A.3.

(i) Let h and H be (respectively) the PDF and CDF of Y. Then we can use the definition of Y to obtain

$$H(y)=\Pr(Y\leq y)=\Pr(Z\leq uy^v)=G(uy^v),$$

which in turn yields $h(y)=uvy^{v-1}g(uy^v)$ and $h'(y)=uv(v-1)y^{v-2}g(uy^v)+u^2v^2y^{2(v-1)}g'(uy^v)$. It follows that

$$\frac{yh'(y)}{h(y)}=v-1+v\frac{uy^vg'(uy^v)}{g(uy^v)}\quad\text{and}\quad\frac{yh(y)}{H(y)}=v\frac{uy^vg(uy^v)}{G(uy^v)}.$$

Given that uy^v is an increasing function of y, the result is a direct consequence of Z satisfying (P).

(ii) For a gamma distribution with rate parameter r and shape parameter s, we have

$$\frac{zg'(z)}{g(z)}=s-1-rz\quad\text{and}\quad\frac{zg(z)}{G(z)}=\frac{(rz)^se^{-rz}}{\gamma(s,rz)},$$

where $\gamma(s,rz)=\int_0^{rz}t^{s-1}e^{-t}\,dt$ is the incomplete gamma function (Johnson et al. 1995). The function $zg'(z)/g(z)$ is trivially decreasing in z. The function $zg(z)/G(z)$ is also decreasing in z because its derivative, $r^sz^{s-1}e^{-rz}[(s-rz)\gamma-(rz)^se^{-rz}]/\gamma^2$, is always negative; this follows because the term in brackets is decreasing in z (its derivative is $-r\gamma(s,rz)\leq 0$) and its value at $z=0$ is zero. Since the exponential, Erlang, and chi-squared distributions are all special cases of the gamma distribution (Johnson et al. 1995), they also satisfy property (P).

Since the chi random variable is the square root of chi-squared random variable and the Weibull random variable W is equal to $sZ^{1/q}$ where Z is the standard exponential random variable, s and q, are respectively, the scale and shape parameters of W (Johnson et al. 1995), it follows from part (i) that both the chi and the Weibull also satisfy property (P).

For the log-normal distribution, $g(z) = \phi(\log z)/z$ and $G(z) = \Phi(\log z)$, where ϕ and Φ are (respectively) the PDF and the CDF of the standard normal distribution (Johnson et al. 1995). Therefore,

$$\frac{zg'(z)}{g(z)} = -(1 + \log z) \quad \text{and} \quad \frac{zg(z)}{G(z)} = \frac{\phi(z)}{\Phi(z)}.$$

The former function is trivially decreasing in z. The latter is decreasing in z because its derivative, $-\phi(z)[z\Phi(z) + \phi(z)]/\Phi(z)^2$, is always negative; the reason is that the term in brackets is increasing in z (its derivative is $\Phi(z) \geq 0$) and its value at $z = 0$ is $\phi(0)$. For the power-law distribution with parameter c such that $g(z) = cz^{c-1}$, both $zg(z)/G(z)$ and $zg'(z)/g(z)$ are constants (and equal to c and $c-1$, respectively). The uniform distribution trivially satisfies property (P). $\square$

Lemma A.4. *Let g and G be the PDF and CDF (respectively) of a positive random variable such that hazard rate $g(z)/(1 - G(z))$ is increasing in z and the function $zg(z)/G(z)$ is decreasing in z. Then:*

(i) *$G(z)$ is log-concave in z;*

(ii) *$(\alpha - G(z))g'(z) + g(z)^2 \geq 0$ for $0 \leq \alpha \leq 1$; and*

(iii) *$(\alpha - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq 0$ for $0 \leq \alpha \leq 1$.*

Proof of Lemma A.4.

(i) This follows by noting that the log-concavity of $G(z)$ is equivalent to decreasing $g(z)/G(z)$.

(ii) That the hazard rate is increasing in z yields $(1 - G(z))g'(z) + g(z)^2 \geq 0$, whereas the log-concavity of CDF yields $G(z)g'(z) - g(z)^2 \leq 0$. For z such that $g'(z) \leq 0$, we have $(\alpha - G(z))g'(z) + g(z)^2 \geq (1 - G(z))g'(z) + g(z)^2 \geq 0$ because $\alpha \leq 1$. Similarly, for z satisfying $g'(z) > 0$ we have $(\alpha - G(z))g'(z) + g(z)^2 \geq -G(z)g'(z) + g(z)^2 \geq 0$ because $\alpha \geq 0$. As a consequence, the result is valid for all z.

(iii) Given our assumption on $zg(z)/G(z)$, it follows that $(zg'(z) + g(z))G(z) - zg(z)^2 \leq 0$. By the increasing hazard rate property, $(1 - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq 0$. On the one hand, if z satisfies the inequality $zg'(z) + g(z) \leq 0$ then $(\alpha - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq (1 - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq 0$ because $\alpha \leq 1$. On the other hand, if z is such that $zg'(z) + g(z) > 0$ then $(\alpha - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq -G(z)(zg'(z) + g(z)) + zg(z)^2 \geq 0$ because $\alpha \geq 0$. Hence the result holds for all z. $\square$

Proof of Proposition 3. The derivative of revenue w.r.t. Q is

$$\frac{\partial R}{\partial Q} = \frac{P}{\Psi^2}\left(\Psi\frac{\partial D}{\partial Q} - D\frac{\partial \Psi}{\partial Q}\right) = \frac{PD}{\Psi^2}\frac{\partial \Psi}{\partial Q}\left(\frac{\Psi}{D}\frac{\partial D/\partial Q}{\partial \Psi/\partial Q} - 1\right) \equiv \frac{PD}{\Psi^2}\frac{\partial \Psi}{\partial Q}(h(Q) - 1).$$

We can use (9) to translate the condition $\beta > (\leq)\, \hat{\beta}$ into $Q < (\geq)\, Q_\beta$ for some unique Q_β. Then, for $Q < Q_\beta$, we have $D(Q) = G(I_1(Q)) - G(I_0(Q))$. Let $i_0 = \sqrt{I_0}$ and $i_1 = \sqrt{I_1}$. We can now rewrite $h(Q)$ as

$$h(Q) = 2\left[\frac{I_1 g(I_1)}{G(I_1) - G(I_0)}\right]\left[\frac{\partial i_1/\partial Q}{\partial \Psi/\partial Q}\right]\left[\frac{\Psi}{i_1}\right] + 2\left[\frac{g(I_0)}{G(I_1) - G(I_0)}\right]\left[-\Psi\frac{\partial i_0/\partial Q}{\partial \Psi/\partial Q}\right][i_0].$$

It now follows from Lemmas A.2 and A.4 that each term in brackets is positive and also decreasing in Q. Therefore, $h(Q)$ is also decreasing in Q for $Q < Q_\beta$.

If $Q \geq Q_\beta$ then $D(Q) = G(I_1(Q))$. By Lemma A.2(ii), Ψ/i_1 is U-shaped in Q; we use $\overline{Q}$ to denote the minimum of this function. (One can easily verify that $\overline{Q} > Q_\beta$.) Then, for $Q \geq Q_\beta$, we have

$$h(Q) = 2\left[\frac{I_1 g(I_1)}{G(I_1)}\right]\left[\frac{\partial i_1/\partial Q}{\partial\Psi/\partial Q}\right]\left[\frac{\Psi}{i_1}\right].$$

Since the domain extends only up to $\overline{Q}$, it follows that all the terms in brackets are both positive and decreasing in Q; therefore, $h(Q)$ is decreasing in $[\underline{Q}, \overline{Q}]$. Moreover, $\lim_{Q\to\underline{Q}} h(Q) = \infty$ and so $h(Q) - 1$ crosses the x-axis at most once (from above); hence $R(Q)$ is unimodal in this domain. $\square$

Proof of Proposition 4. It is easy to verify that $\Psi^x = x\Psi^1$ and that $B^x = x^2 B^1$. For simplicity, we shall use Ψ and B to denote Ψ^1 and B^1, respectively. Then solving $\Delta^x = 0$ yields the following solutions:

$$\sqrt{I_0^x} = x\Psi\left(\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} - \sqrt{\delta^x}\right) \quad \text{and} \quad \sqrt{I_1^x} = \Psi\left(x\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} + \sqrt{x^2\delta^x}\right),$$

where

$$\delta^x = \beta\left(\frac{\hat{B}_k}{\hat{\Psi}_k^2} - \frac{B}{\Psi^2}\right) + \frac{1}{x\Psi}\left(\frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi}\right)$$

is decreasing in x. It is easy to verify that $x^2\delta^x$ is increasing in x.

(i) This result follows once we note that the terms in parentheses in the expressions for $\sqrt{I_0^x}$ and $\sqrt{I_1^x}$ are positive and increasing in x. (Note that if $p_k \geq P/Q$ then $\hat{B}_k/\hat{\Psi}^2 \geq B/\Psi^2$.)

(ii) First, observe that

$$I_1^x - I_0^x = 4\Psi^2 x\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k}\sqrt{x^2\delta^x} \quad \text{and} \quad \frac{I_0^x}{I_1^x} = \left(\frac{\sqrt{\beta\hat{B}_k}\big/\hat{\Psi}_k - \sqrt{\delta^x}}{\sqrt{\beta\hat{B}_k}\big/\hat{\Psi}_k + \sqrt{\delta^x}}\right)^2.$$

The first function above is increasing in x because $x^2\delta^x$ is increasing in x, and the second is increasing in x because δ^x is decreasing in x. So now we have that

$$\frac{\partial I_0^x/\partial x}{\partial I_1^x/\partial x} = \sqrt{\frac{I_0^x}{I_1^x}}\left[\frac{\Psi\sqrt{\beta\hat{B}_k}\big/\hat{\Psi}_k - \partial\sqrt{x^2\delta^x}/\partial x}{\Psi\sqrt{\beta\hat{B}_k}\big/\hat{\Psi}_k + \partial\sqrt{x^2\delta^x}/\partial x}\right]$$

is increasing in x because:

$$\frac{\partial^2\sqrt{x^2\delta^x}}{\partial x^2} = \frac{-(p_k/\hat{\Psi}_k - P/\Psi)^2}{4\Psi^2(x^2\delta^x)^{3/2}} < 0,$$

the term in brackets is positive (from part (i)), and I_0^x/I_1^x is increasing in x. Also, $g(I_0^x)/g(I_1^x)$ is increasing in x because

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{g(I_0^x)}{g(I_1^x)}\right) &= \frac{g(I_0^x)}{g(I_1^x)}\left\{\left(\frac{g'(I_0^x)}{g(I_0^x)} - \frac{g'(I_1^x)}{g(I_1^x)}\right)\frac{\partial I_0^x}{\partial x} - \frac{g'(I_1^x)}{g(I_1^x)}\left(\frac{\partial I_1^x}{\partial x} - \frac{\partial I_0^x}{\partial x}\right)\right\} \\ &\geq \frac{g(I_0^x)}{g(I_1^x)}\left(\frac{\partial(I_1^x - I_0^x)/\partial x}{I_1^x - I_0^x}\right)\left\{\frac{g'(I_0^x)I_0^x}{g(I_0^x)} - \frac{g'(I_1^x)I_1^x}{g(I_1^x)}\right\} \geq 0.\end{aligned}$$

The first inequality follows because g'/g is decreasing and I_0^x/I_1^x is increasing in x, from which it follows that $(I_1^x - I_0^x)\partial I_0^x/\partial x \geq I_0^x \partial(I_1^x - I_0^x)/\partial x$. The second inequality follows because $zg'(z)/G(z)$ is decreasing and $I_1^x - I_0^x$ is increasing in x.

Next we use the preceding results to show that demand D^x is unimodal in x. Note that $I_0^x > 0$ is equivalent to $x > x_\beta$ for $x_\beta = \Psi(p_k/\Psi_k - P/\Psi)/(\beta B)$. Then, for $x \le x_\beta$, we have that $D^x = G(I_1^x)$ is increasing in x. Otherwise, $D^x = G(I_1^x) - G(I_0^x)$ and its derivative w.r.t. x is

$$\frac{\partial D^x}{\partial x} = g(I_1^x)\frac{\partial I_1^x}{\partial x}\left[1 - \frac{g(I_0^x)}{g(I_1^x)}\frac{\partial I_0^x/\partial x}{\partial I_1^x/\partial x}\right] \equiv g(I_1^x)\frac{\partial I_1^x}{\partial x}[1 - h(x)].$$

We can see that $h(x)$ increases with x. Also, $\lim_{x\to x_\beta} h(x) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. Therefore, the function $1 - h(x)$ is decreasing in x and crosses the x-axis exactly once from above, which yields a unimodal D^x. □

Proof of Proposition 5. Put $\rho = Q\mu/(\lambda P)$.

(i) This claim follows from Lemma 1(ii).

(ii) The results w.r.t. μ follow from Lemma 1(ii). The term P/Ψ is increasing in P because

$$\frac{\partial(P/\Psi)}{\partial P} = \frac{1}{\Psi^2}\left\{\Psi - P\frac{\partial \Psi}{\partial P}\right\} = \frac{Q}{\lambda\Psi^2}F(\rho) \ge 0.$$

(iii) First note that the shape of $\beta B/\Psi$ is same as that of B/Ψ. The derivative of latter w.r.t. P is given by

$$\frac{\partial(B/\Psi)}{\partial P} = \frac{1}{\Psi^2}\left\{\Psi\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P}\right\}.$$

The term in braces can alternatively be written as

$$\begin{aligned}
&\left[\frac{Q}{\lambda} + \int_\rho^\infty \left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right) dF(\varepsilon)\right]\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P} \\
&= \frac{Q}{\lambda}F(\rho)\frac{\partial B}{\partial P} + \int_\rho^\infty \frac{P\varepsilon}{\mu}\, dF(\varepsilon)\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P} \\
&\ge \int_\rho^\infty \frac{P\varepsilon}{\mu}\, dF(\varepsilon)\int_\rho^\infty b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{\varepsilon}{\mu}\, dF(\varepsilon) - \int_\rho^\infty b\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right) dF(\varepsilon)\int_\rho^\infty \frac{\varepsilon}{\mu}\, dF(\varepsilon) \\
&= \int_\rho^\infty \frac{\varepsilon}{\mu}\, dF(\varepsilon)\left\{\int_\rho^\infty \left[b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{P\varepsilon}{\mu} - b\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\right] dF(\varepsilon)\right\} \ge 0.
\end{aligned}$$

The first inequality follows from Lemma 1(i). The last inequality follows by noting that the term in brackets is always positive for any given ε because its derivative w.r.t. P is $b''(P\varepsilon/\mu - Q/\lambda)P(\varepsilon/\mu)^2 \ge 0$ and its value at the lowest feasible $P = Q\mu/(\lambda\varepsilon)$ is $b'(0)Q/\lambda \ge 0$.

The result w.r.t. μ follows immediately from the result w.r.t. P because P and μ co-occur in the expression for B/Ψ, with P in the numerator and μ in the denominator. □

Proof of Remark 4. Let $\rho = Q\mu/(\lambda P)$. We first prove the following two inequalities, which are then used to establish the main result:

$$\frac{\partial^3 B}{\partial P^3}\frac{\partial^2 \Psi}{\partial P^2} - \frac{\partial^3 \Psi}{\partial P^3}\frac{\partial^2 B}{\partial P^2} \ge 0; \tag{22}$$

$$\frac{\partial^2 B}{\partial P^2}\frac{\partial \Psi}{\partial P} - \frac{\partial^2 \Psi}{\partial P^2}\frac{\partial B}{\partial P} \ge 0. \tag{23}$$

We prove (22) for $m > 3$; it can easily be proved also for $m = 2$ and $m = 3$. We have $b'(z) = mz^{m-1}$, $b''(z) = m(m-1)z^{m-2}$, and $b'''(z) = m(m-1)(m-2)z^{m-3}$. Now the combination of (14) and (15) yield

$$\frac{\partial^2 \Psi}{\partial P^2} = \frac{\rho^2}{\mu P}f(\rho), \qquad \frac{\partial^3 \Psi}{\partial P^3} = -\frac{\rho^2}{\mu P^2}(3f(\rho) + \rho f'(\rho)),$$

$$\frac{\partial^2 B}{\partial P^2} = \int_\rho^\infty b''\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{\varepsilon^2}{\mu^2}f(\varepsilon)\, d\varepsilon, \qquad \frac{\partial^3 B}{\partial P^3} = \int_\rho^\infty b'''\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{\varepsilon^3}{\mu^3}f(\varepsilon)\, d\varepsilon.$$

Then

$$\begin{aligned}
&\frac{\partial^3 B}{\partial P^3}\frac{\partial^2 \Psi}{\partial P^2} - \frac{\partial^3 \Psi}{\partial P^3}\frac{\partial^2 B}{\partial P^2} \\
&\quad \geq m(m-1)\frac{\rho^2 f(\rho)}{\mu^3 P^2}\left(\frac{Q}{\lambda\rho}\right)^{m-2}\int_\rho^\infty \left\{(m-2)(\varepsilon-\rho)^{m-3}\varepsilon^3 + \left(1+\frac{\rho f'(\rho)}{f(\rho)}\right)(\varepsilon-\rho)^{m-2}\varepsilon^2\right\} f(\varepsilon)\, d\varepsilon \\
&\quad \geq m(m-1)\frac{\rho^2 f(\rho)}{\mu^3 P^2}\left(\frac{Q}{\lambda\rho}\right)^{m-2}\int_\rho^\infty \left\{(m-2)(\varepsilon-\rho)^{m-3}\varepsilon^3 + \left(1+\frac{\rho f'(\varepsilon)}{f(\varepsilon)}\right)(\varepsilon-\rho)^{m-2}\varepsilon^2\right\} f(\varepsilon)\, d\varepsilon;
\end{aligned}$$

here the first inequality follows because $3f(\rho) \geq f(\rho)$, and the second inequality follows from the log-concavity of f (i.e., f'/f is decreasing). It is sufficient to show that the integral on the RHS is positive. This integral can be rewritten as $\rho V(\rho)$, where $V(\rho) = I_1(\rho) + I_2(\rho) + I_3(\rho)$ for

$$I_1(\rho) = \frac{1}{\rho}\int_\rho^\infty (m-2)(\varepsilon-\rho)^{m-3}\varepsilon^3 f(\varepsilon)\, d\varepsilon, \quad I_2(\rho) = \frac{1}{\rho}\int_\rho^\infty (\varepsilon-\rho)^{m-2}\varepsilon^2 f(\varepsilon)\, d\varepsilon,$$
$$\text{and} \quad I_3(\rho) = \int_\rho^\infty (\varepsilon-\rho)^{m-2}\varepsilon^2 f'(\varepsilon)\, d\varepsilon.$$

Using induction, one can easily verify that

$$\frac{\partial^{m-1} I_3}{\partial \rho^{m-1}} = (-1)^{m-1}(m-2)!\,\rho^2 f'(\rho)$$

and that, for $1 \leq k < m$,

$$\begin{aligned}
\frac{\partial^k I_1}{\partial \rho^k} &= -\frac{k}{\rho}\frac{\partial^{k-1} I_1}{\partial \rho^{k-1}} + \frac{(-1)^k}{\rho}\int_\rho^\infty (m-2)\cdots(m-k-2)(\varepsilon-\rho)^{m-k-3}\varepsilon^3 f(\varepsilon)\, d\varepsilon \implies (-1)^k \frac{\partial^k I_1}{\partial \rho^k} \geq 0, \\
\frac{\partial^k I_2}{\partial \rho^k} &= -\frac{k}{\rho}\frac{\partial^{k-1} I_2}{\partial \rho^{k-1}} + \frac{(-1)^k}{\rho}\int_\rho^\infty (m-2)\cdots(m-k-1)(\varepsilon-\rho)^{m-k-2}\varepsilon^2 f(\varepsilon)\, d\varepsilon \implies (-1)^k \frac{\partial^k I_2}{\partial \rho^k} \geq 0.
\end{aligned}$$

Then we have

$$\frac{\partial^{m-1} V}{\partial \rho^{m-1}} = -\frac{m-3}{\rho^3}\frac{\partial^{m-4}}{\partial \rho^{m-4}}(I_1+I_2) + \frac{m-3}{\rho^2}\frac{\partial^{m-3}}{\partial \rho^{m-3}}(I_1+I_2) + \frac{4}{\rho^3}(-1)^{m-1}\int_\rho^\infty (m-2)!\,\varepsilon^3 f(\varepsilon)\, d\varepsilon.$$

If m is even (resp., odd), then $\frac{\partial^{m-1} V}{\partial \rho^{m-1}}$ is less than (resp., greater than) zero. Given that $\lim_{\rho\to\infty} \frac{\partial^k V}{\partial \rho^k} = 0$ for $1 \leq k < m$, we infer that $(-1)^k \frac{\partial^k V}{\partial \rho^k} \geq 0$. Hence $V(\rho)$ is decreasing in ρ with $\lim_{\rho\to\infty} V(\rho) = 0$, which yields the desired result.

To prove (23), we first write $\partial B/\partial P = v(\partial\Psi/\partial P)$, where v is an increasing convex function (and $v(0) = 0$). This relationship follows from (22) and Theorem 1 of Pratt (1964). Then

$$\frac{\partial^2 B}{\partial P^2}\frac{\partial \Psi}{\partial P} - \frac{\partial^2 \Psi}{\partial P^2}\frac{\partial B}{\partial P} = \frac{\partial^2 \Psi}{\partial P^2}\left\{v'\left(\frac{\partial \Psi}{\partial P}\right)\frac{\partial \Psi}{\partial P} - v\left(\frac{\partial \Psi}{\partial P}\right)\right\} \geq 0$$

because Ψ is convex in P and $v'(z)z - v(z) \geq 0$ for $z \geq 0$.

Now we are ready to prove the main result.

(i) Put $\nu = 1/\mu$. Since ν behaves similarly to P in the expressions for B and Ψ, it follows that – much as in (23) – we have

$$\frac{\partial^2 B}{\partial \nu^2}\frac{\partial \Psi}{\partial \nu} - \frac{\partial^2 \Psi}{\partial \nu^2}\frac{\partial B}{\partial \nu} \geq 0.$$

Then

$$\frac{\partial^2 B}{\partial \mu^2}\frac{\partial \Psi}{\partial \mu} - \frac{\partial^2 \Psi}{\partial \mu^2}\frac{\partial B}{\partial \mu} = \left(\frac{\partial^2 B}{\partial \nu^2}\frac{\partial \Psi}{\partial \nu} - \frac{\partial^2 \Psi}{\partial \nu^2}\frac{\partial B}{\partial \nu}\right)\left(\frac{\partial \nu}{\partial \mu}\right)^3 \leq 0. \tag{24}$$

The derivative of C w.r.t. μ is given by

$$\frac{\partial C}{\partial \mu}=\frac{\partial\Psi/\partial\mu}{\Psi^2}\left\{-(I+P)+\beta\frac{\Psi(\partial B/\partial\mu)-B(\partial\Psi/\partial\mu)}{\partial\Psi/\partial\mu}\right\}\equiv\frac{\partial\Psi/\partial\mu}{\Psi^2}\{-(I+P)+\beta l(\mu)\}.$$

By taking the derivative of $l(\mu)$ w.r.t. μ, it is easy to see from (24) that $l(\mu)$ is decreasing in μ. Because $\lim_{\mu\to 0} l(\mu)=\infty$ and $\lim_{\mu\to\infty} l(\mu)=0$, the term in braces is decreasing in μ and crosses the x-axis once from above. Since $\partial\Psi/\partial\mu\le 0$, it follows that C is U-shaped in μ.

(ii) The derivative of C w.r.t. P is given by

$$\frac{\partial C}{\partial P}=\frac{\partial\Psi/\partial P}{\Psi^2}\left[-I+\frac{\Psi-P(\partial\Psi/\partial P)}{\partial\Psi/\partial P}+\beta\frac{\Psi(\partial B/\partial P)-B(\partial\Psi/\partial P)}{\partial\Psi/\partial P}\right]\equiv\frac{\partial\Psi/\partial P}{\Psi^2}[-I+h(P)]. \tag{25}$$

Our aim is to show that $h(P)$ is U-shaped in P. First, we have

$$\frac{\partial h}{\partial P}=\frac{\Psi}{(\partial\Psi/\partial P)^2}\left(\frac{\partial^2 B}{\partial P^2}\frac{\partial\Psi}{\partial P}-\frac{\partial^2\Psi}{\partial P^2}\frac{\partial B}{\partial P}\right)\{\beta-r(P)\}\quad\text{for}\quad r(P)=\frac{\frac{\partial^2\Psi}{\partial P^2}}{\frac{\partial^2 B}{\partial P^2}\frac{\partial\Psi}{\partial P}-\frac{\partial^2\Psi}{\partial P^2}\frac{\partial B}{\partial P}}. \tag{26}$$

Next, from (23) it follows that the sign of $\partial h/\partial P$ depends only on the sign of the term in braces (i.e., of $\beta-r(P)$). By taking the derivative of $r(P)$ w.r.t. P, one can easily see from (22) that $r(P)$ is decreasing in P. Since $\lim_{P\to 0} r(P)=\infty$ and since $\lim_{P\to\infty} r(P)=0$, it follows that the term in braces in (26) crosses the x-axis once from below, which yields a U-shaped $h(P)$. Now observe that $\lim_{P\to 0} h(P)=\lim_{P\to\infty} h(P)=\infty$, and let $\hat{I}=\min_P h(P)$. By (25) we see that, if $I\le\hat{I}$, then the derivative is completely positive and C is increasing in P. Otherwise, the derivative crosses the x-axis twice – first from above and then from below – and so C is N-shaped. □

Proof of Lemma 5. After setting $\rho=Q\mu/(\lambda P)$, we can rewrite L_n as follows:

$$L_n=\frac{P}{\mu}\mathbb{E}[\tilde{e}_n-\rho]^+=\frac{P}{\mu}\left(\mathbb{E}\tilde{e}_n-\rho+\mathbb{E}[\rho-\tilde{e}_n]^+\right).$$

Because $\tilde{e}_n$ is a mean-preserving spread of $\tilde{e}_{n+1}$, we can use Definition 1.5.1 of Müller and Stoyan (2002) to write $\tilde{e}_n\le_{icv}\tilde{e}_{n+1}$ or, equivalently, $-\tilde{e}_{n+1}\le_{icx}-\tilde{e}_n$. Using their Theorem 1.5.7(ii), we now deduce that $\mathbb{E}[\rho-\tilde{e}_{n+1}]^+\le\mathbb{E}[\rho-\tilde{e}_n]^+$. Since $\mathbb{E}\tilde{e}_n=\mathbb{E}\tilde{\varepsilon}$ for all n, it follows that $L_{n+1}\le L_n$. Given this inequality, from Theorem 1.5.7(i) of Müller and Stoyan (2002) we obtain $\tilde{e}_{n+1}\le_{icx}\tilde{e}_n$. Since $b(P[z-\rho]^+/\mu)$ is an increasing convex function in z, it follows that $B_{n+1}\le B_n$. □

Lemma A.5. *In (10), Δ is U-shaped in I. Let Δ_m be the minimum value of Δ. Then – for any given μ, λ, P, Q, and p_k – there exists a threshold $\zeta\ge 0$ such that $\Delta_m\le(>)\,0$ if $P/Q\le(>)\,p_k(1+\zeta)$.*

Proof of Lemma A.5. We first characterize the optimal cost with bulbs and kerosene. Then we describe the shape and minimum value of Δ.

Optimal cost with bulbs. The unconstrained minimization of $C(T)=(I+P+\beta(T-Q/\lambda)^2)/T$ w.r.t. T yields the optimal cycle length T^* and optimal cost $C(T^*)$:

$$T^*=\sqrt{\left(\frac{Q}{\lambda}\right)^2+\frac{I+P}{\beta}};\qquad C(T^*)=2\beta\left(T^*-\frac{Q}{\lambda}\right).$$

We require that the cycle length be greater than P/μ, so if $T^*>P/\mu$ then the optimal cost is $C(T^*)$; otherwise, it is $C(P/\mu)$. We can rewrite the condition $T^*\le(>)\,P/\mu$ as $I\le(>)\,\hat{I}$, where the threshold $\hat{I}=\beta[(P/\mu)^2-(Q/\lambda)^2]-P$.

Optimal cost with kerosene. The Lagrangian for the constrained optimization of $C_k(Q_k, T_k)$ in (10) is given by

$$\mathcal{L}(Q_k, T_k, \chi_1, \chi_2) = C_k(Q_k, T_k) - \chi_1(T_k - p_k Q_k/\lambda) - \chi_2(Q_k M - T_k).$$

Any local minimum satisfies the following Karush–Kuhn–Tucker conditions:

$$\frac{\partial C_k}{\partial Q_k} + \frac{\chi_1 p_k}{\lambda} - \chi_2 M = 0, \quad \frac{\partial C_k}{\partial T_k} - \chi_1 + \chi_2 = 0, \quad \chi_1\left(T_k - \frac{p_k Q_k}{\lambda}\right) = 0, \quad \chi_2(Q_k M - T_k) = 0,$$
$$T_k \geq \frac{p_k Q_k}{\lambda}, \quad Q_k M \geq T_k, \quad Q_k \geq 0, \quad T_k \geq 0, \quad \chi_1 \geq 0, \quad \chi_2 \geq 0.$$

We consider three cases as follows.

1. $\chi_1 = \chi_2 = 0$. This case is not possible because the equations $\partial C_k/\partial Q_k = 0$ and $\partial C_k/\partial T_k = 0$ are inconsistent.
2. $\chi_1 = 0$ and $\chi_2 > 0$. This case results in the optimal values $Q_k = 0$ and $T_k = 0$, which lead to infinite cost.
3. $\chi_1 > 0$ and $\chi_2 = 0$. This case simply reduces to optimizing C_k w.r.t. Q_k, with $T_k = p_k Q_k/\mu$; it is the deterministic version of the problem considered in Section 3.2. Since the corresponding optimal cost is finite, it follows that this is the only feasible solution. The optimal solution is given by

$$Q_k^* = \frac{1}{L_k}\sqrt{\frac{I}{\beta}}, \quad T_k^* = \frac{p_k Q_k^*}{\mu}, \quad \text{and} \quad C_k^* = \mu + \frac{2\mu L_k}{p_k}\sqrt{I\beta} \quad \text{for } L_k = \frac{p_k}{\mu} - \frac{1}{\lambda}.$$

Shape of Δ. First we suppose that $\hat{I} > 0$, in which case $P/\mu > Q/\lambda$. It follows that Δ is equal to $\Delta_<$ for $I \leq \hat{I}$ or is equal to $\Delta_>$ for $I > \hat{I}$; here

$$\Delta_< = \frac{I + P + \beta(P/\mu - Q/\lambda)^2}{P/\mu} - \mu - \frac{2\mu L_k}{p_k}\sqrt{I\beta} \quad \text{and} \quad \Delta_> = 2\beta(T^* - Q/\lambda) - \mu - \frac{2\mu L_k}{p_k}\sqrt{I\beta}. \tag{27}$$

Note that Δ is continuous at $\hat{I}$. The corresponding derivatives w.r.t. I are given by

$$\frac{\partial \Delta_<}{\partial I} = \frac{\mu}{P} - \frac{\mu L_k}{p_k}\sqrt{\frac{\beta}{I}} \quad \text{and} \quad \frac{\partial \Delta_>}{\partial I} = \frac{1}{\sqrt{I}}\left\{\sqrt{\frac{I}{(Q/\lambda)^2 + (I+P)/\beta}} - \frac{\mu L_k}{p_k}\sqrt{\beta}\right\}. \tag{28}$$

Since $\partial\Delta_</\partial I$ is increasing in I, $\partial\Delta_>/\partial I$ crosses the x-axis at most once from below (because the term in braces is increasing in I), and $\lim_{I\to\hat{I}} \partial\Delta_</\partial I = \lim_{I\to\hat{I}} \partial\Delta_>/\partial I$, it follows that $\partial\Delta/\partial I$ also crosses the x-axis at most once from below. Finally, since $\lim_{I\to 0} \partial\Delta_</\partial I < 0$ and $\lim_{I\to\infty} \partial\Delta_>/\partial I = 0$ (from above, since the term in braces in (28) is positive as $I \to \infty$), $\partial\Delta/\partial I$ crosses the x-axis exactly once and so Δ is U-shaped in I.

Now we consider the case when $\hat{I} \leq 0$; then $\Delta = \Delta_>$ for all I. Because $\lim_{I\to 0} \partial\Delta_>/\partial I < 0$ and $\lim_{I\to\infty} \partial\Delta_>/\partial I = 0$ (from above), Δ is again U-shaped in I.

Minimum value of Δ. As before, we first consider the case $\hat{I} > 0$. Let Δ_m denote the minimum value of Δ, which is achieved at I_m. Since Δ has only one minimum, it is either from $\Delta_<$ or $\Delta_>$ depending on the sign of $\lim_{I\to\hat{I}} \partial\Delta/\partial I$.

On the one hand, if $\lim_{I\to\hat{I}} \partial\Delta/\partial I \geq 0$, then I_m is obtained by setting $\partial\Delta_</\partial I = 0$. It follows from (27) and (28) that Δ_m is given by

$$\Delta_m = I_m\left(\frac{\mu}{P} - \frac{\mu L_k}{p_k}\sqrt{\frac{\beta}{I_m}}\right) + \beta\mu P\left(\frac{(P/\mu - Q/\lambda)^2}{P^2} - \frac{L_k^2}{p_k^2}\right).$$

By (28), the first term is equal to zero – and the second term is less than zero – iff $P/Q \leq p_k$.

On the other hand, if $\lim_{I\to\hat{I}} \partial\Delta/\partial I < 0$ then I_m is obtained by setting $\partial\Delta_>/\partial I = 0$. We can now use (27) and (28) to obtain

$$\Delta_m = 2\beta\sqrt{\left[1-\left(\frac{\mu L_k}{p_k}\right)^2\right]\left[\left(\frac{Q}{\lambda}\right)^2 + \frac{P}{\beta}\right]} - \frac{2\beta Q}{\lambda} - \mu.$$

Then $\Delta_m \leq 0$ iff

$$\frac{P}{Q} < \frac{\frac{\beta Q}{\lambda^2}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu^2}{4\beta Q} + \frac{\mu}{\lambda}}{1-\left(\frac{\mu L_k}{p_k}\right)^2} = p_k\left\{\frac{\frac{\beta Q}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu\lambda}{4\beta Q} + 1}{2 - \frac{\mu}{\lambda p_k}}\right\}. \tag{29}$$

The term in the braces is greater than one because

$$\frac{\beta Q}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu\lambda}{4\beta Q} - 1 + \frac{\mu}{\lambda p_k} \geq 2\sqrt{\frac{\beta}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2\frac{\mu\lambda}{4\beta}} - 1 + \frac{\mu}{\lambda p_k} = 0.$$

We can therefore rewrite the term in braces in (29) as $1+\zeta$ for some $\zeta \geq 0$, so the condition in (29) reduces to $P/Q \leq p_k(1+\zeta)$.

Finally, we consider the case $\hat{I} \leq 0$; then $\Delta = \Delta_>$ for all I. It follows that $\Delta_m \leq 0$ iff $P/Q \leq p_k(1+\zeta)$. So in all possible cases (as just described), $\Delta_m \leq 0$ if and only if $P/Q \leq p_k(1+\zeta)$ for some $\zeta \geq 0$. $\square$