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Competitive Bundling in a Bertrand Duopoly

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Key words: industrial Organization; Bundling; Bertrand Competition; Non-cooperative Game Theory

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1. Introduction

Bundling, or tying, is the practice of selling several products (or services) as a combination for a single price (Adams and Yellen 1976). It is prevalent in many industry sectors, including physical goods (e.g., gift baskets, car options, fast-food menu combos), services (insurance, fast food, telecommunications, retail banking), and digital platforms (Google, Amazon).

Bundling offers several benefits (Eppen et al. 1991). First, it can lead to economies of scope by reducing the costs of production, transaction, and administration. Second, bundling can expand demand by improving product performance or combining complementary products. Bundling also enables firms to extract more customer surplus by reducing heterogeneity in customer valuations—a form of price discrimination.

Although bundling offers multiple benefits in principle, competing firms do not appear to employ a uniform bundling strategy. For instance, telecom companies compete through bundling, unbundling, and re-bundling their current services. Cable companies providing broadband Internet, TV, and telephone services once offered double-play or triple-play bundles of several sizes and types. Facing intense competition from Internet-based streaming services (e.g., Netflix, Amazon Video), they now offer smaller, targeted "skinny" bundles in which fewer TV channels are bundled for a lower price. In retail banking, incumbent banks typically offer bundles of services—such as checking and saving accounts, credit cards, and mortgages—whereas new entrants (e.g., N26, TransferWise), which are mostly digital, often focus on a particular product or customers segment (Bakos et al. 2005). In entertainment streaming services, Disney has responded to Netflix's growing importance in the segment of "stories and entertainment" by offering Hulu both as a standalone product and as part of a bundle with Disney+ (brand and family) and ESPN+ (sports).[1]

As these examples suggest, different bundling strategies are adopted under competition and they evolve over time. To take a first step to explain the numerous strategies observed in practice, we develop a theory of competitive bundling that exploits a symmetric Bertrand competition model. Within the framework of that stylized model, we investigate several questions. What bundling strategies emerge in equilibrium? If bundling is optimal for a monopolist to smooth the heterogeneity in customer valuations, is it still optimal in a dupoloy? Can asymmetric bundling strategies be expected irrespective of firms' symmetry? Do firms always seek to avoid head-to-head competition? How are equilibrium outcomes affected by the extent of heterogeneity in customer valuations?

We address these questions by considering a symmetric Bertrand duopoly in which firms can offer two undifferentiated components (e.g., washer and dryer; burgers and fries) either separately or as a bundle. Given two components A and B, a product may consists of a single component

[1] `https://stratechery.com/2019/disney-and-the-future-of-tv/`, accessed October 24, 2019.

(i.e., either {A} or {B}) or the bundle ({A, B}). Under *pure* bundling, each firm offers at most one product; under *mixed* bundling, each firm may offer any set of products.[2] Firms make two simultaneous (and non-cooperative) decisions: first their product offerings and then the prices of those offered products. So that our model will be as streamlined as possible, we make similar assumptions to Adams and Yellen (1976). On the supply side, we assume zero costs not only of including products in an offering but also of transaction and delivery—as is common to many information goods. On the demand side, we assume that customers purchase the products that yield them the greatest surplus, that customers purchase at most one unit of each component, and that customers' valuations of the bundle are equal to the sum of their valuations of the respective components. When we analyze the case of mixed bundling, we assume also that the market is segmented into two customer groups with identical valuations of the bundle, but distinct valuations of the components. Those valuations may be positively or negatively correlated; for tractability, we address the extreme cases of *perfect* (positive or negative) correlation.

We obtain the following results. First, under pure bundling (and regardless of the number of customer groups and their relative valuations of the different components), there always exist equilibria in which one firm bundles and the other firm offers only a single-component product. It follows that asymmetric bundling strategies can emerge in equilibrium even when firms are ex ante identical. The rationale underlying this outcome is that firms seek to avoid head-to-head competition. On the one hand, a firm that faces a bundling competitor is better-off offering a single-component product and thereby capturing customers who value that component highly than offering the bundle and entering a price war. On the other hand, bundling is a lucrative value proposition because it reduces the heterogeneity in customer valuations. Thus, it is in a firm's best interest, when facing a competitor that offers a single-component product, to offer the bundle instead. Our equilibrium characterization may explain why new entrants in retail banking

[2] Although the term "mixed bundling" is sometimes used with reference to an offering that consists of *all* products (i.e., {{A}, {B}, {A, B}}, we use it to refer to any subset—including the null set—of {{A}, {B}, {A, B}}; this approach is also known as "partial mixed bundling" (Bhargava 2013).

or entertainment compete with large bundlers by focusing on a niche product or service. Yet the bundling firm earns higher profits, so there could well be a first-mover advantage to bundling.

Second, under mixed bundling, almost any bundling strategy (provided that all components are offered) can emerge in equilibrium when customer valuations of the components are relatively homogeneous. Hence the diverse set of bundling strategies observed in practice can be explained by way of a simple competitive model. Furthermore, strategies that involve any form of direct or indirect competition tend to be Pareto-dominated by strategies that avoid competition. So if firms could coordinate then either (a) they would operate without bundling, each offering a distinct single-component product or (b) one of them would offer the bundle as a monopolist, denying market entry to the other.

Third, a much smaller set of equilibria (consisting of only three types of equilibria) emerges under mixed bundling when customers have very heterogeneous valuations of the different components.

- *One firm offers the bundle and the other firm offers only a single-component product, which is not offered independently by the bundler.* In this situation, bundling helps firms differentiate their offerings to soften price competition (Chen 1997).
- *One firm offers the full set of products as a monopolist and the other firm stays out of the business.* Here bundling is used to pre-empt the other firm's entry (Nalebuff 2004).
- *Both firms offer both single-component products (possibly along with the bundle) and compete on price, driving their profits down to zero.* There are clearly no benefits to bunding under such circumstances. In fact, this type of equilibria is Pareto-dominated by the former two and could be avoided if firms coordinated their actions.

In short, our simple competitive model demonstrates that (i) various bundling strategies can be sustained in equilibrium, (ii) asymmetric bundling equilibria can emerge even if firms are symmetric, (iii) the more homogeneous the customer valuations, the larger the set of equilibria, including several in which no firm bundles (iv) under competition, bundling is essentially used to soften price competition or to create barriers to entry, and not to price discriminate, and (v) there are first-mover advantages to bundling.

The rest of this paper proceeds as follows. Section 2 briefly reviews the literature on bundling. In Section 3, we model bundling and pricing as a two-stage non-cooperative game. We characterize the equilibrium bundling and pricing strategies under pure bundling in Section 4 and under mixed bundling in Section 5. Section 6 summarizes our results and discusses their managerial implications. All proofs and supporting results are given in the appendices.

2. Literature Review

Since the seminal work by Stigler (1963), the economics and management literature has explored the numerous benefits and pitfalls of bundling—first from a monopolist's perspective and more recently in oligopolistic settings. For surveys of the literature, see Stremersch and Tellis (2002), Kobayashi (2005), and Venkatesh and Mahajan (2009).

For a monopolist, bundling offers several benefits in terms of product performance as well as economies of scope in production, distribution, and promotional activities (Eppen et al. 1991, Evans and Salinger 2005). Bundling may also expand demand in response to complementarities among the bundle components (Venkatesh and Kamakura 2003) or to its greater perceived value (Sharpe and Staelin 2010). More subtly, bundling is a form of price discrimination because it renders customer valuations less heterogeneous (Stigler 1963, Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989, Salinger 1995, Bakos and Brynjolfsson 2000, Raghunathan and Sarkar 2016). In particular: a monopolist can extract more surplus from its customers—especially when their valuations of the different components are negatively correlated—by offering a bundle of components, for which there is little heterogeneity in valuations, than by offering the components separately. This insight is especially relevant for goods that have zero marginal costs, such as information goods (Bakos and Brynjolfsson 1999). However, bundling is less attractive in the presence of "double marginalization" in the supply chains for physical goods (Bhargava 2012, Girju et al. 2013), of congestion for physical services (Wu and Yang 2019), of ample demand relative to the inventory (Abdallah et al. 2019), of digital piracy for information goods (Wu et al. 2019), and of asymmetric network externalities (Prasad et al. 2010).

Bundling also gives the monopolist leverage in other markets. Thus a firm that is a monopolist on one component but competes with other firms on another component can leverage its monopolist position by bundling the two components together and foreclosing rivals' sales, thereby increasing its market power, in a competitive market (Whinston 1990). In fact, bundling can even pre-empt the entry of potential competitors or force the exit of current ones (Carlton and Waldman 2002, Nalebuff 2004, Peitz 2008). In addition, bundling enables the monopolist to shift "slack" resources from one market to the other, giving it control over the rate, direction, and timing of innovation across markets (Choi 1996). Bakos and Brynjolfsson (2000) find that bundling allows large bundlers of information goods to outbid smaller ones in securing upstream content, discourages competitors' entry in the bundler's market while favoring entry of the bundler in adjacent markets, and discourages innovation by niche players. However, leverage has its limits: if the market for the second component is perfectly competitive (instead of a differentiated oligopoly), then there is no benefit to bundling (Schmalensee 1982).

An emerging stream of research has explored how bundling affects the intensity of competition in oligopolistic markets, which is the focus of our study. Yet, the literature has so far studied specific settings. Considering a firm that is a monopolist in one market and competes with another firm in another market, Carbajo et al. (1990) observe that bundling softens price competition; even if customers' valuations are perfectly positively correlated, bundling is profitable because it leads rivals to price less aggressively. So bundling favors competitors in this case, which is at odds with the leverage effect. Building on the setup described by Schmalensee (1982), Chen (1997) considers two symmetric firms that offer a component in a duopoly (rather than in a monopoly) and another component in a perfectly competitive market. In equilibrium, one firm offers only the first component while the other firm offers a bundle of the two components. Hence bundling can be used to differentiate the firms' offerings and reduce price competition, even though it is not optimal for a monopolist in the focal setting (Schmalensee 1982). We consider two symmetric duopolies and obtain a similar equilibrium result. In our setting, however, the bundling firm always earns more profit than the firm that offers a single-component product (i.e., contra Chen 1997). Much as in our study, Zhou et al. (2019) consider a setting in which both components are sold in duopolies; these authors likewise find that bundling can soften competition. Yet they assume that firms are asymmetric, allowing one firm to make its product offering (bundling) decision before the other firm. Hence much of the emerging literature on the effects of bundling on competition presupposes some form of asymmetry between firms (with respect to market power or market presence), or distinct market structures among components. In contrast, we consider two symmetric firms operating in two duopolies.

In addition, the literature on competitive bundling has often assumed that firms offered *all* components either separately or as a bundle; see, e.g., Economides (1993), Liao and Tauman (2002), and Raghunathan and Sarkar (2016). Instead, we consider the full set of offering strategies, which corresponds to the so-called "power set" of products, and we find that, in equilibrium, one firm may choose to offer only one component or no component at all; hence these works' ex ante assumption that firms offer all components may be too restrictive.

The literature on how bundling affects competitive intensity has been expanded by the consideration of differentiated oligopolies. Matutes and Regibeau (1988), Economides et al. (1989), Matutes and Regibeau (1992), Nalebuff (2000), Ghosh and Balachander (2007), Thanassoulis (2007), and Vamosiu (2018) work with spatial models of horizontal differentiation between firms. Armstrong and Vickers (2010) and Gwon (2015) extend those spatial models by considering not only firm-specific but also product-specific preferences. Instead of using a spatial model of differentiation, Anderson and Leruth (1993) use a logit model to study the competitive dynamics of bundling. Moving beyond duopolies, Zhou (2017) uses a random utility framework to address bundling competition in an oligopolistic market. Unlike all of these cited works, which focus on horizontal differentiation, Ahn and Yoon (2012) study bundling under vertical differentiation. Although Zhou (2017) argues that "introducing product differentiation is necessary for studying competitive bundling if firms have similar cost conditions, [for] otherwise, prices would settle at marginal costs and there would be no meaningful scope for bundling" (footnote 9), we establish that this implication may not hold when firms are free to choose their product offerings.

In sum, the literature on competitive bundling has either assumed some form of differentiation among firms and/or products or assumed some form of asymmetry in the markets in which they operate—or it has imposed restrictions on the types of offerings. Yet in its purest form, Bertrand competition presupposes complete symmetry (Bertrand 1883). Given the foundational nature of Bertrand competition, a natural first step in any study of competitive bundling should be to investigate its practice under complete symmetry of firms, products, and markets while considering the exhaustive set of offering strategies. Taking that step is precisely the objective of our study.

3. Model

We consider a Bertrand duopoly with two symmetric firms, indexed by $i \in \{1,2\}$, that cater to two customer segments, indexed by $j \in \{1,2\}$, with two components (e.g., washer and dryer; burgers and fries), indexed by $k \in \{1,2\}$, that can be sold separately or as a bundle. Thus each firm can offer three products, indexed by $l \in \{1,2,3\}$: product 1, consisting of component 1; product 2, consisting of component 2; and/or product 3, consisting of the bundle of components 1 and 2. Both firms operate under perfect information and are fully rational. We shall use $-i \doteq 3-i$ to denote the firm other than firm i, $-j \doteq 3-j$ to denote the customer other than customer j, and $-l \doteq 3-l$ to denote the single-component product other than product l ($l \in \{1,2\}$). We first introduce the supply side, then the demand side, and finally the resulting equilibria.

Supply. Bundling decisions tend to be irreversible: once a firm decides on a product offering, it must usually remain committed to that decision for a substantial amount of time. We accordingly model this scenario as a two-stage game. In the first stage, which we call the *bundling game*, firms choose their product offering (or bundling) strategy simultaneously and non-cooperatively. In the second stage, the *pricing game*, firms choose their pricing decisions (again simultaneously and non-cooperatively). For any $i \in \{1,2\}$, let $\mathbf{z}_i = (z_{il})_{l=1,2,3}$ be firm i's offering decision; here $z_{il} = 1$ if firm i offers product l in its offering and $z_{il} = 0$ otherwise. We assume that products can be included in a firm's offering at no cost. Under pure bundling, each firm may offer only one product (i.e., either one of the individual components or the bundle); under mixed bundling, each firm may offer any set of products. Let $\mathcal{Z}$ denote the set of feasible offerings. Then $\mathcal{Z} = \left\{\mathbf{z}_i \in \{0,1\}^3 \mid \sum_{l=1}^{3} z_{il} \le 1\right\}$ under pure bundling and $\mathcal{Z} = \{\mathbf{z}_i \in \{0,1\}^3\}$ under mixed bundling. Here $\{0,1\}^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$.

Now let $\mathbf{p}_i = (p_{il})_{l=1,2,3}$ denote firm i's vector of pricing decisions, which is assumed to be nonnegative. In the event that firm i randomizes its pricing strategy, let $F_i(p)$ denote the cumulative distribution function (CDF) of its pricing decisions; that function is constrained to lie in $\mathcal{F} = \{F_i(p) \mid F_i(0) = 0,\ \lim_{p\to\infty} F_i(p) = 1, \text{ and } F_i(p) \text{ nondecreasing}\}$. To streamline the notation, we put $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$ and $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$. We assume that products—as is typical of many information goods (Bakos and Brynjolfsson 1999)—have zero marginal cost of production and delivery.[3]

Demand. The demand side consists of two homogeneous segments of surplus-maximizing customers. For simplicity, we assume that both segments have the same size (normalized to 1). Customer valuations for the bundle are assumed to be the sum of their valuations for each component of the bundle (Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989); hence there is no complementarity or substitutability in customer valuations. Let u_{jk} denote customer j's utility from component k for any $j \in \{1,2\}$ and $k \in \{1,2\}$, and let v_{jk} denote customer j's utility from product l for any $l \in \{1,2,3\}$. Then $v_{j1} = u_{j1}$, $v_{j2} = u_{j2}$, and $v_{j3} = u_{j1} + u_{j2}$.

Let $x_{ijl}(\mathbf{p}, \mathbf{z})$ denote customer j's decision to purchase product l from firm i when presented with the two firms' offerings $\mathbf{z}$ and pricing decisions $\mathbf{p}$. We assume that customers buy at most one unit of each component—in other words, that there is full satiation in consumption—and that customers can randomize their purchase decisions; thus, $x_{ijl} \in [0,1]$. We assume also that customers

[3] Accounting for marginal costs that are nonzero and uniform would amount, in essence, to reducing customer valuations. If production costs are nonuniform across firms, then the lowest-cost firm has an advantage and the Bertrand paradox disappears (Fudenberg and Tirole 1991). Although an equilibrium characterization is in theory feasible for this case, its exacerbation of the problem's combinatorial nature would complicate expressing that characterization and so make it difficult to generate useful insights. Finally, given that one of our paper's goals is to demonstrate the existence of asymmetric equilibria even when firms are symmetric, this case lies outside the scope of our study.

have zero reservation utility, which means that customer j's surplus from product l purchased from firm i is $s_{ijl} = (v_{jl} - p_{il}) \cdot x_{ijl}$. Customers purchase the produce set that maximizes their surplus. In case customers obtain the same surplus from two distinct sets of products, we consider the three sequential tie-breaking rules described next.

1. *More Is Better*: If a customer obtains the same surplus from two sets of products, then she will choose the set with the greatest number of components.
2. *Lower Transaction Cost Is Better*: If a customer obtains the same surplus from two sets of products with the same number of components then she will choose the set with the fewest number of products.
3. *Even Split*: If a customer obtains the same surplus from two sets of products with the same number of components and products, then one of those sets is chosen randomly and with equal probability.

The customers' purchasing decisions that are consistent with these tie-breaking rules are formally presented in Appendix EC.1.

Price Equilibria. Given firms' offerings $\mathbf{z}$ and the other firm's prices, each firm's profit function is a (discontinuous) piecewise linear function of its own prices:

$$\Pi_i(\mathbf{p}_i; \mathbf{p}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = \sum_{j,l} p_{il} x_{ijl}(\mathbf{p}, \mathbf{z}).$$

In its pricing strategy, a firm can choose to capture either none, one, or both customers with each product it offers. In particular, a firm can exclude some customers from all markets if doing so enables it to earn more profit by catering to only the higher-value customers (Armstrong 1996).

In principle, firms can randomize their pricing decisions. Then firm i's expected profit, given its own random pricing strategy F_i and the other firm's pricing strategy F_{-i}, can be written as

$$\Pi_i(F_i; F_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = \int_0^\infty \int_0^\infty \Pi_i(p_i; p_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) \, dF_{-i}(p_{-i}) \, dF_i(p_i).$$

In the pricing game, a Nash equilibrium (in mixed strategies) is defined as follows:

$$\hat{F}_i = \arg\max_{F_i \in \mathcal{F}} \Pi_i(F_i; \hat{F}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) \quad \forall i \in \{1, 2\}. \tag{1}$$

Despite the lack of continuity in firm i's profit function, there is always a pure- or mixed-strategy Nash equilibrium in the cases we consider.

In most cases there will be a unique pure-strategy equilibrium, on which we will focus. We shall use $(\hat{p}_i, \hat{p}_{-i})$ to denote this equilibrium, in which $\hat{F}_i(\hat{p}_i - \delta) = 0$ and $\hat{F}_i(\hat{p}_i + \delta) = 1$ for an arbitrarily small $\delta > 0$. In those instances where no pure-strategy equilibrium exists, we will select the mixed-strategy equilibrium as the limit point of a sequence of mixed-strategy equilibria associated with a sequence of games defined on discretized action spaces (e.g., when prices are expressed in monetary units, e.g., dollars or in cents) by sequentially considering gradually finer action sets by using a logic similar to that employed by Myerson (1991, Sec. 3.13).[4]

Bundling Equilibria. Given the equilibrium $(\hat{F}_i, \hat{F}_{-i})$ in the pricing game, the firms' profits are

$$\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \Pi_i(\hat{F}_i; \hat{F}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}).$$

In the bundling game, a Nash equilibrium (in pure strategies) is defined as

$$\hat{\mathbf{z}}_i = \arg\max_{\mathbf{z}_i \in \mathcal{Z}} \hat{\Pi}_i(\mathbf{z}_i; \hat{\mathbf{z}}_{-i}) \quad \text{for } \; i \in \{1, 2\}. \tag{2}$$

Because the bundling game is finite, it always has a mixed-strategy Nash equilibrium (Nash 1950). In fact, it has always has a pure-strategy Nash equilibrium in the cases we consider (and so we ignore any mixed-strategy equilibria). Yet the bundling game might have *multiple* pure-strategy equilibria, in which case we employ such equilibrium selection rules as Pareto dominance, risk dominance (Harsanyi and Selten 1988), or max-min (von Neumann and Morgenstern 1952). We also consider the impact of infinitesimal costs of including products to an offering.

[4] Specifically, we consider a sequence of nested price grids $\{\mathcal{G}_m\}_{m \geq 0,\, m \in \mathbb{Z}}$ such that $\mathcal{G}_m \subseteq \mathcal{G}_{m+1}$. For each price grid $\mathcal{G}_m$ we consider a price equilibrium in which firms put positive probability weight on $n = 2^m$ points, denoted by $(\hat{F}_1^{(n)}, \hat{F}_2^{(n)})$, and show that: (i) the grids are dense in the sense that, for any price p, there is a price point arbitrarily close to p on all grids that are sufficiently fine; and (ii) the distributions $(\hat{F}_1^{(n)}, \hat{F}_2^{(n)})$ converge to a mixed strategy $(\hat{F}_1, \hat{F}_2)$ on the continuous action set. See Appendix EC.3.

Correlation Structure in Customer Valuations. For tractability in the case of mixed bundling, we assume that the two customer groups have the same valuation of the bundle (viz., $v_{13} = v_{23}$) and that their valuations of the components are perfectly correlated, either negatively or positively.

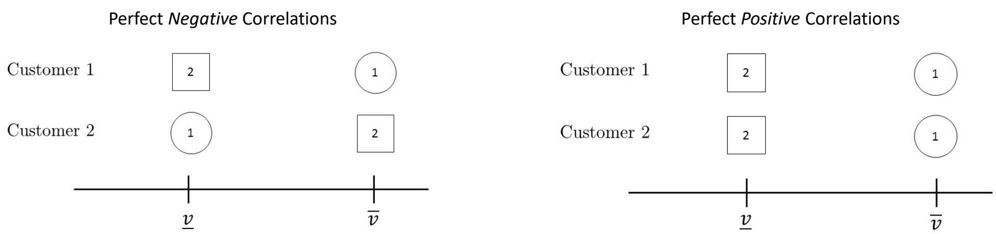


Figure 1 Customer valuations that are perfectly *negatively* (left) or *positively* (right) correlated

Assumption 1 **(PNC)**. *Customer valuations are perfectly negatively correlated (PNC) if* $u_{11} = u_{22} = \overline{v}$ *and* $u_{12} = u_{21} = \underline{v}$*; see Figure 1 (left).*

Assumption 2 **(PPC)**. *Customer valuations are perfectly positively correlated (PPC) if* $u_{11} = u_{21} = \overline{v}$ *and* $u_{12} = u_{22} = \underline{v}$*; see Figure 1 (right).*

In both cases, we assume without loss of generality (w.l.o.g.) that $\overline{v} > \underline{v} > 0$.

Benchmark: Monopoly. As a benchmark, consider a monopolistic firm under PNC valuations (this setup is similar to the one in Adams and Yellen 1976). A firm that offers the two components separately can choose to capture either the high-value customer, in which case it earns $\overline{v}$ per component, or both customers, in which case it earns $2\underline{v}$ per component. The firm's profit is accordingly $2\max\{\overline{v}, 2\underline{v}\}$. In contrast, if the firm offers only the bundle then it can capture both customers at a price $\underline{v} + \overline{v}$ and thereby earn a profit of $2(\underline{v} + \overline{v})$. Comparing the two options reveals that, in a monopoly, offering the bundle is always optimal under PNC valuations.

Yet under PPC valuations, bundling offers no particular benefit. A firm offering the two components separately may indeed set the price of the first (i.e., the most-valued) component to $\overline{v}$ and the price of the second one to $\underline{v}$. But doing so would simply generate a profit of $2(\overline{v} + \underline{v})$, which is identical to that obtained by offering the bundle at a price of $\overline{v} + \underline{v}$.

With competition, however, bundling may no longer be optimal even under PNC valuations. We explore this topic next, first under pure bundling and then under mixed bundling.

4. Pure Bundling

We first consider the case of pure bundling—that is, when each firm can offer at most one product (either the bundle or one of the individual components). Hence we write $\mathcal{Z} = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$. We find that the set of the potential equilibria always contains four asymmetric equilibria such that one firm bundles and the other offers a single-component product.

Our first result is stated in quite general terms because it does not depend on the number of customer segments (which could exceed two), on their respective sizes (which could be unequal), or on their valuations (which could be arbitrary).[5]

PROPOSITION 1. *The pure bundling game has at least four equilibria in which one firm offers the bundle and the other offers a single-component product. In such equilibria, the bundling firm earns more profit than does the non-bundling firm.*

According to this proposition, the equilibrium may be asymmetric despite the firms' symmetry. Once a firm offers the bundle, the other firm prefers to avoid engaging in a price war and instead offers only a single component. Conversely, if one firm offers a single-component product then the other firm finds it attractive to offer the bundle—that is, to expand horizontally. Thus the asymmetry in bundling strategies results from the firms' attempt to soften price competition by differentiating their product offerings (as noted, in a different setting, by Chen 1997).

In any of these four equilibria, the firm that offers the bundle earns more profit than does the other firm.[6] This result suggests a first-mover advantage to bundling because followers are forced to operate in niche markets and earn less profit. One illustration of this equilibrium is Uber's announcement that it would compete against Lyft by offering access to bikes and scooters, public transportation, and self-driving cars (i.e., in addition to human drivers).[7]

[5] The result holds even with component complementarities, i.e., if $v_{ij3} \geq v_{ij1} + v_{ij2}$ for all i and j.

[6] This outcome is in contrast to Chen (1997), who reports the opposite result when one of the components is supplied from a perfectly competitive market.

[7] `https://stratechery.com/2018/ubers-bundles/`, accessed July 9, 2019.

5. Mixed Bundling

Next we consider the case of mixed bundling. We solve the game (1)–(2) by backward induction. First we identify the Nash equilibria of the pricing game for each of the $2^3 \times 2^3 = 64$ possible outcomes of the bundling game (since $\mathcal{Z} = \{0,1\}^3$). We then identify the equilibrium strategies in the bundling games while incorporating the equilibrium outcomes of the pricing subgames.

Our next proposition, which holds for any structure of customer valuations, generalizes the classical result that Bertrand competition in homogeneous markets drives prices—and therefore profits—down to zero.

Proposition 2. *If the firms' offerings are identical* (*i.e., when* $\mathbf{z}_1 = \mathbf{z}_2$) *or if both firms offer both components either separately or as a bundle* (*i.e., when either* $z_{i3} = 1$ *or* $z_{i1} = z_{i2} = 1$ *for all* $i \in \{1,2\}$), *then neither firm makes any profit in equilibrium.*

It follows that each firm finds symmetric offerings to be especially unattractive, although they might still constitute equilibria. In particular: when both firms offer both components, either separately or as a bundle, they earn zero profit regardless of whether (or not) their offerings are identical. That is to say, the effects of symmetric bundling overwhelm those of any differentiation in product offerings.

We now address the opposite case in which no firm offers the bundle. The following proposition states that if a firm is a monopolist on a single-component product $l \in \{1,2\}$, then it must decide either to capture the highest-valuation customer only (by setting its price at $\max_j v_{jl}$) or to capture both customers (by setting its price at $\min_j v_{jl}$).

Proposition 3. *If no firm offers the bundle* (*i.e., when* $z_{13} = z_{23} = 0$), *then* $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \sum_{l=1}^{2} \max\{2\min_j v_{jl}, \max_j v_{jl}\} z_{il}(1 - z_{-i,l})$ *for all* $i \in \{1,2\}$.

Finally, we consider the case where one firm offers the bundle but the other firm offers nothing. The bundling firm thus acts as a monopolist. We posit (in line with Adams and Yellen 1976) that it is optimal for the bundling firm to capture both customer segments because doing so reduces heterogeneity in customer valuations.

Proposition 4. *If one firm offers the bundle and the other firm has a null offering (i.e., when* $z_{i3}=1$ *and* $\mathbf{z}_{-i}=(0,0,0)$*), then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=v_{13}+v_{23}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i)=0$*.*

As a result, the only cases left to analyze are those in which only one firm offers the bundle, perhaps together with other products, while the other firm offers only one single-component product. In those cases, the equilibrium outcomes depend on the structure of customer valuations. We shall now characterize the outcomes first under PNC valuations and then under PPC valuations.

5.1. Perfectly Negatively Correlated Valuations

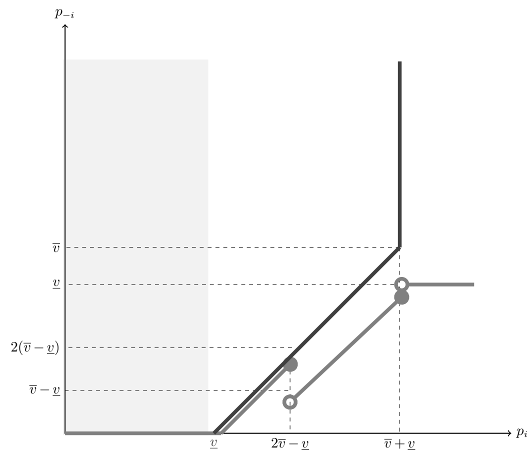
5.1.1. The Pricing Game. In light of Propositions 2–4, we can restrict our attention to cases in which only one firm offers the bundle and the other firm offers only a single-component product. We start by considering the case where the bundling firm offers only the bundle and the other firm offers only one single-component product. We show that there might not exist a pure-strategy equilibrium in the pricing game if customer valuations are too heterogeneous.

Proposition 5. *Under PNC valuations, if one firm offers only the bundle and the other firm offers only a single component (i.e., when* $\mathbf{z}_i=(0,0,1)$ *and either* $\mathbf{z}_{-i}=(1,0,0)$ *or* $\mathbf{z}_{-i}=(0,1,0)$*), then there exists a pure-strategy Nash equilibrium in the pricing game provided that* $\overline{v}\le 2\underline{v}$*. Otherwise, if* $\overline{v}>2\underline{v}$*, then there exists only a mixed-strategy Nash equilibrium in the pricing game. With these equilibria, the payoffs are as follows.*

- *For* $\overline{v}\le 2\underline{v}$: $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i)=0$*.*
- *For* $2\underline{v}<\overline{v}<3\underline{v}$: $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=2(\overline{v}-\underline{v})$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i)=\overline{v}-2\underline{v}$*.*
- *For* $\overline{v}\ge 3\underline{v}$: $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=\overline{v}+\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i)=(\overline{v}-\underline{v})/2$*.*

When $\overline{v}\le 2\underline{v}$, there exists a pure-strategy Nash equilibrium. The equilibrium is such that the bundling firm prices at $\underline{v}$ and the other firm prices at zero. To appreciate the nature of this

Figure 2 Firms' pricing best-response correspondences under PNC when $\overline{v} \leq 2\underline{v}$.



Note. Firm i's best response is shown in black; firm $-i$'s best response is shown in gray, including the shaded rectangle. Here $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$.

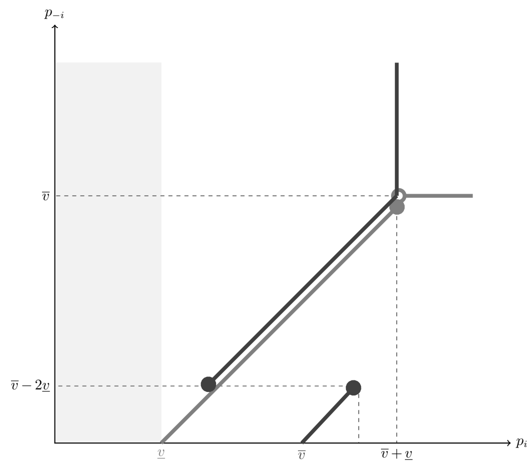
equilibrium, let us assume that prices must be set in increments of $\delta > 0$ (e.g., of cents). As shown in the proof of Proposition 5, the firms' best responses can be expressed as

$$\hat{p}_i(p_{-i}) = \begin{cases} \overline{v} + \underline{v} & \text{if } p_{-i} > \overline{v}, \\ p_{-i} + \underline{v} & \text{if } 0 \leq p_{-i} \leq \overline{v}; \end{cases} \qquad \hat{p}_{-i}(p_i) = \begin{cases} \underline{v} & \text{if } p_i > \overline{v} + \underline{v}, \\ p_i - \overline{v} - \delta & \text{if } 2\overline{v} - \underline{v} < p_i \leq \overline{v} + \underline{v}, \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \leq 2\overline{v} - \underline{v}, \\ [0, \infty) & \text{if } 0 \leq p_i \leq \underline{v}. \end{cases} \tag{3}$$

These responses are plotted in Figure 2.

Consider the following price trajectory when $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$. Suppose that firm i, the bundling firm, sets its initial price at $p_i = \overline{v} + \underline{v}$ with the intention of capturing both customers. Then firm $-i$ responds by undercutting firm i's price, setting its own price to $p_{-i} = \underline{v} - \delta$ so as to capture both customers. In turn, firm i responds by setting $p_i = 2\underline{v} - \delta$ in an attempt to regain them. The firms continue undercutting each other's prices until firm $-i$ abandons the idea of capturing the customer with the lowest valuation of its product. Even so, there is no abatement in the price war—but from that point on, the firms compete for only one customer: whomever has the highest valuation of the stand-alone component. At some point, firm $-i$ must reduce its price to zero and so can no longer compete, ceding both customers to the bundling firm.

Figure 3 Firms' pricing best-response correspondences under PNC when $2\underline{v} < \overline{v} < 3\underline{v}$



Note. Firm i's best response is shown in black; firm $-i$'s best response is shown in gray, including the shaded rectangle. As in Figure 2, $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$.

When $\overline{v} > 2\underline{v}$, however, there is no pure-strategy equilibrium. To see this, suppose that $2\underline{v} < \overline{v} < 3\underline{v}$. In that case, and assuming again that prices must be set in increments of $\delta > 0$ (e.g., cents), the firms' best responses can be expressed as

$$\hat{p}_i(p_{-i}) = \begin{cases} \underline{v} + \overline{v} & \text{if } p_{-i} > \overline{v}, \\ p_{-i} + \underline{v} & \text{if } \overline{v} - 2\underline{v} \le p_{-i} \le \overline{v}, \\ p_{-i} + \overline{v} & \text{if } 0 \le p_{-i} \le \overline{v} - 2\underline{v}; \end{cases} \quad \hat{p}_{-i}(p_i) = \begin{cases} \overline{v} & \text{if } p_i > \overline{v} + \underline{v}, \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \le \overline{v} + \underline{v}, \\ [0, \infty) & \text{if } 0 \le p_i \le \underline{v}. \end{cases} \tag{4}$$

These best responses are illustrated in Figure 3.

Again we consider the price trajectory when $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$. Suppose that firm i, the bundling firm, sets its initial price at $p_i = \overline{v} + \underline{v}$—intending, as before, to capture both customers. In this case, firm $-i$ cannot capture the customer with the lowest valuation of its product because of the large gap in customer valuations; hence it will attempt to capture only the customer with the highest valuation of its product. In response, firm $-i$ sets its price at $p_{-i} = \overline{v} - \delta$ so as to lure that customer away from firm i, which responds by setting $p_i = \overline{v} + \underline{v} - \delta$ in an attempt to regain that customer. The firms keep undercutting each other's prices until firm i's price drops below $\overline{v} - \underline{v}$. At that price, it is no longer profitable for the bundling firm to capture both customers. So in a radical strategic pivot, it seeks instead to extract as much profit as possible from the customer with the lowest valuation of the stand-alone component by *doubling* its price to $2(\overline{v} - \underline{v})$. As a result of that price increase, the customer with the highest valuation for the stand-alone component is now captive to firm $-i$. Yet firm $-i$ cannot resist the urge to charge that customer a higher price and so it, too, sharply increases its price: from $\overline{v} - 2\underline{v}$ to $2\overline{v} - 3\underline{v} - \delta$, which is just enough to retain this customer. But at that price, firm i is once again interested in capturing that customer via lowering its own price by δ. This maneuvering brings the firms back into a price war over the customer with the highest valuation for the stand-alone component, which was the starting point of this price trajectory. The cycle keeps repeating without any end. Similar dynamics arise when $\overline{v} \geq 3\underline{v}$.

Although there is no pure-strategy equilibrium when $\overline{v} > 2\underline{v}$, there does exist a mixed-strategy equilibrium, which we construct (in Appendix EC.3) following the method outlined in Section 3.

Hence both firms compete fiercely over the customer with the highest valuation for the stand-alone component when $\overline{v} \leq 2\underline{v}$, whereas firm i considers an alternative strategy when $2\underline{v} < \overline{v} < 3\underline{v}$—namely, capturing only the customer with the lowest valuation for that product and, in effect, abandoning the competitive field. Competitive intensity is thus softer when $2\underline{v} < \overline{v} < 3\underline{v}$ than when $\overline{v} \leq 2\underline{v}$, resulting into higher shares of value capture; indeed, as $\overline{v}$ increases while $\underline{v}$ remains fixed, both firms' profits increase at a faster rate when $2\underline{v} < \overline{v} < 3\underline{v}$ than when $\overline{v} \leq 2\underline{v}$.

Competitive intensity is also softer when $2\underline{v} < \overline{v} < 3\underline{v}$ than when $\overline{v} \geq 3\underline{v}$. In the latter case, the customer with the highest valuation for the stand-alone component is too profitable for firm i to abandon too soon. Hence the two firms' price war over this particular customer lasts longer. As a result, less value is captured; indeed, as $\overline{v}$ increases while $\underline{v}$ remains fixed, both firms' profits increase at a faster rate when $2\underline{v} < \overline{v} < 3\underline{v}$ than when $\overline{v} \geq 3\underline{v}$.

Next, we consider cases where the bundling firm also includes a single-component product in its offering. We first examine a situation in which the other firm offers the same single-component product. In this case, firms engage in a price war with regard to the common product; the result is zero profits on that product. Thus only the bundling firm makes a positive profit in equilibrium.

That said, the other firm's mere presence is enough to force the bundling firm to lower the price of its bundle to $\underline{v}$.

PROPOSITION 6. *Under PNC valuations, if one firm offers the bundle and a single component while the other firm offers only the same component (i.e., when either* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$*), then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = \max\{2\underline{v}, \overline{v}\}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Next we explore the case where each firm offers a distinct single-component product. Even though the firms' offerings may seem to be non-overlapping, they do include one common component that results in indirect competition. It turns out that the bundling firm is not interested in selling the single-component product and therefore sets its price so high that no customer purchases it. Thus the set of offerings is no different than if the bundling firm offered only the bundle, and the equilibrium is identical to the one characterized by Proposition 5.

PROPOSITION 7. *Under PNC valuations, if one firms offers the bundle and a single component while the other firm offers only the other component (i.e., when either* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$ *or* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$*), then the equilibrium is the same as described in Proposition 5.*

Finally, we consider a scenario in which the bundling firm offers all three products and the other firm offers only one single-component product. Much as in Proposition 6, the firms engage in a price war on the common component, driving its price down to zero. And just as in Proposition 7, the bundling firm sets the price of the single-component product that is *not* offered by the other firm so high that no customer purchases it. So the set of offerings is similarly the same, in effect, as if the bundling firm offered only the bundle and the common single-component product; here the equilibrium is identical to the one characterized by Proposition 6.

PROPOSITION 8. *Under PNC valuations, if one firm offers all products and the other firm offers only a single component (i.e., when* $\mathbf{z}_i = (1,1,1)$ *and when either* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_{-i} = (0,1,0)$*), then the equilibrium is the same as described in Proposition 6.*

5.1.2. The Bundling Game. We now analyze the bundling game by building on the equilibrium outcomes in the pricing games characterized by Propositions 2–8. Tables 1, 2, and 3 present equilibrium outcomes of the pricing games when (respectively) $\overline{v} \leq 2\underline{v}$, $2\underline{v} < \overline{v} < 3\underline{v}$, and $\overline{v} \geq 3\underline{v}$ for each possible combination of product offerings. Thus each table represents a finite strategic-form game in which each player has eight strategies from which to choose.

In each game, we find that there always exists a pure-strategy Nash equilibrium; however, that equilibrium may not be unique. In the tables, all cells corresponding to pure-strategy equilibria are shaded.

When the gap in customer valuations is *small*—that is, when $\overline{v} \leq 2\underline{v}$—which corresponds to a situation where customer preferences are relatively *homogeneous*, the game has the 55 equilibria shown in Table 1. What do not qualify as equilibria are situations in which at least one component is not offered by either firm (which creates opportunities for entry) and in which a monopolistic firm offers both components separately without bundling them (which is suboptimal under PNC valuations, per Adams and Yellen 1976).

Among all equilibria, for $i \in \{1,2\}$ the Pareto-dominant equilibria are: $\mathbf{z}_i = (1,0,0)$ and $\mathbf{z}_{-i} = (0,1,0)$, which yield symmetric payoffs of $(2\underline{v}, 2\underline{v})$; and $z_{i3} = 1$ and $\mathbf{z}_{-i} = \mathbf{0}$, which yield a payoff of $2(\overline{v} + \underline{v})$ for the bundling firm and zero payoff to the other firm. In the first type of equilibrium, no firm bundles; both firms avoid head-to-head competition by focusing on separate single-component product markets. In the second type of equilibrium, one firm bundles (with or without offering additional single-component products) and the other firm declines to enter the market; thus bundling is associated with predation, as noted by Nalebuff (2004). A pairwise comparison of these two equilibria establishes that they are equivalent from the perspective of risk dominance (Harsanyi and Selten 1988) as well as from a security or max-min perspective (von Neumann and Morgenstern 1952). Among these equilibria, the only ones that survive in case of infinitesimal costs for offering a component are when $\mathbf{z}_i = (1,0,0)$ and $\mathbf{z}_{-i} = (0,1,0)$ and when $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = \mathbf{0}$.

In contrast, if the gap in customer valuations is *intermediate* or *large* (i.e., when $2\underline{v} < \overline{v}$), which corresponds to a situation where customer preferences are relatively *heterogeneous*, then the number

Table 1 Firms' payoff matrix in the mixed bundling game under PNC valuations when $\overline{v} \le 2\underline{v}$

	Firm 2's offering							
Firm 1's offering	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,4\underline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
$(1,0,0)$	$2\underline{v},0$	$0,0$	$2\underline{v},2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
$(0,1,0)$	$2\underline{v},0$	$2\underline{v},2\underline{v}$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
$(1,1,0)$	$4\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$

Note. The shaded cells correspond to pure-strategy Nash equilibria.

Table 2 Firms' payoff matrix in the mixed bundling game under PNC valuations when $2\underline{v} < \overline{v} < 3\underline{v}$

	Firm 2's offering							
Firm 1's offering	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	$0,0$	$0,\overline{v}$	$0,\overline{v}$	$0,2\overline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
$(1,0,0)$	$\overline{v},0$	$0,0$	$\overline{v},\overline{v}$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$0,\overline{v}$
$(0,1,0)$	$\overline{v},0$	$\overline{v},\overline{v}$	$0,0$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$0,\overline{v}$	$0,\overline{v}$
$(1,1,0)$	$2\overline{v},0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$

Note. The shaded cells correspond to pure-strategy Nash equilibria.

of equilibria shrinks substantially to 12; see Tables 2 and 3. In eight equilibria, one firm offers only one single-component product and the other firm offers the bundle (which may or may not include the other component). The other equilibria are either monopolies, which amount to one firm

Table 3 Firms' payoff matrix in the mixed bundling game under PNC valuations when $\overline{v} \geq 3\underline{v}$

		Firm 2's offering							
		$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
Firm 1's offering	$(0,0,0)$	$0,0$	$0,\overline{v}$	$0,\overline{v}$	$0,2\overline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
	$(1,0,0)$	$\overline{v},0$	$0,0$	$\overline{v},\overline{v}$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$0,\overline{v}$
	$(0,1,0)$	$\overline{v},0$	$\overline{v},\overline{v}$	$0,0$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$0,\overline{v}$	$0,\overline{v}$
	$(1,1,0)$	$2\overline{v},0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$

Note. The shaded cells correspond to pure-strategy Nash equilibria.

offering the full set of products and the other firm offering nothing, or perfect competition, in which each firm offers both single-component products (with or without the bundle). All except for the latter type of equilibrium are such that firms, by making sure their offerings do not overlap, avoid head-to-head competition in any of the single components. In fact, the latter type of equilibrium is Pareto-dominated by the others.

Eliminating those Pareto-dominated equilibria from consideration, we obtain that the equilibria resulting in monopolies (i.e., $\mathbf{z}_i = (1,1,1)$ and $\mathbf{z}_{-i} = (0,0,0)$) are risk-dominated. Therefore, the Pareto-dominant and risk-dominant equilibria are such that one firm includes the bundle in its offering and the other firm offers only a single-component product. Among those, the surviving equilibria in the presence of infinitesimal costs for offering a component are such that one firm offers only the bundle and the other firm offers only a single-component product—consistently with the case of pure bundling (Section 4). These equilibria are, morever, more secure than those in which the bundler also offers a single-component product—conditional on some role assignment (e.g., firm 1 offering the bundle and firm 2 offering only the single-component product).

It is instructive to compare these two outcomes: (i) when one firm offers the bundle and a single-component product and the other firm offers only the other component (e.g., $\mathbf{z}_1 = (1, 0, 1)$ and $\mathbf{z}_2 = (0, 1, 0)$); and (ii) when both firms bundle in addition to offering their respective single-component products ($\mathbf{z}_1 = (1, 0, 1)$ and $\mathbf{z}_2 = (0, 1, 1)$). One might suppose that it is preferable to have more options, which is indeed the case in a monopolistic setting (Adams and Yellen 1976, McAfee et al. 1989). Yet in a duopoly, *both* firms earn more profit if one of them decides to omit the bundle from its offering. We remark that, because the pricing decision is noncommittal, this decision is not equivalent to setting the price of the bundle so high that no customer purchases it; given that the firm actively selling the bundle makes more profit, any hard commitment to a pricing scheme would not be sustainable; it would indeed be too tempting for the firm who is not selling the bundle to lower its price and capture its competitor's profit. In other words, including the bundle in a product offering without any intention of selling it will inevitably lead firms to engage in a price war and hence to earn zero profit. In line with the saying that "strategy is not only about which projects to choose but also about which projects *not* to choose", a firm that faces a bundling competitor might be better-off deliberately excluding the bundle from its offering so as to preclude any later temptation to fight for value capture.

Similarly, if one firm offers the bundle and the other firm offers only a single-component product, then both firms are worse-off when the bundler offers the same single-component product in its offering. In principle, that inclusion may cost the bundler nothing because both components are already included in its bundle; yet doing so would lead to a price war on that product, reducing the bundler's pricing advantage. Thus once the bundling firm includes the single-component product in its offering, it can no longer commit *not* to compete on price with the other firm. For instance, Microsoft's Teams software is integrated into the firm's Office 365 suite. Even though "Slack [its competitor] may be 'better', overcoming 'free' raises the bar considerably."[8] According to our

[8] `https://stratechery.com/2019/microsofts-earnings-teams-passes-slack-the-partner-advantage/`, accessed July 23, 2019.

analysis—and assuming PNC and heterogeneous valuations—it would be a strategic mistake for Microsoft to offer Teams separately from Office 365.

Irrespective of gaps in customer valuations, the bundling firm earns more profit in equilibrium than does the non-bundling firm. This result mirrors the case of pure bundling in suggesting the existence of a first-mover advantage to bundling.

We conclude that, under PNC valuations, customer valuations need to be very heterogeneous for bundling to emerge as a distinctive strategy. When customer valuations are relatively *homogeneous* (i.e., when $\overline{v} \leq 2\underline{v}$), almost any combination of offerings (provided that both components are offered) may be an equilibrium, including many cases that do not involve bundling. In fact, one of the Pareto-optimal equilibria is such that both firms offer distinct single-component products. When customers have *very heterogeneous* preferences (i.e., when $\overline{v} > 2\underline{v}$), then bundling emerges more as a natural strategy, but it is used more to soften price competition or to raise barriers to entry, and less to price discriminate. Head-to-head competition may still emerge in equilibrium in case both firms offer both of the single-component products, with or without bundle.

5.2. Perfectly Positively Correlated Valuations

In this section we consider the case of PPC valuations, which is equivalent to assuming only one customer group. Under PPC, bundling does not benefit the firm in a monopoly setting. In a duopoly, however, bundling may still be used to soften price competition or pre-empt entry.

5.2.1. The Pricing Game. Here we characterize the outcome of the pricing game for each possible combination of offerings. In light of Propositions 2–4, we need only consider situations in which just one firm offers the bundle while the other firm offers only a single-component product. We shall start by considering the case where the bundling firm offers only the bundle and the other firm offers only one single-component product. Under PNC valuations, this combination of product offerings results in a pure-strategy Nash equilibrium in the pricing game—but only if the gap in customer valuations is sufficiently small (i.e., when $\overline{v} \leq 2\underline{v}$), in which case the bundling firm prices the other firm out of business by setting a price of $\underline{v}$ that captures both customers (Proposition 5). Under PPC valuations we similarly find that there is always a pure-strategy Nash equilibrium, although now irrespective of any gap in customer valuations. In equilibrium, the bundling firm prices the other firm out of business by setting its price either to $\underline{v}$ or to $\overline{v}$ (according as whether the other firm offers the high- or low-value component) and thereby capturing both customers.

PROPOSITION 9. *Under PPC valuations, if one firm offers only the bundle and if the other firm offers only a single component (i.e., when* $\mathbf{z}_i = (0,0,1)$ *and either* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_{-i} = (0,1,0)$*), then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *for* $z_{-i,1} = 1$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\overline{v}$ *for* $z_{-i,2} = 1$*, and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

We next examine the situation in which the bundler offers a single-component product in addition to the bundle. We study first the case where both firms offer the same single-component product and then the case where their offerings differ. As described similarly in Proposition 6, firms offering the same single-component product end up in a price war, which ultimately results in products being "sold" for nothing. Only the bundler makes a positive profit in equilibrium, though its profit is constrained by that free component.

PROPOSITION 10. *Under PPC valuations, if one firm offers the bundle and a single component while the other firm offers only the same component (i.e., when either* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$*), then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *for* $z_{-i,1} = 1$ *and* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\overline{v}$ *for* $z_{-i,2} = 1$*, and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Similarly to Proposition 7's statement, if firms offer distinct single-component products then the bundler prices its single-component product so high that no customer purchases it; thus all sales are geared toward the bundle. As a result, the equilibrium outcome is identical to that described in Proposition 9.

PROPOSITION 11. *Under PPC valuations, if one firms offers the bundle and a single component and if the other firm offers only the other component (i.e., when either* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$ *or* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$*), then the equilibrium is the same as the one characterized by Proposition 9.*

Table 4 Firms' payoff matrix in the mixed bundling game under PPC valuations

Firm 1's offering \ Firm 2's offering	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	$0,0$	$0,2\overline{v}$	$0,2\underline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
$(1,0,0)$	$2\overline{v},0$	$0,0$	$2\overline{v},2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
$(0,1,0)$	$2\underline{v},0$	$2\underline{v},2\overline{v}$	$0,0$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$
$(1,1,0)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$
$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$	$0,0$

Note. The shaded cells correspond to pure-strategy Nash equilibria.

Finally, we consider the case where the bundler offers all three products. Much as in the preceding two propositions, the firms engage in a price war on their common single-component product and the bundler sets the price of its other single-product component so high that no customers purchases it.

Proposition 12. *Under PPC valuations, if one firm offers all products while the other firm offers only a single component (i.e., when* $\mathbf{z}_i = (1,1,1)$ *and either* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_{-i} = (0,1,0)$*), then the equilibrium is the same as in Proposition 10.*

5.2.2. The Bundling Game. We now address the bundling game by building upon the equilibrium outcomes in the pricing games described in Propositions 2–4 and Propositions 9–12. Table 4 presents the equilibrium outcomes of the pricing games for each possible combination of product offerings.

Here, as in the PNC case, we find that the bundling game has multiple pure-strategy equilibria (indicated by the table's shaded cells). And similarly to PNC valuations with small valuation gaps (Table 1), almost any combination of offerings can be sustained as an equilibrium *unless* at least one component is not offered by either firm, which would create opportunities for entry. Comparing Tables 1 and 4 reveals two additional equilibria: when a monopolist offers both single-component products separately, which is suboptimal under PNC valuations but equivalent to offering the bundle under PPC valuations.

Among all equilibria, the Pareto-dominant ones are: (a) $\mathbf{z}_i = (1,0,0)$ and $\mathbf{z}_{-i} = (0,1,0)$ for $i \in \{1,2\}$, which yield asymmetric payoffs of $(2\overline{v}, 2\underline{v})$ and (b) $z_{i3} = 1$ (or $z_{i1} = z_{i2} = 1$) and $\mathbf{z}_{-i} = \mathbf{0}$, yielding a payoff of $2(\overline{v} + \underline{v})$ to the monopolist firm and zero payoff to the other firm. In the first type of equilibrium, no firm bundles; both firms avoid head-to-head competition by focusing on separate single-component product markets. In the second type of equilibrium, one firm bundles or offers the two single-component products separately and the other firm does not enter the market. The similarity of equilibria between the case with PPC valuations and the case with PNC valuations when the gap in valuations is small corroborates our previous result that the set of bundling equilibria depends on the heterogeneity in customer valuations.

6. Conclusions

In this paper we take a first step to understand the competitive bundling dynamics by studying a stylized model of competitive bundling in a Bertrand duopoly. We consider two symmetric firms that simultaneously and non-cooperatively make offering (bundling) decisions followed by pricing decisions. Within this setting, we characterize the equilibrium bundling strategies in the cases of pure and mixed bundling.

With pure bundling, we show that there always exists an equilibrium with differentiated product offerings, namely in which one firm bundles and the other firm offers a single component, resulting in softer price competition. Because the bundling firm earns more profit, we surmise that there are first-mover advantages to bundling. These results are robust to the number, size, and preferences of customer groups.

With mixed bundling, we consider two heterogeneous groups of customers in terms of their valuations of the components. If customers have relatively similar valuations of both components then almost any combination of offerings (provided that all components are supplied) results in an equilibrium, an outcome that helps explain the variety of bundling strategies observed in real-world competitive markets. The Pareto-dominant equilibria are of two types: those in which firms make distinct offers, each offering only a single-component product with no bundling; or those where one firm operates as a monopolist, offering both components as a bundle (or separately) while the other firm makes no profit—presumably resulting in exit or even no entry. In this case, bundling—which arises in the second type of equilibrium—is used for creating barriers to entry.

However, if customers are more heterogeneous in terms of their preferences for specific components, then the set of equilibria is much smaller. In this case, three types of equilibria emerge: (i) only one firm bundles and the other firm offers a single-component product; (ii) one firm offers the full set of products, pre-empting entry by the other firm; (iii) both firms offer both single-component products (potentially together with the bundle), competing head to head and earning zero profit. In the first type of equilibrium, both firms make nonnegative profit but only by randomizing their pricing strategy. As a result, this equilibrium is associated with volatile prices and frequent customer switches. Yet this equilibrium is both Pareto-dominant and risk-dominant, so it is likely to be salient in practice if firms can somehow coordinate. Even if the bundler's offering includes the single-component product not offered by its competitor, it would price that component so high that no customer would purchase it. In contrast, the bundler would never want to include the other firm's product in its offering: doing so would inevitably lead to a price war on that product, eroding the profits of both firms.

Our analysis yields five key insights: (i) various strategies can be sustained in equilibrium; (ii) asymmetric bundling equilibria can emerge even if firms are symmetric; (iii) the more homogeneous the customer valuations, the larger the set of equilibria, including several in which no firm bundles; (iv) under competition, bundling is essentially used to soften price competition or to create barriers to entry, and not to price discriminate; and (v) there are first-mover advantages to bundling. A promising direction for future research would be to explore other market settings—for example, quantity competition with nonzero production and entry costs—so as to contrast the results with those reported in this paper.

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Appendix

EC.1. Customer Purchasing Decisions

In this appendix, we formally introduce the customers' purchasing decisions that are consistent with the tie-breaking rules outlined in §3. Customer j buys the bundle from firm i (i.e., $x_{ij3}(\mathbf{p},\mathbf{z})=1$) if and only if (a) firm i offers the bundle and (b) customer j's surplus from the bundle is nonnegative, weakly dominates the surplus she may obtain with any other product offered by either of the two firms, and strictly dominates the surplus obtained from the bundle offered by the other firm. Formally,

$$x_{ij3}=1 \iff z_{i3}=1,\, v_{j3}-p_{i3}\geq \sum_{l=1}^{2}\max_{\iota\in\{1,2\}}\{(v_{jl}-p_{\iota l})^{+}z_{\iota l}\},\, v_{j3}-p_{i3}>(v_{j3}-p_{-i3})z_{-i,3}; \tag{A-1}$$

here $(y)^{+}\doteq\max\{0,y\}$. The last inequality in (A-1) is not strict when $z_{-i,3}=0$. Suppose that both firms offer the bundle and so deliver the same surplus to customer j; then, ceteris paribus, the market is evenly split according to the third tie-breaking rule:

$$x_{ij3}=\frac{1}{2} \iff z_{i3}=z_{-i,3}=1,\, v_{j3}-p_{i3}=v_{j3}-p_{-i,3},\, v_{j3}-p_{i3}\geq \sum_{l=1}^{2}\max_{\iota\in\{1,2\}}\{(v_{jl}-p_{\iota l})^{+}z_{\iota l}\}. \tag{A-2}$$

Now consider any single-component product $l\in\{1,2\}$ that customer j buys from firm i (i.e., $x_{ijl}(\mathbf{p},\mathbf{z})=1$)—but if and only firm i offers that product. Customer j's surplus from the product must be nonnegative, and it must strictly dominate not only the surplus obtained from the same product offered by the other firm but also the "incremental" surplus obtained from the bundle relative to the surplus obtained from the other single-component product $-l$ (if the latter is offered):

$$\begin{aligned} x_{ijl}=1 \iff & z_{il}=1,\, v_{jl}-p_{il}\geq 0,\\ & v_{jl}-p_{il}>\max\Big\{(v_{jl}-p_{-i,l})^{+}z_{-i,l},\\ & \qquad \max_{\iota\in\{1,2\}}\{(v_{j3}-p_{\iota 3})^{+}z_{\iota 3}\}-\max_{\iota\in\{1,2\}}\{(v_{ji'}-p_{\iota,-l})^{+}z_{\iota,-l}\}\Big\}. \end{aligned} \tag{A-3}$$

If $z_{-i,l}=0$ and $z_{\iota 3}=z_{\iota l}=0$ for both $\iota\in\{1,2\}$ and $l\in\{1,2\}$, then the last inequality in (A-3) applies nonstrictly. If both firms offer the same single-component product l (yielding the same surplus to customer j) but all conditions hold otherwise, then the market is evenly split according to the third tie-breaking rule:

$$\begin{aligned} x_{ijl}=\frac{1}{2} \iff & z_{il}=z_{-i,l}=1,\, v_{jl}-p_{il}=v_{jl}-p_{-i,l},\, v_{jl}-p_{il}\geq 0,\\ & v_{jl}-p_{il}>\max_{\iota\in\{1,2\}}\{(v_{j3}-p_{\iota 3})^{+}z_{\iota 3}\}-\max_{\iota\in\{1,2\}}\{(v_{j\iota}-p_{\iota,-l})^{+}z_{\iota,-l}\}. \end{aligned} \tag{A-4}$$

EC.2. Proofs and Supplementary Results

LEMMA A-1. *For any product $l \in \{1,2,3\}$, suppose that $z_{1l} = z_{2l} = 1$. Then $\hat{p}_{il}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ijl}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ for both firms $i \in \{1,2\}$. Similarly, under mixed bundling suppose that for any firm $i \in \{1,2\}$ we have $z_{i3} = z_{-i,1} = z_{-i,2} = 1$. Then $\hat{p}_{i3}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ij3}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ and $\sum_{l=1}^{2} \hat{p}_{-i,l}(\mathbf{z}_{-i}, \mathbf{z}_i) \sum_j x_{-i,jl}(\hat{\mathbf{p}}, \mathbf{z}) = 0$.*

Proof. Fix l and suppose that $z_{1l} = z_{2l} = 1$ for some $l \in \{1,2,3\}$. We can obtain a contradiction by first supposing that, in equilibrium, firm i earns a positive profit from product l; that is, $p_{il} > 0$ and $\sum_j x_{ijl} > 0$. Since $\sum_j x_{ijl} > 0$, it follows from (A-1)–(A-4) that firm i must sell product l at a lower price than does firm $-i$; that is, $p_{-i,l} \geq p_{il}$. But then firm $-i$ could lower its price below p_{il} (since it is positive), steal market share from firm i (since it, too, is positive), capture firm i's profit on the sale of that product, and thus strictly increase its own profit—a contradiction. So if there does exist an equilibrium, then $\hat{p}_{il}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ijl}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ for both firms $i \in \{1,2\}$. Finally, we show that there indeed exists an equilibrium that achieves this outcome. Suppose $p_{il} = 0$. Then firm $-i$'s best response is $\hat{p}_{-i,l}(p_{il}) \in [0, \infty)$; that is, $p_{il} = p_{-i,l} = 0$ is an equilibrium.

Next suppose that $z_{i3} = z_{-i,1} = z_{-i,2} = 1$. To obtain a contradiction in this case we suppose that, in equilibrium, firm i earns a positive profit from the bundle; that is, $p_{i3} > 0$ and $\sum_j x_{ij3} > 0$. Since $\sum_j x_{ij3} > 0$, again by (A-1)–(A-4) we must have that $p_{-i,1} + p_{-i,2} \geq p_{i3}$. But then firm $-i$ could lower $p_{-i,1} + p_{-i,2}$ below p_{i3} (since it is positive), steal market share from firm i (since it is positive), capture firm i's profit, and thereby strictly increase its own profit—a contradiction. As a consequence, if there exists an equilibrium then $\hat{p}_{i3}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ij3}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. Using a symmetric argument, we can show that $\sum_{l=1}^{2} \hat{p}_{-i,l}(\mathbf{z}_{-i}, \mathbf{z}_i) \sum_j x_{-i,jl}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. Finally, we establish that there does exist an equilibrium that achieves this outcome. Suppose that $p_{i3} = 0$. Then firm $-i$'s best response is $\hat{p}_{-i,l}(p_{i3}) \in [0, \infty)$ for any $l \in \{1,2\}$. Similarly, if $p_{-i,l} = 0$ for both $l \in \{1,2\}$ then $\hat{p}_{i3}(p_{-i,1}, p_{-i,2}) \in [0, \infty)$. That is, $p_{i3} = p_{-i,1} = p_{-i,2} = 0$ is an equilibrium. □

EC.2.1. Pure Bundling

Proof of Proposition 1. We analyze the best response of firm 1 to the bundling strategy of firm 2; symmetry ensures that a similar analysis applies to the reverse case. We use the shorthand notation p_1 and p_2 when referring to the prices of products offered by (respectively) firms 1 and 2. Let $\hat{\mathbf{z}}_1(\mathbf{z}_2)$ be the set of firm 1's best responses to $\mathbf{z}_2$.

First we show that, when firm 2 offers either nothing or a single component, a weakly dominant strategy for firm 1 is to offer the bundle. Thereafter we show that, if firm 2 offers the bundle, then a weakly dominant strategy for firm 1 is to offer a single component.

- When $\mathbf{z}_2 = (0,0,0)$, we consider four different responses by firm 1:

  1. if $\mathbf{z}_1 = (0,0,0)$, then $\Pi_1(p_1; p_2, \mathbf{z}) = 0$ for all p_1;
  2. if $\mathbf{z}_1 = (1,0,0)$, then $\Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j1}$;
  3. if $\mathbf{z}_1 = (0,1,0)$, then $\Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j2}$;
  4. if $\mathbf{z}_1 = (0,0,1)$, then $\Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j3}$.

  Customers' valuation of the bundle is the sum of their valuations of each component (i.e., $v_{j3} = v_{j1} + v_{j2}$ for any customer j); hence, for any given p_1 (for either a component or the bundle), we must have $\sum_j x_{ij3} \geq \sum_j x_{ij1}$ and $\sum_j x_{ij3} \geq \sum_j x_{ij2}$. Therefore, $(0,0,1) \in \hat{\mathbf{z}}_1(0,0,0)$.

- When $\mathbf{z}_2 = (1,0,0)$, we consider these responses by firm 1:

  1. if $\mathbf{z}_1 = (0,0,0)$, then $\Pi_1(p_1; p_2, \mathbf{z}) = 0$ for all p_1;
  2. if $\mathbf{z}_1 = (1,0,0)$ then, by Lemma A-1, $\hat{\Pi}_1(\hat{p}_1; p_2, \mathbf{z}) = \hat{\Pi}_2(\mathbf{z}_1; \mathbf{z}_2) = 0$;
  3. if $\mathbf{z}_1 = (0,1,0)$ then $\Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j2} p_1$, where it follows from (A-3) that, for any customer $j \in \{1,2\}$,
  $$x_{1j2} = 1 \iff v_{j2} - p_1 \geq 0; \tag{A-5}$$
  4. if $\mathbf{z}_1 = (0,0,1)$ then $\Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j3} p_1$, where now from (A-1) we have that, for any customer $j \in \{1,2\}$,
  $$x_{1j3} = 1 \iff v_{j1} + v_{j2} - p_1 \geq \max\{v_{j1} - p_2, 0\}. \tag{A-6}$$

  The first and the second strategies are always dominated by the other two and so will never be best responses. Consider the following price equilibria, which correspond to different offerings associated with the third and fourth strategies:

  $$\tilde{p}_i = \arg\max_{p_i \geq 0} \Pi_i(p_i; \tilde{p}_{-i}, (0,1,0), (1,0,0)) \quad \forall i \in \{1,2\},$$

  $$\hat{p}_i = \arg\max_{p_i \geq 0} \Pi_i(p_i; \hat{p}_{-i}, (0,0,1), (1,0,0)) \quad \forall i \in \{1,2\}.$$

  From (A-5) and (A-6) we obtain that

  $$x_{1j2} = 1 \iff v_{j2} - p_1 \geq 0 \implies \begin{cases} v_{j1} + v_{j2} - p_1 \geq v_{j1} \geq v_{j1} - p_2, \\ v_{j1} + v_{j2} - p_1 \geq 0 \end{cases} \implies x_{1j3} = 1.$$

Thus $\sum_j x_{1j3}p_1 \geq \sum_j x_{1j2}p_1$ for all p_1 and p_2. Therefore,

$$\hat{\Pi}_1((0,0,1);(1,0,0)) = \sum_j x_{1j3}\hat{p}_1 \geq \sum_j x_{1j3}\tilde{p}_1 \geq \sum_j x_{1j2}\tilde{p}_1 = \hat{\Pi}_1((0,1,0);(1,0,0)) \geq 0.$$

Hence $(0,0,1) \in \hat{\mathbf{z}}_1(1,0,0)$.

- When $\mathbf{z}_2 = (0,1,0)$, we can use a symmetric argument to derive that $(0,0,1) \in \hat{\mathbf{z}}_1(0,1,0)$.
- When $\mathbf{z}_2 = (0,0,1)$, we consider the following four responses by firm 1.

  1. If $\mathbf{z}_1 = (0,0,0)$, then $\Pi_1(p_1;p_2,\mathbf{z}) = 0$ for all p_1.
  2. If $\mathbf{z}_1 = (1,0,0)$, then $\Pi_1(\hat{p}_1;p_2,\mathbf{z}) = \max_{p_2} \sum_j x_{1j1}p_1$; here, by (A-3), we have
  $$x_{1j1} = 1 \iff v_{j1} - p_1 > v_{j1} + v_{j2} - p_2 \text{ and } v_{j1} - p_1 \geq 0.$$
  3. If $\mathbf{z}_1 = (0,1,0)$, then $\Pi_1(\hat{p}_1;p_2,\mathbf{z}) = \max_{p_2} \sum_j x_{1j2}p_1$; here, by (A-3), we can write
  $$x_{1j2} = 1 \iff v_{j2} - p_1 > v_{j1} + v_{j2} - p_2 \text{ and } v_{j2} - p_1 \geq 0.$$
  4. If $\mathbf{z}_1 = (0,0,1)$ then, by Lemma A-1, $\hat{\Pi}_1(\mathbf{z}_1,\mathbf{z}_2) = \hat{\Pi}_2(\mathbf{z}_1,\mathbf{z}_2) = 0$.

  The first and the fourth strategies are always weakly dominated by the second and third strategies. Both $\mathbf{z}_1 = (1,0,0)$ and $\mathbf{z}_1 = (0,1,0)$ could be dominant strategies depending on the relative sizes of customer segments and their valuations of the products. As a result, either $(1,0,0) \in \hat{\mathbf{z}}_1(0,0,1)$ or $(0,1,0) \in \hat{\mathbf{z}}_1(0,0,1)$.

In sum, we have shown that (a) offering a bundle is a best response if the other firm offers a single-component product (or offers nothing) and (b) offering a single-component product is a best response if the other firm offers the bundle.

We establish that the bundling firm always earns more profit by observing that the focal firm, when responding to a firm offering a single-component product, could choose to offer the other single-component product; however, the best response always includes the bundle. In other words, the profits from bundling are always (weakly) greater than those from offering a single-component product. $\square$

EC.2.2. Mixed Bundling

EC.2.2.1. Preliminaries

LEMMA A-2. *Under mixed bundling and for any firm $i \in \{1,2\}$, if $z_{i3} = 1$ then firm i is always (weakly) better-off selling the bundle; that is, $\sum_j x_{ij3}(\hat{\mathbf{p}},\mathbf{z}) > 0$.*

Proof. For any firm $i \in \{1,2\}$, suppose that $z_{i3} = 1$. To obtain a contradiction, suppose that $\sum_j x_{ij3}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. Then, by (A-1), either: (a) $p_{i3} > v_{j3}$ for all j or (b) $p_{i3} > p_{-i,3}$ or (c) there exists a product $l \in \{1,2\}$ and a firm $\iota \in \{1,2\}$ such that $x_{\iota 1 l} > 0$ as well as a product $l' \in \{1,2\}$ and a firm $\iota' \in \{1,2\}$ such that $x_{\iota' 2 l'} > 0$; that is, both customer groups prefer buying a stand-alone product over the bundle. The first two cases can be easily addressed by setting $p_{i3} \le \min\{\max_j v_{j3}, p_{-i,3}\}$, or $p_{i3} \le \min\{v_{13}, p_{-i,3}\}$ (since $v_{13} = v_{23}$). Hence we shall focus on the latter case while assuming that $p_{i3} \le \min\{v_{13}, p_{-i,3}\}$. If there are multiple firms $\iota \in \{1,2\}$, customers $j \in \{1,2\}$, and products $l \in \{1,2\}$ for which $x_{\iota j l} > 0$, then we choose $(\iota^*, j^*, l^*) = \arg\min_{(\iota,j,l): x_{\iota j l} > 0}\{v_{jl} - p_{\iota l} + \max_{\iota' \in \{1,2\}}\{(v_{j,-l} - p_{\iota',-l})^+ z_{\iota',l}\}\}$. To streamline the notation, we assume w.l.o.g. that $\iota^* = i$ and $l^* = 1$; we also omit the asterisk from j^*. By (A-3), the inequality $x_{\iota^*, j^*, l^*} > 0$ (i.e., $x_{ij1} > 0$) implies that (i) $z_{i1} = 1$ and (ii) $v_{j1} - p_{i1} + \max_{\iota \in \{1,2\}}\{(v_{j2} - p_{\iota 2})^+ z_{\iota 2}\} > v_{j1} + v_{j2} - p_{i3}$; that is, $p_{i3} > p_{i1} + v_{j2} - \max_{\iota \in \{1,2\}}\{(v_{j,2} - p_{\iota 2})^+ z_{\iota 2}\}$.

Now we consider an alternate scenario, $\mathbf{p}'$, in which firm i sets the price of its bundle to $p'_{i3} = p_{i1} + v_{j2} - \max_{\iota \in \{1,2\}}\{(v_{j2} - p_{\iota 2})^+ z_{\iota 2}\}$ and keeps all other prices unchanged. We use $x'_{ijl} \doteq x_{ijl}(\mathbf{p}', \mathbf{z})$ to denote the sales of any product $l = 1$, 2, or 3 to customer j under this alternate scenario. Since $p_{i3} \le \min\{v_{j3}, p_{-i,3}\}$ by assumption, it follows that $p'_{i3} < p_{-i,3}$ and $p_{i3'} < v_{j3}$. Moreover, $v_{j3} - p'_{i3} = v_{j1} + v_{j2} - p'_{i3} = v_{j1} - p_{i1} + \max_{\iota \in \{1,2\}}\{(v_{j2} - p_{\iota 2})^+ z_{\iota 2}\} \ge v_{j1} - p_{i1}$. According to (A-1), customer j now prefers the bundle over product 1 (i.e., product l^*). Hence $x'_{ij3} \ge x_{ij1}$; that is, firm i captures at least the same amount of sales from customer j at a price that is at least as large: $p'_{i3} \ge p_{i1}$, and the inequality is strict unless $z_{\iota 2} = 1$ and $p_{\iota 2} = 0$ for some $\iota \in \{1,2\}$. Assessing the impact of this alternate pricing scenario on the total profit obtained from customer j requires that we distinguish between two cases. Suppose first that $x_{ij2} = 0$. Then $\sum_{l=1}^{3} p_{il} x_{ijl} = p_{i1} x_{ij1} \le p_{i3} x'_{ij3} = \sum_{l=1}^{3} p'_{il} x'_{ijl}$. Suppose next that $x_{ij2} > 0$, which implies that $z_{i2} = 1$ and that $p_{i2} \le p_{-i,2}$ if $z_{-i,2} = 1$. In that case, $p'_{i3} = p_{i1} + v_{j2} - (v_{j2} - p_{i2}) = p_{i1} + p_{i2}$. By (A-1), customer j now prefers the bundle over purchasing products 1 and 2 separately. Hence $x'_{ij3} \ge x_{ij2}$. Therefore, $\sum_{l=1}^{3} p_{il} x_{ijl} = p_{i1} x_{ij1} + p_{i2} x_{ij2} \le p'_{ij3} x'_{ij3} = \sum_{l=1}^{3} p'_{il} x'_{ijl}$. In both cases, firm i's profit from customer j increases when the bundle price is reduced, and that increase is strict unless $z_{\iota 2} = 1$ and $p_{\iota 2} = 0$ for some $\iota \in \{1,2\}$.

However, profit generated from the other customer does not change given the way (ι^*, j^*, l^*) is chosen—that is, since this other customer derives more surplus from the individual components than from the bundle and since the component prices do not change. It is therefore beneficial for firm i to lower the price of its bundle from p_{i3} to p'_{i3}, which results in a situation where $\sum_j x'_{ij3}(\mathbf{p}', \mathbf{z}) > 0$. $\square$

LEMMA A-3. *Under mixed bundling, let $z_{i3}=1$, $z_{il}=1$, and $z_{-i,l}=0$ for any firm $i\in\{1,2\}$ and any product $l\in\{1,2\}$. Then $\sum_j x_{ijl}(\hat{\mathbf{p}},\mathbf{z})=0$.*

Proof. This proof uses Lemma A-2. For any firm $i\in\{1,2\}$ and product $l\in\{1,2\}$, suppose that $z_{i3}=z_{il}=1$ and $z_{-i,l}=0$. We obtain a contradiction by supposing that $x_{ijl}>0$ for any customer $j\in\{1,2\}$. Then $x_{ij3}=0$ by (A-3). But since $\sum_j x_{ij3}(\hat{\mathbf{p}},\mathbf{z})>0$ (by Lemma A-2), it follows that $x_{i,-j,3}>0$. So given that $z_{il}=1$ and $z_{-i,l}=0$, both customers assess the surplus they obtain from the bundle offered by firm i with the surplus from product l provided by firm i and from product $-l$ provided by any firm. In particular, customer j purchases product l because $v_{jl}-p_{il}+\max_{\iota\in\{1,2\}}\{(v_{j,-l}-p_{\iota,-l})^{+}z_{\iota,-l}\}>v_{j3}-p_{i3}$ whereas customer $-j$ purchases the bundle because $v_{-j,3}-p_{i3}\ge v_{-j,l}-p_{il}+\max_{\iota\in\{1,2\}}\{(v_{-j,-l}-p_{\iota,-l})^{+}z_{\iota,-l}\}$. Firm i could then increase p_{il} so that the first inequality no longer holds—while keeping customer $-j$'s decision unchanged—and sell the bundle to both customers, thereby increasing its profit. In equilibrium, then, $x_{ijl}=0$ for both customers $j\in\{1,2\}$. $\square$

Proof of Proposition 2. If $\mathbf{z}_1=\mathbf{z}_2$ then, by Lemma A-1, $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=\sum_l \hat{p}_{il}(\mathbf{z}_i,\mathbf{z}_{-i})\sum_j x_{ijl}(\hat{\mathbf{p}},\mathbf{z})=0$ for both firms $i\in\{1,2\}$. To establish the proposition's second claim, fix $i\in\{1,2\}$ and suppose first that $z_{i3}=1$ and either $z_{-i,3}=1$ or $z_{-i,1}=z_{-i,2}=1$. We can then obtain a contradiction by assuming that $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})>0$. By Lemma A-1, $\hat{p}_{i3}(\mathbf{z}_i,\mathbf{z}_{-i})\sum_j x_{ij3}(\hat{\mathbf{p}},\mathbf{z})=0$ and so, for some product $l\in\{1,2\}$, we must have $\hat{p}_{il}(\mathbf{z}_i,\mathbf{z}_{-i})\sum_j x_{ijl}(\hat{\mathbf{p}},\mathbf{z})>0$. In particular, $z_{il}=1$. If $z_{-i,l}=1$ then, by Lemma A-1, firm i would make zero profit on product l; therefore, $z_{-i,l}=0$. But then, by Lemma A-3, $\sum_j x_{ijl}(\hat{\mathbf{p}},\mathbf{z})=0$—a contradiction. Suppose next that $z_{i3}=0$ and $z_{i1}=z_{i2}=1$ and that either $z_{-i,3}=1$ or $z_{-i,1}=z_{-i,2}=1$. By Lemma A-1, firm i makes zero profit on both products $l\in\{1,2\}$; hence $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=0$. $\square$

Proof of Proposition 3. Fix $i\in\{1,2\}$ and $l\in\{1,2\}$, and suppose that $z_{il}=1$. If $z_{-i,l}=1$, then the result follows from Lemma A-1. Otherwise, firm i captures two customers when $p_{il}\le\min_j v_{jl}$, one customer when $\min_j v_{jl}<p_{jl}\le\max_j v_{jl}$, and no customer when $p_{jl}>\max_j v_{jl}$. $\square$

Proof of Proposition 4. Fix i, and suppose that $z_{i3}=1$ and $\mathbf{z}_{-i}=(0,0,0)$. We know from Lemma A-3 that, for any product $l\in\{1,2\}$, if $z_{il}=1$ then $x_{ijl}(\hat{\mathbf{p}},\mathbf{z})=0$. Therefore, $\Pi_i(p_i;p_{-i},\mathbf{z})=p_{i3}\sum_j x_{ij3}=2p_{i3}$ if $p_{i3}\le\min_j v_{j3}$, $\Pi_i(p_i;p_{-i},\mathbf{z})=p_{i3}$ if $\min_j v_{j3}<p_{i3}\le\max_j v_{j3}$, and $\Pi_i(p_i;p_{-i},\mathbf{z})=0$ if $p_{i3}>\max_j v_{j3}$. By assumption, $v_{13}=v_{23}$. Hence $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i})=\hat{p}_{i3}\sum_j x_{ij3}(\hat{\mathbf{p}},\mathbf{z})=v_{13}+v_{23}$. $\square$

EC.2.2.2. Perfectly Negatively Correlated (PNC) Valuations

Proof of Proposition 5. Since $\mathbf{z}_i$ and $\mathbf{z}_{-i}$ are kept fixed, hereafter we shall omit these arguments from the profit functions and best-response correspondences; we also use the simplified notation $p_i = p_{i3}$, $p_{-i} = p_{-il}$ and $s_{ij} = s_{ij3}$, $s_{-i,j} = s_{-ijl}$ for any customer $j \in \{1,2\}$ and product $l \in \{1,2\}$. We focus on the case where $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$. First we analyze the best-response correspondence of firm i and then that of firm $-i$. Throughout this characterization of best responses, we assume that prices must be set in infinitesimal increments of $\delta > 0$.

Firm i. We begin by analyzing the best-response correspondence for different ranges of firm i's prices—namely, when $p_{-i} > \overline{v}$, $\underline{v} \le p_{-i} \le \overline{v}$, and $0 \le p_{-i} < \underline{v}$.

- If $p_{-i} > \overline{v}$, then customers' surplus from firm $-i$ is negative; that is, $s_{-i,1}(p_{-i}) = s_{-i,2}(p_{-i}) < 0$. As long as firm i offers them a nonnegative surplus (i.e., when $s_{i1}(p_i) = s_{i2}(p_i) = \overline{v} + \underline{v} - p_i \ge 0$), it can capture both customers and earn $\Pi_i(p_i; p_{-i}) = 2p_i$. Therefore, firm i's best response is

$$\hat{p}_i(p_{-i}) = \overline{v} + \underline{v} \quad \text{and} \quad \hat{\Pi}_i(p_i; p_{-i}) = 2(\underline{v} + \overline{v}).$$

- If $\underline{v} \le p_{-i} \le \overline{v}$, then $s_{-i,1}(p_{-i}) = \underline{v} - p_{-i} \le 0$ and $s_{-i,2}(p_{-i}) = \overline{v} - p_{-i} > 0$. For firm i to capture customer 1, it is necessary and sufficient that $s_{i1}(p_i) \ge 0$, which means that $p_i \le \overline{v} + \underline{v}$. Similarly, to capture customer 2 it is necessary and sufficient that $s_{i2}(p_i) \ge 0$ and $s_{i2}(p_i) \ge s_{-i,2}(p_{-i})$ (i.e., $\overline{v} + \underline{v} - p_i \ge \overline{v} - p_{-i} > 0$). Hence firm i captures both customers if $p_i \le \underline{v} + p_{-i}$, it captures customer 1 only if $\underline{v} + p_{-i} < p_i \le \overline{v} + \underline{v}$, and it captures no customers if $p_i > \underline{v} + \overline{v}$. As a result,

$$\hat{\Pi}_i(p_i; p_{-i}) = \max\{2(p_{-i} + \underline{v}), \overline{v} + \underline{v}\} \quad \text{and} \quad \hat{p}_i(p_{-i}) = \begin{cases} p_{-i} + \underline{v} & \text{if } p_{-i} \ge (\overline{v} - \underline{v})/2, \\ \overline{v} + \underline{v} & \text{if } p_{-i} \le (\overline{v} - \underline{v})/2. \end{cases}$$

- If $0 \le p_{-i} < \underline{v}$, then $s_{-i,1}(p_{-i}) = \underline{v} - p_{-i} > 0$ and $s_{-i,2}(p_{-i}) = \overline{v} - p_{-i} > 0$. For firm i to capture customer 1 it is necessary and sufficient that $s_{i1}(p_i) \ge s_{-i,1}(p_{-i})$ (i.e., $\overline{v} + \underline{v} - p_i \ge \underline{v} - p_{-i}$, or $p_i \le p_{-i} + \overline{v}$). Similarly, to capture customer 2 it is necessary and sufficient that $s_{i2}(p_i) \ge s_{-i,2}(p_{-i})$ (i.e., $\overline{v} + \underline{v} - p_i \ge \overline{v} - p_{-i}$, or $p_i \le p_{-i} + \underline{v}$). Therefore,

$$\hat{\Pi}_i(p_i; p_{-i}) = \max\{2(p_{-i} + \underline{v}), p_{-i} + \overline{v}\} \quad \text{and} \quad \hat{p}_i(p_{-i}) = \begin{cases} p_{-i} + \underline{v} & \text{if } p_{-i} \ge \overline{v} - 2\underline{v}, \\ p_{-i} + \overline{v} & \text{if } p_{-i} \le \overline{v} - 2\underline{v}. \end{cases}$$

Firm $-i$. Next we identify firm $-i$'s best-response correspondence. Toward that end, we consider the following four distinct sets of values for p_i.

- If $p_i > \overline{v} + \underline{v}$, then $s_{i1}(p_i) = s_{i2}(p_i) < 0$. Hence for firm $-i$ to capture customer 1 it is necessary and sufficient that $p_{-i} \leq \underline{v}$; the corresponding requirement to capture customer 2 is $p_{-i} \leq \overline{v}$. Hence firm $-i$ captures both customers if $p_{-i} \leq \underline{v}$, captures only one customer if $\underline{v} < p_{-i} \leq \overline{v}$, and captures no customers if $p_{-i} > \overline{v}$. It follows that

$$\hat{\Pi}_{-i}(p_{-i}; p_i) = \max\{2\underline{v}, \overline{v}\} \quad \text{and} \quad \hat{p}_{-i}(p_i) = \begin{cases} \underline{v} & \text{if } 2\underline{v} \geq \overline{v}, \\ \overline{v} & \text{if } 2\underline{v} \leq \overline{v}. \end{cases}$$

- If $\overline{v} \leq p_i \leq \overline{v} + \underline{v}$, then $s_{i1}(p_i) = s_{i2}(p_i) \geq 0$. So in order to capture customer 1, firm $-i$ must price such that $\underline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$ (i.e., $p_{-i} < p_i - \overline{v}$). Similarly, capturing customer 2 requires $\overline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$ (i.e., $p_{-i} < p_i - \underline{v}$). Hence firm $-i$ captures both customers if $p_{-i} < p_i - \overline{v}$, captures only one customer if $p_i - \overline{v} \leq p_{-i} < p_i - \underline{v}$, and captures no customers if $p_{-i} \geq p_i - \underline{v}$. As a result, for some infinitesimal $\delta > 0$ we have

$$\hat{\Pi}_{-i}(p_{-i}; p_i) = \sup_{\delta > 0}\{2(p_i - \overline{v} - \delta)^+, (p_i - \underline{v} - \delta)^+\} \quad \text{and} \quad \hat{p}_{-i}(p_i) = \begin{cases} p_i - \overline{v} - \delta & \text{if } p_i > 2\overline{v} - \underline{v}, \\ p_i - \underline{v} - \delta & \text{if } p_i \leq 2\overline{v} - \underline{v}. \end{cases}$$

- If $\underline{v} < p_i < \overline{v}$, then $\underline{v} - p_{-i} < \overline{v} + \underline{v} - p_i$ (i.e., $s_{-i,1}(p_{-i}) < s_{i1}(p_i)$ for all $p_{-i} \geq 0$. Thus firm $-i$ can never capture customer 1 at any nonnegative price $p_{-i} \geq 0$. To capture customer 2, it is necessary and sufficient that $s_{-i,2}(p_{-i}) > s_{i2}(p_i)$ and $s_{-i,2}(p_{-i}) \geq 0$; these conditions are equivalent to, respectively, $\overline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$ and $\overline{v} - p_{-i} \geq 0$ (i.e., $p_{-i} < p_i - \underline{v}$). It follows that, for some infinitesimal $\delta > 0$,

$$\hat{\Pi}_{-i}(p_{-i}; p_i) = \hat{p}_{-i}(p_i) = p_i - \underline{v} - \delta.$$

- If $0 \leq p_i \leq \underline{v}$, then $s_{-i,1}(p_{-i}) < s_{i1}(p_i)$ and $s_{-i,2}(p_{-i}) < s_{i2}(p_i)$ for any $p_{-i} \geq 0$. Hence firm $-i$ does not capture any customer at any nonnegative price $p_{-i} \geq 0$, so we may write

$$\hat{\Pi}_{-i}(p_{-i}; p_i) = 0 \quad \text{and} \quad \hat{p}_{-i}(p_i) = \{p_{-i} \mid p_{-i} \geq 0\}.$$

Existence of Pure-Strategy Nash Equilibria. For both firms we conclude that there are three separate best-response correspondences depending on the relative values of $\overline{v}$ and $\underline{v}$.

1. If $\overline{v} \le 2\underline{v}$ then, assuming prices must be set in increments of $\delta > 0$ (e.g., cents), we obtain that the best responses are given by (3). For any $\delta > 0$, this game has a pure-strategy Nash equilibrium at $(\underline{v}, 0)$ (see Figure 2), which yields $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. Firm i does not deviate from this strategy because it would earn strictly less profit by either increasing or decreasing its price. At the same time, setting $p_{-i} = 0$ is a weakly dominant strategy for firm $-i$ because it would still earn zero profit for any other p_{-i}. This equilibrium is not unique, but it is payoff-equivalent to any other equilibria for this game.

2. If $2\underline{v} < \overline{v} < 3\underline{v}$ then, assuming that prices must be set in increments of $\delta > 0$, we find that the best responses are given by (4). In this case, the game does not have a pure-strategy Nash equilibrium because the best-response correspondences do not intersect.

3. If $\overline{v} \ge 3\underline{v}$ then, assuming that prices must be set in increments of $\delta > 0$, we obtain that the best responses are given by

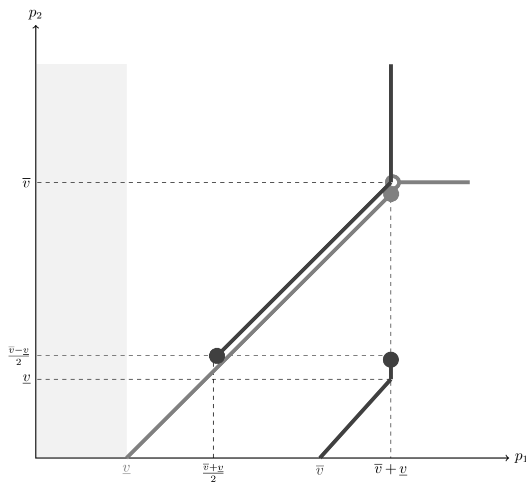
$$\hat{p}_i(p_{-i}) = \begin{cases} \underline{v} + \overline{v} & \text{if } p_{-i} > \overline{v}, \\ p_{-i} + \underline{v} & \text{if } (\overline{v} - \underline{v})/2 \le p_{-i} \le \overline{v}, \\ \underline{v} + \overline{v} & \text{if } \underline{v} \le p_{-i} \le (\overline{v} - \underline{v})/2, \\ p_{-i} + \overline{v} & \text{if } 0 \le p_{-i} < \underline{v}; \end{cases} \qquad \hat{p}_{-i}(p_i) = \begin{cases} \overline{v} & \text{if } p_i > \overline{v} + \underline{v}, \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \le \overline{v} + \underline{v}, \\ [0, \infty) & \text{if } 0 \le p_i \le \underline{v}. \end{cases}$$

The best responses are illustrated in Figure EC.1. In this case as well, the game does not have a pure-strategy Nash equilibrium because the best-response correspondences do not intersect.

Support of Non-Dominated Strategies. In what follows we assume that $\overline{v} > 2\underline{v}$; thus we focus on cases where no pure-strategy Nash equilibrium exists. We first identify the support of non-dominated strategies based on (4) and (A-7) by undertaking a process of elimination of strictly dominated strategies. Just as in our characterization of the best responses, we assume that prices must be set in increments of $\delta > 0$.

1. $2\underline{v} < \overline{v} \le 3\underline{v}$: In the first iteration, firm $-i$ starts off with the full set of pricing strategies (i.e., $\hat{p}_{-i} \in [0, \infty)$). According to (4), $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v}]$ for any $p_i \in [0, \infty)$. In the next iteration, firm $-i$'s best response to $p_i \in [\overline{v} - \underline{v}, \overline{v} + \underline{v}]$ lies in the interval $[\overline{v} - 2\underline{v} - \delta, \overline{v} - \delta]$. In turn, firm i sets $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v} - \delta]$, to which firm $-i$ responds with $\hat{p}_{-i}(p_i) \in [\overline{v} - 2\underline{v} - \delta, \overline{v} - 2\delta]$. It is easy to show by induction that, after $t \ge 1$ such iterations, firm i sets its price within $[\overline{v} - \underline{v}, \overline{v} + \underline{v} - t\delta]$ and firm $-i$ sets

Figure EC.1 Firms' pricing best-response correspondences under PNC when $3\underline{v} \le \overline{v}$



Note. Firm i's best response is shown in black; firm $-i$'s best response is shown in gray, including the shaded rectangle. As before, $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$.

its price within $[\overline{v} - 2\underline{v} - \delta, \overline{v} - (t+1)\delta]$ as long as $\overline{v} + \underline{v} - t\delta \ge 2\overline{v} - 2\underline{v}$. At this point, the support of non-dominated strategies will stop shrinking and be equal to

$$\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, 2(\overline{v} - \underline{v})] \quad \text{and} \quad \hat{p}_{-i}(p_i) \in [\overline{v} - 2\underline{v} - \delta, 2\overline{v} - 3\underline{v} - 2\delta]. \tag{A-7}$$

2. $\overline{v} \ge 3\underline{v}$: In the first iteration, firm $-i$ starts off with the full set of pricing strategies (i.e., $\hat{p}_{-i} \in [0,\infty)$). By (A-7), $\hat{p}_i(p_{-i}) \in [(\overline{v}+\underline{v})/2, \overline{v}+\underline{v}]$. In the next iteration, firm $-i$ sets $\hat{p}_{-i}(p_i) \in [(\overline{v}-\underline{v})/2 - \delta, \overline{v} - \delta]$. In turn, firm i sets $\hat{p}_i(p_{-i}) \in [(\overline{v}+\underline{v})/2, \overline{v}+\underline{v}]$. So unlike the case where $2\underline{v} < \overline{v} \le 3\underline{v}$, the process converges after one iteration. At that point, the support of non-dominated strategies is equal to

$$\hat{p}_i(p_{-i}) \in \left[\frac{\overline{v}+\underline{v}}{2}, \overline{v}+\underline{v}\right] \quad \text{and} \quad \hat{p}_{-i}(p_i) \in \left[\frac{\overline{v}-\underline{v}}{2} - \delta, \overline{v} - \delta\right]. \tag{A-8}$$

Mixed-Strategy Equilibrium Characterization. Although there are no pure-strategy Nash equilibria when $2\underline{v} < \overline{v}$ for any $\delta > 0$, here we show that the following CDFs constitute a mixed-strategy Nash equilibrium when $\delta \to 0$:

$$F_i(p_i) = \begin{cases} \frac{p_i - a}{p_i + c - a} & \text{for } a \le p_i < b, \\ 1 & \text{for } p_i = b; \end{cases} \tag{A-9}$$

$$F_{-i}(p_{-i}) = 2 - \frac{b}{p_{-i} + a - c} \quad \text{for } c \le p_{-i} \le c + b - a. \tag{A-10}$$

Here

$$a \doteq \overline{v} - \underline{v}, \qquad b \doteq 2(\overline{v} - \underline{v}) - \delta, \qquad \text{and} \quad c \doteq \overline{v} - 2\underline{v} - \delta \qquad \text{when} \quad 2\underline{v} < \overline{v} < 3\underline{v}, \tag{A-11}$$

$$a \doteq \frac{\overline{v} + \underline{v}}{2}, \qquad b \doteq \overline{v} + \underline{v}, \qquad \text{and} \quad c \doteq \frac{\overline{v} - \underline{v}}{2} - \delta \qquad \text{when} \quad \overline{v} \geq 3\underline{v}. \tag{A-12}$$

For any $p_i \in [a,b]$, let $\Pi_i(p_i; F_{-i})$ denote firm i's profit when firm $-i$ employs the randomizing profile F_{-i} over $c \leq p_{-i} \leq c+b-a$. Similarly, for any $p_{-i} \in [c, c+b-a]$ we use $\Pi_{-i}(p_{-i}; F_{-i})$ to denote firm $-i$'s profit when firm i employs the randomizing profile $F_i(p_i)$ over $a \leq p_i \leq b$. The distributions (A-9) and (A-10) define a Nash equilibrium if $\Pi_i(p_i; F_{-i}) = \Pi_i(a; F_{-i})$ for all $p_i \in [a,b]$ and $\Pi_i(p_i; F_{-i}) \leq \Pi_i(a; F_{-i})$ for all $p_i \notin [a,b]$ *and* if $\Pi_{-i}(p_{-i}; F_{-i}) = \Pi_{-i}(c; F_{-i})$ for all $p_i \in [c, c+b-a]$ and $\Pi_{-i}(p_{-i}; F_{-i}) \leq \Pi_{-i}(c; F_{-i})$ for all $p_i \notin [c, c+b-a]$.

Firm i offers a surplus of $s_{ij3}(p_i) = \overline{v} + \underline{v} - p_i$ to any customer $j \in \{1,2\}$, whereas firm $-i$ offers a surplus of $s_{-i,12}(p_{-i}) = \underline{v} - p_{-i}$ to customer 1 and a surplus of $s_{-i,22}(p_{-i}) = \overline{v} - p_{-i}$ to customer 2. We establish this result by considering first the case where $2\underline{v} < \overline{v} < 3\underline{v}$ and then the case where $\overline{v} \geq 3\underline{v}$.

1. If $2\underline{v} < \overline{v} < 3\underline{v}$, then $s_{i13}(p_i) \geq 0$ for any $p_i \in [a,b]$ and $s_{i13}(p_i) \geq s_{-i,12}(p_{-i})$ for all $p_{-i} \in [c, c+b-a]$. Hence firm i captures customer 1 for any $p_i \in [a,b]$. Similarly, $s_{i23}(p_i) \geq 0$ for any $p_i \in [a,b]$ whereas $s_{i23}(p_i) \geq s_{-i,22}(p_{-i})$ if only if $p_{-i} \geq p_i - \underline{v}$. That is to say, firm i also captures customer 2 as long as $p_i \leq p_{-i} + \underline{v}$. It follows that, for any $p_i \in [a,b]$:

$$\begin{aligned}\Pi_i(p_i; F_{-i}) &= \lim_{\delta \to 0} \int_c^{c+b-a} \Pi_i(p_i; p_{-i})\, dF_{-i}(p_{-i}) = \lim_{\delta \to 0} \left(p_i \left(1 + \int_{p_i - \underline{v}}^{c+b-a} dF_{-i}(p_{-i}) \right) \right) \\ &= \lim_{\delta \to 0} \left(p_i (1 + F_{-i}(c+b-a) - F_{-i}(p_i - \underline{v})) \right) \\ &= \lim_{\delta \to 0} \left(p_i \left(\frac{2(\overline{v} - \underline{v}) - \delta}{p_i + \delta} \right) \right) = 2(\overline{v} - \underline{v}).\end{aligned} \tag{A-13}$$

Following our previously outlined iterative process of elimination of dominated strategies, we find that $\Pi_i(p_i; F_{-i}) < \Pi_i(a; F_{-i})$ for all $p_i < a$ or $p_i > b + \delta$. Finally, for any $p_i \in (b, b+\delta]$, firm i captures customer 1 whenever $p_i \leq p_{-i} + \overline{v}$ and captures customer 2 whenever $p_i \leq p_{-i} + \underline{v}$. So for any $p_i \in (b, b+\delta]$, we have

$$\begin{aligned}\Pi_i(p_i; F_{-i}) &= \lim_{\delta \to 0} \int_c^{c+b-a} \Pi_i(p_i; p_{-i})\, dF_{-i}(p_{-i}) = \lim_{\delta \to 0} \left(p_i \left(\int_{p_i - \overline{v}}^{c+b-a} dF_{-i}(p_{-i}) + \int_{p_i - \underline{v}}^{c+b-a} dF_{-i}(p_{-i}) \right) \right) \\ &= \lim_{\delta \to 0} \left(p_i (2F_{-i}(c+b-a) - F_{-i})(p_i - \overline{v} - F_{-i}(p_i - \underline{v})) \right) \\ &= \lim_{\delta \to 0} \left(p_i \left(\frac{2(\overline{v} - \underline{v}) - \delta}{p_i + \delta} + \frac{4(\overline{v} - \underline{v}) - 2p_i - 3\delta}{p_i - (\overline{v} - \underline{v}) + \delta} \right) \right) = 2(\overline{v} - \underline{v}) = F(a; F_{-i}).\end{aligned}$$

As a consequence, when $\delta \to 0$ we see that $\Pi_i(p_i;F_{-i}) = \Pi_i(a;F_{-i})$ for all $p_i \in [a,b]$ and $\Pi_i(p_i;F_{-i}) \leq \Pi_i(a;F_{-i})$ for all $p_i \notin [a,b]$.

Firm $-i$ captures customer 2 if firm i sets its price such that $p_i \in (p_{-i} + \underline{v}, b]$. Accordingly, for any $p_i \in [c, c+b-a]$ we have

$$\begin{aligned}
\Pi_{-i}(p_{-i};F_i) &= \lim_{\delta\to 0}\left(\int_a^b \Pi_{-i}(p_{-i};p_i)\,dF_i(p_i)\right) = \lim_{\delta\to 0}\left(p_{-i}\int_{p_{-i}+\underline{v}}^b dF_i(p_i)\right)\\
&= \lim_{\delta\to 0}\left(p_{-i}(F_i(b) - F_i(p_{-i}+\underline{v}))\right)\\
&= \lim_{\delta\to 0}\left(p_{-i}\left(1 - \frac{p_{-i} - (\overline{v} - 2\underline{v})}{p_{-i} - \delta}\right)\right) = \overline{v} - 2\underline{v}. \qquad \text{(A-14)}
\end{aligned}$$

Again following the iterative process of elimination of dominated strategies, we obtain $\Pi_{-i}(p_i;F_i) < \Pi_{-i}(c;F_i)$ for all $p_{-i} < c$ or $p_{-i} > c+b-a$. So as $\delta \to 0$ we have $\Pi_{-i}(p_{-i};F_{-i}) = \Pi_{-i}(c;F_{-i})$ for all $p_i \in [c, c+b-a]$ and $\Pi_{-i}(p_{-i};F_{-i}) \leq \Pi_{-i}(c;F_{-i})$ for all $p_i \notin [c, c+b-a]$.

2. If $\overline{v} \geq 3\underline{v}$, then $s_{i13}(p_i) \geq 0$ for any $p_i \in [a,b]$. Furthermore, $s_{i13}(p_i) \geq s_{-i,12}(p_{-i})$ if and only if $p_i - p_{-i} \leq \overline{v}$. We assume in the following that $\delta \leq (\overline{v} - 3\underline{v})/2$. Hence for any $p_i \in [a,b]$ and $p_{-i} \in [c, c+b-a]$ we have $p_i - p_{-i} \leq b - c = \overline{v} + \underline{v} - ((\overline{v} - \underline{v})/2 - \delta) \leq \overline{v}$. It follows that $s_{i13} \geq s_{-i,12}$ for all $p_i \in [a,b]$ and $p_{-i} \in [c, c+b-a]$; that is, firm i captures customer 1 for every $p_i \in [a,b]$. Similarly, $s_{i23}(p_i) \geq 0$ for any $p_i \in [a,b]$ whereas $s_{i23}(p_i) \geq s_{-i,22}(p_{-i})$ if only if $p_{-i} \geq p_i - \underline{v}$. We conclude that firm i captures customer 2 provided that $p_i \leq p_{-i} + \underline{v}$. So for any $p_i \in [a,b]$,

$$\begin{aligned}
\Pi_i(p_i;F_{-i}) &= \lim_{\delta\to 0}\int_c^{c+b-a} \Pi_i(p_i;p_{-i})\,dF_{-i}(p_{-i}) = \lim_{\delta\to 0}\left(p_i\left(1 + \int_{p_i-\underline{v}}^{c+b-a} dF_{-i}(p_{-i})\right)\right)\\
&= \lim_{\delta\to 0}\left(p_i(1 + F_{-i}(c+b-a) - F_{-i}(p_i - \underline{v}))\right)\\
&= \lim_{\delta\to 0}\left(p_i\left(\frac{\overline{v} + \underline{v}}{p_i + \delta}\right)\right) = \overline{v} + \underline{v}. \qquad \text{(A-15)}
\end{aligned}$$

Following the same iterative process of elimination of dominated strategies as before gives that $\Pi_i(p_i;F_{-i}) < \Pi_i(a;F_{-i})$ for all $p_i < a$ or $p_i > b$. Therefore, as δ approaches zero, $\Pi_i(p_i;F_{-i}) = \Pi_i(a;F_{-i})$ for all $p_i \in [a,b]$ and $\Pi_i(p_i;F_{-i}) \leq \Pi_i(a;F_{-i})$ for all $p_i \notin [a,b]$.

Firm $-i$ captures customer 2 if firm i sets its price such that $p_i \in (p_{-i} + \underline{v}, b]$. For any $p_i \in [c, c+b-a]$,

$$\begin{aligned}
\Pi_{-i}(p_{-i};F_i) &= \lim_{\delta\to 0}\left(\int_a^b \Pi_{-i}(p_{-i};p_i)\,dF_i(p_i)\right)\\
&= \lim_{\delta\to 0}\left(p_{-i}\int_{p_{-i}+\underline{v}}^b dF_i(p_i)\right) = \lim_{\delta\to 0}\left(p_{-i}(F_i(b) - F_i(p_{-i}+\underline{v}))\right)\\
&= \lim_{\delta\to 0}\left(p_{-i}\left(1 - \frac{p_{-i} - (\overline{v} - \underline{v})/2}{p_{-i} - \delta}\right)\right) = \frac{\overline{v} - \underline{v}}{2}. \qquad \text{(A-16)}
\end{aligned}$$

Again Following our iterative process of elimination of dominated strategies, we obtain that $\Pi_{-i}(p_i;F_i) < \Pi_{-i}(c;F_i)$ for all $p_{-i} < c$ or $p_{-i} > c+b-a$. So as $\delta \to 0$, we have $\Pi_{-i}(p_{-i};F_{-i}) = \Pi_{-i}(c;F_{-i})$ for all $p_i \in [c, c+b-a]$ and $\Pi_{-i}(p_{-i};F_{-i}) \le \Pi_{-i}(c;F_{-i})$ for all $p_i \notin [c, c+b-a]$. $\square$

Proof of Proposition 6. We focus on the case where $\mathbf{z}_i = (1,0,1)$ and $\mathbf{z}_{-i} = (1,0,0)$. By symmetry, a similar proof holds for the other case; we omit the details. According to Lemma A-1, $\hat{p}_{i1}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ij1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ and $\hat{p}_{-i,1}(\mathbf{z}_{-i}, \mathbf{z}_i) \sum_j x_{-i,j1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. As a result, the only profitable sale that occurs in equilibrium is through the bundle. We next consider three possible ranges for p_{i3} over which firm i maximizes its profit.

First, if $p_{i3} > \underline{v} + \overline{v}$ then firm i captures no customers.

Second, suppose that $\underline{v} < p_{i3} \le \underline{v} + \overline{v}$. In that case, if $p_{i1} > 0$ then, by (A-3), firm $-i$ can set its price to $\hat{p}_{-i,1} = \min\{p_{i1} - \delta, p_{i3} - \underline{v} - \delta\} > 0$ for some infinitesimal $\delta > 0$ and thereby capture at least customer 1 at a profit—that is, since customer 1's surplus from product 1 (which is equal to $\overline{v} - \hat{p}_{-i,1} \ge \overline{v} - p_{i3} + \underline{v} + \delta$) would then be larger than her surplus with the bundle, contradicting firm $-i$'s inability to make a profit in equilibrium. Hence $\hat{p}_{i1} = 0$ in equilibrium. In this case, firm i can capture customer 2 with the bundle only when $\underline{v} + \overline{v} - p_{i3} \ge \underline{v} - \hat{p}_{i1}$ (i.e., as long as $p_{i3} \le \overline{v}$. Under that condition, $\Pi_i(\hat{p}_{i1}, p_{i3}; p_{-i,1}, \mathbf{z}) = p_{i3}$. Therefore, $\hat{p}_{i3} = \overline{v}$, $\hat{p}_{i1} = 0$, and $\sum_j x_{ij3}(\hat{\mathbf{p}}, \mathbf{z}) = 1$.

Finally, suppose $p_{i3} \le \underline{v}$. Then, for any $p_{i1}, p_{-i,1} \ge 0$, firm i captures both customer groups; that is, $\Pi_i(p_{i1}, p_{i3}; p_{-i,1}, \mathbf{z}) = 2p_{i3}$. Hence firm i's profit is maximized when $\hat{p}_{i3} = \underline{v}$ for any $\hat{p}_{i1} \ge 0$ and $\hat{p}_{-i,1} \ge 0$. As a result, $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = 2\underline{v}$ and $\sum_j x_{ij3}(\hat{\mathbf{p}}, \mathbf{z}) = 2$.

Combining these results establishes that $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = \max\{2\underline{v}, \overline{v}\}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}, \mathbf{z}_i) = 0$. $\square$

Proof of Proposition 7. Here we address the case where $\mathbf{z}_i = (1,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$; symmetry ensures that an analogous proof holds for the other case. By Lemma A-3, $\sum_j x_{ij1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ and so the equilibrium characterization is identical to that given in Proposition 5. $\square$

Proof of Proposition 8. We focus on the case where $\mathbf{z}_i = (1,1,1)$ and $\mathbf{z}_{-i} = (1,0,0)$; as before, a similar proof holds (by symmetry) for the other case. We know from Lemma A-3 that $\sum_j x_{ij2}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. The rest of the proof is similar to the proof of Proposition 6. $\square$

EC.2.2.3. Perfectly Positively Correlated (PPC) Valuations

Proof of Proposition 9. Suppose $\mathbf{z}_i = (0,0,1)$ and $\mathbf{z}_{-i} = (1,0,0)$; the other case can be treated in a symmetric fashion. Suppose that firm $-i$ makes a positive profit on the sale of component 1, which implies that $p_{-i,1} > 0$. Under PPC, firm $-i$ captures both customers. In that case, firm i can increase its profit by setting its price to $\hat{p}_{i3}(p_{-i,1}) = p_{-i,1} + \underline{v}$ and thus stealing firm $-i$'s market share—a contradiction. Conversely, suppose that firm i makes a positive profit with $p_{i3} > \underline{v}$. Then firm $-i$ can improve its profit by setting its price $\hat{p}_{-i,1}(p_{i3}) = p_{i3} - \underline{v} - \delta > 0$ for some infinitesimal $\delta > 0$, capture both customers, and make positive profit, a contradiction. As a result, $\hat{p}_{-i,1} = 0$ and $\hat{p}_{i3} \leq \underline{v}$ in equilibrium. For any $p_{i3} \leq \underline{v}$, firm i captures both customers and its profits are maximized when $\hat{p}_{i3} = \underline{v}$. In any equilibrium, then, $\hat{p}_{i3} = \underline{v}$, which yields the given profits. □

Proof of Proposition 10. Here we direct our attention to the case in which $\mathbf{z}_i = (1,0,1)$ and $\mathbf{z}_{-i} = (1,0,0)$. By symmetry, a similar proof holds for the other case (we omit the details for brevity). According to Lemma A-1, $\hat{p}_{i1}(\mathbf{z}_i, \mathbf{z}_{-i}) \sum_j x_{ij1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ and $\hat{p}_{-i,1}(\mathbf{z}_{-i}, \mathbf{z}_i) \sum_j x_{-i,j1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. Hence the only profitable sale that occurs in equilibrium is through the bundle. Because customers have identical valuations of the bundle under PPC, they both purchase the bundle in equilibrium.

Suppose that firm i sells the bundle at a price $p_{i3} > \underline{v}$. Then, by (A-3), firm $-i$ can set its price to $\hat{p}_{-i,1} = p_{i3} - \underline{v} - \delta > 0$ for some infinitesimal $\delta > 0$—and thus capture both customers at a profit, since their surplus with component 1 (i.e., $\overline{v} - \hat{p}_{-i,1} = \overline{v} - p_{i3} + \underline{v} + \delta$) would then be larger than their surplus with the bundle; this outcome would contradict firm $-i$'s being unable to make a profit in equilibrium. Therefore, $\hat{p}_{i3} \leq \underline{v}$ in equilibrium.

For any $p_{i3} \leq \underline{v}$, firm i captures both customer groups. Hence firm i's profit is maximized when $\hat{p}_{i3} = \underline{v}$ for any $\hat{p}_{i1}, \hat{p}_{-i,1} \geq 0$. Accordingly, $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = 2\underline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}, \mathbf{z}_i) = 0$. □

Proof of Proposition 11. We focus on the case where $\mathbf{z}_i = (1,0,1)$ and $\mathbf{z}_{-i} = (0,1,0)$; symmetry ensures that a similar proof holds for the other case. By Lemma A-3, $\sum_j x_{ij1}(\hat{\mathbf{p}}, \mathbf{z}) = 0$ and so the equilibrium characterization is identical to that described in Proposition 9. □

Proof of Proposition 12. Here we address the case in which $\mathbf{z}_i = (1,1,1)$ and $\mathbf{z}_{-i} = (1,0,0)$. By Lemma A-3, $\sum_j x_{i2l}(\hat{\mathbf{p}}, \mathbf{z}) = 0$. The rest of the proof is similar to the proof of Proposition 10. □

EC.3. Construction of a Mixed Strategy in the Pricing Game

In this appendix, we construct a mixed-strategy Nash equilibrium when a pure-strategy Nash equilibrium does not exist. We study the game under PNC valuations where $\mathbf{z}_1 = (0,0,1)$ and $\mathbf{z}_2 = (0,1,0)$ as an example. Throughout the analysis, we fix $\delta > 0$ and consider the quantities a, b, and c as defined in Equations (A-11) and (A-12).

We consider successive refinement grids of the firms' action spaces; these spaces are anchored on the points a and b for firm 1 and on c and $c+b-a$ for firm 2. For any $m \in \mathbb{Z}$ $(m \geq 0)$, let $\mathcal{G}_m$ be the mth refinement grid:

$$\mathcal{G}_m \doteq \left\{a + \frac{t}{2^m}(b-a)\right\}_{t\in\mathbb{Z}} \times \left\{c + \frac{t}{2^m}(b-a)\right\}_{t\in\mathbb{Z}}. \tag{A-17}$$

By definition, $\mathcal{G}_m \subseteq \mathcal{G}_{m+1}$ for any $m \in \mathbb{Z}$. In other words: as m grows larger, the grid's previous points are preserved.

We first show that, for any discretized action space $\mathcal{G}_m$, there exists a unique mixed-strategy equilibrium.

Proposition A-1. *Under PNC valuations with $\overline{v} > 2\underline{v}$, let $\mathbf{z}_1 = (0,0,1)$ and $\mathbf{z}_2 = (0,1,0)$. Fix $m \in \mathbb{Z}$ $(m \geq 0)$ and $\delta > 0$ such that $\delta < (b-a)/2^m$, where a, b, and c are as defined in* (A-11) *and* (A-12). *Let $n \doteq 2^m$. On $\mathcal{G}_m$ as defined in* (A-17), *there is a unique mixed-strategy equilibrium under which firms put positive probability on $n+1$ price points. The equilibrium CDFs are as follows:*

$$F_1^{(n)}(p_{1t}) = \begin{cases} 0 & \text{for } t < 0, \\ \frac{(p_{1t}-a)n+b-a}{(p_{1t}+c-a)n+b-a} & \text{for } 0 \leq t \leq n-1, \\ 1 & \text{for } t \geq n; \end{cases} \tag{A-18}$$

$$F_2^{(n)}(p_{2t}) = \begin{cases} 0 & \text{for } t < 0, \\ 2 - \frac{b}{p_{2t}-c+a} & \text{for } 0 \leq t \leq n, \\ 1 & \text{for } t > n. \end{cases} \tag{A-19}$$

Here $p_{1t} \doteq a + t(b-a)/n$ and $p_{2t} = c + t(b-a)/n$, and the equilibrium profits are $\Pi_1(p_{1t}; \mathbf{z}, F_2^{(n)}) = b$ for all $t = 0, \ldots, n$ and $\Pi_2(p_{2t}; \mathbf{z}, F_1^{(n)}) = c$ for all $t = 0, \ldots, n$.

Proof. Let $f_1^{(n)}(p_{1t})$ and $f_2^{(n)}(p_{2t})$ be the firms' discrete probability mass functions. Firm 1 offers a surplus of $s_{1j3}(p_i) = \overline{v} + \underline{v} - p_{1t}$ to any customer $j \in \{1,2\}$, whereas firm 2 offers a surplus of $s_{212}(p_{2t}) = \underline{v} - p_{2t}$ to customer 1 and a surplus of $s_{222}(p_{2t}) = \overline{v} - p_{2t}$ to customer 2. We can use the process of eliminating strictly dominated strategies to show, much as in the proof of Proposition 5, that firm 1 will price only within the range $[a, b+\delta]$—which is equivalent to $[a,b]$ given that $\delta < (b-a)/n$ and that firm 2 will price only within $[c, c+a-b]$. Hence $f_1^{(n)}(p_{1t}) = 0$ for all $t<0$ or $t>n$ and $f_2^{(n)}(p_{2t}) = 0$ for all $t<0$ or $t>n$.

Similarly to the proof of Proposition 5, we can show that $s_{113}(p_{1t}) \geq 0$ and $s_{113}(p_{1t}) \geq s_{212}(p_{2\tau})$ for any $0 \leq t, \tau \leq n$; that is, firm 1 always captures customer 1. Moreover, $s_{123}(p_{1t}) \geq 0$ and $s_{222}(p_{2\tau}) \geq 0$ for any $0 \leq t, \tau \leq n$, whereas $s_{123}(p_{1t}) \geq s_{222}(p_{2\tau})$ if only if $p_{2\tau} \geq p_{1t} - \underline{v}$. Therefore, firm 1 captures customer 2 if and only if

$$p_{2\tau} \geq p_{1t} - \underline{v} \iff c + \tau\frac{b-a}{n} \geq a + t\frac{b-a}{n} - \underline{v} \qquad \iff \tau \geq t + \delta\frac{n}{b-a} \iff \tau \geq t+1.$$

Otherwise, customer 2 is captured by firm 2. So for $t = 0, \ldots, n$, we have

$$\Pi_1(p_{1t}; \mathbf{z}_1, \mathbf{z}_2, F_2^{(n)}) = p_{1t}\left(\sum_{\tau=t+1}^{n} f_2^{(n)}(p_{2\tau}) + 1\right),$$
$$\Pi_2(p_{2t}; \mathbf{z}_1, \mathbf{z}_2, F_1^{(n)}) = p_{2t}\left(\sum_{\tau=t}^{n} f_1^{(n)}(p_{1\tau})\right).$$

We next derive $f_1^{(n)}(p_{1t})$ and $f_2^{(n)}(p_{2t})$ by construction. Since we focus our attention on dense strategies, we must have $f_1^{(n)}(p_{1t}) > 0$ for all $t = 0, \ldots, n$ and $f_2^{(n)}(p_{2t}) > 0$ for all $t = 0, \ldots, n$. By the definition of equilibrium mixed strategies, the profit functions must be equal on the support of the mixed strategy. Hence $\Pi_1(p_{1t}) = \Pi_1(p_{1t-1})$ for any $t = 1, \ldots, n$; in addition, $\Pi_1(p_{1n}) = p_{1n} = b$ and so $\Pi_1(p_{1t}) = b$ for any $t = 0, \ldots, n$. Similarly, we have $\Pi_2(p_{2t}) = \Pi_2(p_{2t+1})$ for any $t = 0, \ldots, n-1$ and that $\Pi_2(p_{20}) = p_{20}\sum_{\tau=0}^{n} f_1^{(n)}(p_{1\tau}) = p_{20} \cdot 1 = c$. We conclude that $\Pi_2(p_{2t}) = c$ for any $t = 0, \ldots, n$.

Hence we may write, for any $t = 1, \ldots, n$,

$$\begin{aligned} f_2^{(n)}(p_{2t}) &= \sum_{\tau=t}^{n} f_2^{(n)}(p_{2\tau}) - \sum_{\tau=t+1}^{n} f_2^{(n)}(p_{2\tau}) \\ &= \frac{\Pi_1(p_{1,t-1})}{p_{1,t-1}} - \frac{\Pi_1(p_{1,t})}{p_{1,t}} = \frac{b(b-a)}{n(p_{2t} - c + a)(p_{2,t} - c + a - (b-a)/n)} \end{aligned}$$

and

$$f_2^{(n)}(p_{20}) = 1 - \sum_{\tau=1}^{n} f_2^{(n)}(p_{2\tau}) = 2 - \frac{b}{a}.$$

Similarly, for any $t = 0, \ldots, n-1$ we have

$$\begin{aligned} f_1^{(n)}(p_{1t}) &= \sum_{\tau=t}^{n} f_1^{(n)}(p_{1t}) - \sum_{\tau=t+1}^{n} f_1^{(n)}(p_{1t}) = \frac{\Pi_2(p_{2t})}{p_{2t}} - \frac{\Pi_2(p_{2,t+1})}{p_{2,t+1}} \\ &= \frac{c}{c + \frac{(t)(b-a)}{n}} - \frac{c}{c + \frac{(t+1)(b-a)}{n}} = \frac{c(b-a)}{n(p_{1t} + c - a)\left(p_{1t} + c - a + \frac{b-a}{n}\right)} \end{aligned}$$

and

$$f_1^{(n)}(p_{1n}) = 1 - \sum_{t=0}^{n-1} f_1^{(n)}(p_{1t}) = \frac{c}{c+b-a}.$$

Summing up the probability mass functions yields the proposed cumulative distribution functions. $\square$

Next we show that the proposed distributions (A-18) and (A-19) converge, as $\mathcal{G}_m$ becomes more refined, to the distributions characterized in (A-9) and (A-10) for continuous action spaces. We start by establishing @¿OK?—*else* please rewrite to clarify...that the limiting set of the discrete grids $\mathcal{G}_m$ is dense. Toward that end we define, for any $p \in [\alpha, \beta]$ and for any m, the least upper bound on p on grid $\mathcal{G}_m$:

$$p^m \doteq \alpha + (\beta - \alpha)\frac{t_m}{2^m};$$

here $t_m = \arg\min \left\{ t \in \mathbb{Z} \mid \alpha + (\beta-\alpha)\frac{t}{2^m} \geq p \right\}$. Then we show that this least upper bound can be arbitrarily close to p as the grids are more refined.

Lemma A-4. *For all $p \in [0,1]$ and all $\gamma > 0$, there exists an N such that*

$$0 \leq p^m - p < \gamma \quad \forall m \geq N.$$

Proof. Fix m. From the definition of p^m it follows that $p^m \geq p$ and $p^m - p \leq 1/2^m$. Therefore, $p^m - p < \gamma$ for all $m > -\log_2 \gamma$. $\square$

Lemma A-5. *For all $\varepsilon > 0$ and all $p \in [a,b]$: there exists an N such that, for all $m > N$, $|F_1^{(2^m)}(p^m) - F_1(p)| < \varepsilon$. Here $F_1(p)$ is defined as in* (A-9) *and $F_1^{(n)}$ is defined as in* (A-18).

Proof. For any ε we set $\gamma < \frac{c\varepsilon}{2}$ and $N > \log_2^{\frac{2(b-a)}{c\varepsilon}}$. We choose the largest N such that $0 \leq p^m - p < \gamma$ for all $m > N$, which is guaranteed to exist by Lemma A-4, and also $N > \log_2^{\frac{2(b-a)}{c\varepsilon}}$. Now, by (A-9) and (A-18), for all $m > N$ we have

$$\begin{aligned} |F_1^{(2^m)}(p^m) - F_1(p)| &= \left| \frac{(p^m - a)2^m + b - a}{(p^m + c - a)2^m + b - a} - \frac{p-a}{p+c-a} \right| = \left| \frac{c(p^m - p) + c\frac{b-a}{2^m}}{\left(p^m - a + c + \frac{b-a}{2^m}\right)(p+c-a)} \right| \\ &< \left| \frac{c(p^m - p)}{(p+c-a)^2} \right| + \left| \frac{c(b-a)}{2^m (p+c-a)^2} \right| < \frac{c\gamma}{c^2} + \frac{c(b-a)}{2^m c^2} = \frac{\gamma}{c} + \frac{b-a}{2^m c} < \varepsilon. \end{aligned}$$

In this expression, the first inequality reflects that $0 \leq p^m - p < \gamma$ and $0 < (p+c-a) < (p^m + c - a) + \frac{b-a}{2^m}$; the second inequality follows from $0 \leq p^m - p < \gamma$ and $0 < c \leq p + c - a$ for all $p \geq a$. $\square$

Lemma A-6. *Forall $\varepsilon > 0$ and all $p \in [c, c+b-a)$, there exists an N such that*

$$|F_2^{(2^m)}(p^m) - F_2^*(p)| < \varepsilon \quad \forall m > N;$$

here $F_2(p)$ and $F_1^{(n)}$ are as defined in (A-10) *and* (A-19), *respectively.*

Proof. For any ε we set $\gamma < a^2\varepsilon/b$. Let N be such that $0 \le p^m - p < \gamma$ for all $m > N$, which is guaranteed to exist by Lemma A-4. Then, for all $m > N$,

$$\begin{aligned} |F_2^{(2^m)}(p^m) - F_2(p)| &= \left|2 - \frac{b}{p^m - c + a} - \left(2 - \frac{b}{p - c + a}\right)\right| = \left|\frac{b(p^m - p)}{(p^m - c + a)(p - c + a)}\right| \\ &< \left|\frac{b\gamma}{(p - c + a)^2}\right| < \frac{b\gamma}{a^2} < \varepsilon. \end{aligned}$$

The first inequality follows from $0 \le p^m - p < \gamma$ and $0 < (p + a - c) \le (p^m + a - c)$, the second from $0 < a \le p - c + a$ for all $p \ge c$. $\square$