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Revenue Management with Repeated Customer Interactions

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Motivated by online advertising, we model and analyze a revenue management problem where a platform interacts with a set of customers over a number of periods. Unlike traditional network revenue management, which treats the interaction between platform and customers as one-shot, we consider stateful customers who can dynamically change their goodwill towards the platform depending on the quality of their past interactions. Customer goodwill further determines the amount of budget that they allocate to the platform in the future. These dynamics create a trade-off between the platform myopically maximizing short-term revenues, versus maximizing the long-term goodwill of its customers to collect higher future revenues. We identify a set of natural conditions under which myopic policies that ignore the budget dynamics are either optimal or admit parametric guarantees; such simple policies are particularly desirable since they do not require the platform to learn the parameters of each customer dynamic and only rely on data that is readily available to the platform. We also show that, if these conditions do not hold, myopic and finite look-ahead policies can perform arbitrarily poorly in this repeated setting. From an optimization perspective, this is one of a few instances where myopic policies are optimal or have parametric performance guarantees for a dynamic program with non-convex dynamics. We extend our model to the cases where supply varies over time and where customers may not interact with the platform in every period.

Keywords: Revenue management; Analysis of algorithms; Dynamic Programming.

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1. Introduction

In many revenue management domains, including online advertising, on-demand services, hospitality and airlines, a platform sells a limited volume of products to a portfolio of customers. The associated pricing and allocation decisions can be broadly grouped under the umbrella term network revenue management (NRM). NRM has been given significant attention in the operations literature (see Talluri and Van Ryzin 2005, Phillips 2005, Gallego and Topaloglu 2019, for surveys of the field) and deployed successfully in a wide range of commercial pricing and yield management systems.

Much of the NRM related research assumes that there is a single, short interval in which customers interact with the platform to potentially purchase a product. Hence, the NRM problem is formulated as a one-shot resource allocation problem. While this approach is well-understood and highly tractable, the one-shot treatment of the underlying system is not consistent with the concept of customer lifetime value. The idea of customer lifetime value has come to prevail in marketing (Gupta et al. 2006, Gupta and Lehmann 2008, Reinartz and Venkatesan 2008), where the key goal is to optimize decisions with respect to the aggregate value of the customer over his/her *entire* lifetime rather than during a one-shot interaction.

The customer lifetime value is relevant to numerous revenue management applications where customer interactions occur *repeatedly* over a long period of time, where the aggregate value of a profitable customer might dwarf the revenues that can be derived from a single interaction (providing the platform is able to retain its profitable customers). Thus, a decision that is seemingly optimal over the current period could in fact decrease customer goodwill towards the platform and reduce the value of their future business with the platform; such a decision could be highly suboptimal over the entire customer's lifetime. At the same time, new technologies enable platforms to track customer trajectories and gather data on how goodwill is affected by past allocation decisions; this has made it practically feasible to incorporate such goodwill effects into a revenue management system. Our work can thus be seen as an attempt to model and then operationalize customer lifetime value considerations.

At an operational level, customer goodwill is the lever through which customer lifetime value can be managed. The customer's goodwill drives future interactions between customer and platform in addition to being dependent on their past history of interactions. We model goodwill as a *state* that evolves dynamically over time, whereby repeated interactions between platform and customers become a sequence of distinct network revenue management problems, coupled by this dynamic state variable. We call this the *repeated network revenue management (RNRM)* problem, as opposed to the classic one-shot problem.

Repeated RM in online advertising. Although our model is generic, our structural assumptions are grounded in the online advertising domain. The customers are advertisers who run campaigns via online advertising platforms. A platform can repeatedly interact with a given customer over hundreds of different campaign cycles. The corresponding RNRM problem evolves in discrete time. There are T periods over which the platform interacts with a fixed pool of heterogeneous customers indexed by j. At the start of each period, the platform has a limited but known supply of heterogeneous products (impressions), indexed by i, to be allocated to the customers. Each customer j has a valuation v_{ij} and there is a pre-specified price p_{ij} for each product i that can be allocated to customer j. At the start of a single period t, each customer decides the budget $b_{j,t}$ they are willing to spend on their period t campaign. Subsequently, the platform decides how to allocate products to customers taking into account their budget constraints. In the next period, the campaign budgets are revised and the process repeats with a renewed supply of products.

A customer's state is specified by the budget $b_{j,t}$. In online advertising, where it is relatively frictionless for customers to switch between competing advertising platforms depending on their expectations of campaign fulfillment quality, the budget they commit to an advertising campaign serves as a proxy for their goodwill towards any specific platform. There is anecdotal evidence that advertisers change their budgets from one campaign to another as a function of the quality of their past experiences with the platform (Wilkens et al. 2017). Mathematically, we assume that customers update their budgets over time according to a deterministic state update function which increases or decreases the budget for the next period depending on the quality of the impression allocation received in the last campaign period. We denote the quality of the allocation that customer j receives in time t by $q_{j,t}$, which we formally define in Section 3. Informally, the budget of customer j at time $t+1$ is some function

$$b_{j,t+1} = \phi_j(b_{j,t}, q_{j,t}).$$

Given the budget dynamics specified by $\{\phi_j\}_{1\leq j\leq m}$, the task is to find a sequence of allocations that maximizes the total platform revenues over T periods.

Quantifying customer goodwill effects on the platform's bottom-line adds considerable complexity to the underlying network revenue management problem:

- Firstly, while one-shot network revenue management is very well understood and offers several prescriptions and policies, such as Talluri and Van Ryzin (1998), that are both theoretically near-optimal and practically tractable, one would not expect these to remain sound in a multi-period setting. In particular, when customer goodwill effects are large, such myopic policies which are inherently short-term looking, have the potential to leave significant revenue on the table in the aggregate.
- At the same time, more sophisticated policies aimed at capturing these effects could be impractical to implement, or even compute. One concern is the amount of customer information that more sophisticated policies would require, such as the shape of individual customer goodwill dynamics, or customer valuation information. To obtain this would involve either designing incentive-compatible mechanisms to solicit the information from customers themselves, or a procedure to estimate them from previously collected data at the customer level. While technologies such as web browser cookies allow platforms to match the identity of a customer over many time-differentiated interactions, this approach brings its own difficulties, including the

cost of tracking individual customers, and the increasing governmental scrutiny of what data is collected and how it is used. [1]

In light of these challenges, the central issue we address is *how to design policies for the RNRM problem that (i) admit revenue performance guarantees, and (ii) are similar, from an implementability standpoint, to existing one-shot network revenue management policies.*

1.1. Main Contributions

For the model broadly described above, our paper makes the following contributions:

Parametrized performance guarantees for the myopic policy. By 'myopic' we mean a policy which ignores customer dynamics and solves each individual NRM problem separately and without regard to its impact towards future periods (in other words, it treats the problem as one-shot). As alluded to in the above, such policies are desirable in terms of tractability and simplicity.

We identify sufficient regularity conditions on the RNRM problem structure such that the myopic policy is a parametric approximation of the optimum. Intuitively, the necessary conditions require similarity in the 'bang-per-buck' across all the products that a given customer desires: for the goods valued by customer j, all $(v_{ij} - p_{ij})/p_{ij} = v_{ij}/p_{ij} - 1$ bang-per-buck ratios lie within a constant range that is specific to that customer j. More precisely, there exists a constant $0 \leq \gamma < 1$ such that

$$\min_i \left\{ \frac{v_{ij}}{p_{ij}} \right\} - 1 \geq (1-\gamma) \cdot \left(\max_i \left\{ \frac{v_{ij}}{p_{ij}} \right\} - 1 \right),$$

where the minimum is taken over i such that $v_{ij} > 0$. In this case, we show a *tight bound* which guarantees that *the myopic policy garners at least a* $(1-\gamma)$ *fraction of what the optimal policy can achieve*, regardless of the horizon length, number of products and customers, etc. We emphasize that the platform does not need to know the exact ratio v_{ij}/p_{ij} for all items and customers for our result to hold. Also, through numerical experiments, we show that the performance of the myopic policy is often significantly higher than the $(1-\gamma)$ bound.

Sufficient conditions for the optimality of the myopic policy. We show that if $\gamma = 0$, i.e. if the bang-per-buck ratios are a constant specific to each customer, then our results imply that the myopic policy is guaranteed to be *optimal* for the RNRM problem. While this assumption restricts the generality of our model, it is consistent with contracts in the online advertising industry where advertisers treat a certain population with specific features as fully homogeneous. Moreover, these results hold for *heterogeneous* budget updates chosen from a general class of functions. For example, one customer may be very sensitive to the service quality received, while another could be entirely insensitive to it.

[1] An example of this are the European Union's recent General Data Protection Regulation (GDPR), governing the usage of algorithms that employ customer-level data.

We emphasize that both of these results are surprising, since they imply that the platform can ignore goodwill effects and apply the same prescriptions suggested by one-shot network revenue management models with a limited or no loss of optimality.

From an optimization perspective, our results on the optimality or parametric performance guarantees for the myopic policy contribute to the dynamic optimization literature and are, to the best of our knowledge, one of the few instances where one can prove such results for a dynamic program with non-convex dynamics. The technical insight that underlies our result is that, when our assumptions hold, the problem satisfies what one could call a "dynamic complementary slackness" property (cf. Proposition 4).

Hardness results. In contrast, we show that if we remove all the regularity assumptions on bang-per-buck ratios, then both the myopic as well as any finite look-ahead heuristics can accrue an *arbitrary* loss of performance with respect to the optimal policy in the RNRM problem. To prove this we develop a family of problem instances for which we can construct feasible policies which garner arbitrarily more revenues than myopic or look-ahead, while at the same time being lower bounds for the true optimum. These lower bound policies accrue revenues in a highly non-smooth fashion, purposefully depleting customer budgets over many periods only to set themselves up for a one-time, large increase in budgets (and accordingly, revenues) at the end of the time horizon. Our bang-per-buck regularity conditions stated above, which allow performance guarantees for the myopic policy, can be interpreted as ensuring that the per-period revenue function remains smooth enough over the course of the time horizon to invalidate this type of behavior.

Although, in the worst case, look-ahead policies can perform as poorly as myopic, their ability to account for customer dynamics allows them to outperform myopic on some problem instances, though at the cost of additional informational and computational burdens.

Existence of efficient policies that use limited customer data. A crucial advantage of the myopic policy is practicality. Its implementation only requires knowledge of the current budget configuration $\mathbf{b}_t$, but not of the budget update functions $\phi_j(\cdot,\cdot)$ nor of the customer valuation vectors $\mathbf{v}_j$. Consequently, even though the underlying customer-level dynamics can be complicated and heterogeneous, the algorithm itself is completely oblivious to their specification. Since, in practice, customer budget update functions and valuations would likely be estimated starting from historical data on customer attributes and past interactions, the simplicity of the myopic policy indirectly reduces the platform's need to track customer-level data.[2] In contrast, more sophisticated policies which track customer goodwill dynamics (for example, look-ahead policies) would presumably require estimation from data which may be difficult for the platform to acquire.

[2] We observe that while running our myopic policy does *not* require such data, estimating the parametric lower bound given by γ does require knowledge of valuations.

Extensions to handle uncertainty. We also provide lower bounds on the myopic policy for two important extensions of our model: (a) uncertain customer arrivals, where customers may not interact with the platform in some time periods and (b) uncertain supply, where the available supply of goods may vary from one time period to the next.

More generally, our model asks whether having more granular data on customer behavior, such as the history of customer interactions with the platform and information on how customers value these interactions, can lead to better decision-making. We identify conditions such that using policies which are completely agnostic to this finer grained data induces a limited loss from the perspective of the platform's long-term revenues. Moreover, we also show that in the absence of these conditions such policies can perform arbitrarily badly. We believe that the question of whether such simple and partially data-agnostic policies can perform well for complex systems is of broad interest to the operations management literature and that our work will inspire additional research on this topic.

2. Literature Review

Our work has links to several existing streams of literature. On the one hand, our problem is solvable by dynamic programming (DP), a method that has received intense attention in recent decades. On the other hand, our problem is a network revenue management or resource allocation problem familiar to the operations management community. Lastly, there is a burgeoning literature on customer behavior in systems where there are repeated interactions. Our literature review is therefore organized around these three streams.

Dynamic programming. Our problem is formulated as a shortest path dynamic program and could, in principle, be solved via dynamic programming techniques. Solving DPs via the standard Bellman recursion suffers from the well-known "curse of dimensionality", hence research on this topic focuses on either finding heuristics to arrive at good approximations of optimal policies, or on identifying certain structural properties that guarantee simple policies perform well.

Good heuristics for solving such problems are often designed via approximate dynamic programming (ADP), where the general approach is to construct an approximation architecture to the value function that is amenable to efficient computation. Among work on ADP methods, we point the reader to surveys by Bertsekas (1995) and Powell (2007), as well as to some recent papers on linear-optimization-based approaches to ADP, such as De Farias and Van Roy (2003, 2004), Desai et al. (2012). A particular sub-stream in the ADP literature is the idea of a weakly-coupled dynamic program, a special class of Markov Decision Processes (MDPs) which can be viewed as a collection of "easy" sub-problems linked together by a constraint on a single state variable. Hawkins (2003) and Adelman and Mersereau (2008) provide ADP-based heuristics for such problems. Bertsimas and Mišić (2016) explore this sub-problem structure but use a fluid based heuristic. Our problem has a similar weakly-coupled structure in the sense that each period is an individual sub-problem and the various periods are linked together by the vector of customer goodwills. However, the heuristics in this literature stream, such as in Adelman and Mersereau (2008), do not yield performance guarantees or computationally tractable approaches for our specific problem.[3] Instead, we focus on simple policies which can be shown to have guaranteed performance for our particular problem structure.

An alternative approach to solving DPs is to impose certain structural assumptions that guarantee the optimality of simple policies as hinted at above. Denardo and Rothblum (1983) and Sobel (1990a,b) examine DPs with affine structure and provide conditions for the optimality of myopic policies. More recently, Ning and Sobel (2018) characterize the class of decomposable affine MDPs. Similar to our model, these MDPs have continuous multidimensional endogenous states and actions. Assuming polyhedral properties of the decomposed sets of feasible actions and affine dynamics and rewards, Ning and Sobel (2018) show that decomposable affine MDPs have an affine value function and an extremal optimal policy, determined by solving a system of auxiliary equations. In contrast, the state transition in our model evolves in non-linear, non-convex fashion. To the best of our knowledge, ours are the first such results for non-linear systems.

Revenue management and online resource allocation. The classical literature on NRM and resource allocation problems considers a setting in which a platform optimizes the allocation of a finite inventory of resources to a pool of heterogeneous buyers. However, the bulk of this work models a stateless, one-shot interaction between the platform and the buyers, while we consider dynamic customers who can change their behavior from one period to the next. Another important difference is that when the supply of products to be allocated over the period is known ahead of time (or in other words, deterministic), the one-shot NRM problem can be solved to optimality via an integer program, or even a linear program, under the assumption that customer budgets are large compared to unit prices. Thus, most of the work within this research stream has focused on the online case, when the supply of products evolves in an uncertain fashion and the decision maker must sequentially make irrevocable allocation decisions as the products arrive. In contrast, our model focuses on the deterministic case, which becomes non-trivial to solve because of the statefulness of the customers.

For network revenue management, the fluid approximations of Gallego and van Ryzin (1997) and Talluri and Van Ryzin (1998) are of particular note. In the case where products arrive as a

[3] For example, the Lagrangian relaxation developed in Adelman and Mersereau (2008) can be shown to not be tight unless one uses time dependent dual variables.

point process with known rates, these yield tractable "bid-price" control policies based on linear programming. Within this stream, Reiman and Wang (2008) and Jasin and Kumar (2012) provide more refined approximations and heuristics. Several alternative supply uncertainty models which fluid approximations cannot handle have been considered. Such examples include adversarial arrivals (Karp et al. 1990, Mehta et al. 2005, Golrezaei et al. 2014), random order arrivals (Devanur and Hayes 2009, Agrawal and Devanur 2015) and non-stationary fluid arrivals (Ciocan and Farias 2012, Bateni et al. 2016). We emphasize Bateni et al. (2016), who focus on optimizing allocation fairness instead of platform revenues. While their problem is different from ours, the motivation is similar: if the interaction with customers is repeated, guaranteeing them fairness benefits the platform in the long run.

In the broader revenue management literature, the last decade has brought forth some significant results on pricing in the presence of strategic customers. In this type of work, one considers agents who may strategize when to purchase a product in anticipation of future discounts offered by the seller, as in Aviv and Pazgal (2008), Liu (2007), Borgs et al. (2014), Besbes and Lobel (2015), Chen and Farias (2015), Lobel et al. (2015). While also an attempt to understand how platform allocation and pricing policies affect customer behavior, this line of work focuses on how customers shift a purchase temporally; we do not model such effects and instead model how the platform can alter the level of customer goodwill and, ultimately, the long term profits derived from customers. Recently, Agrawal et al. (2018) investigate mechanisms for repeated auctions when an auctioneer interacts with a buyer of limited rationality, such as a finite look-ahead buyer; while elegant, their results only apply to a single buyer and unit auctions. Lastly, Chawla et al. (2016) consider a pricing problem where a customer repeatedly pays for using a good, gradually learning its valuation while reacting myopically to prices; they provide guarantees for simple "pay-per-play" mechanisms.

Customer behavior and operations interface. The lifetime value of a customer is a key metric evaluating the impact of a customer acquisition or incentive plan. As platform-based models increasingly permeate traditional service industries, there has been an interest among both practitioners and academics to incorporate this metric into a company's day-to-day operations.

A stream of papers closely related to ours focus on customers who change their goodwill towards a platform over time through an exponentially smoothed update function that weighs current and past experiences. Aflaki and Popescu (2014) examine a stylized model of a service provider interacting with independent customers who remember the quality of past service, and whose retention probability depends on the history of said service, and establish structural properties of the optimal service rate. Recently, Kanoria et al. (2018) examine a related model where a firm chooses how to exercise two different quality service modes to minimize customer churn. Adelman and Mersereau

(2013) model a supplier who must allocate a finite capacity of a single type of product to multiple customers, whose demand is modulated by past fill rates; they provide ADP-based heuristics for the supplier's allocation policy. Technically, our paper differs from these in that we consider an allocation problem where both customers and products are heterogeneous, which makes characterizing the optimal policy substantially harder; also, we consider a completely deterministic model and make no large market assumptions as in classical fluid approximations where uncertainty can help the analysis by essentially smoothing the problem. Additionally, Adelman and Mersereau (2013) consider the long-run average criterion in their objective function, while we explicitly focus on maximizing the platform's revenue in a finite horizon problem,[4] allowing us to model transient effects rather than just steady state behavior of the system. Lastly, L'Ecuyer et al. (2017) consider an sponsored search platform which optimizes search result rankings to trade-off between extracting instantaneous revenues from its users, and improving their user experience to improve future search engine traffic.

There are other interesting streams of work at the behavioral and operations interface, such as on loyalty programs and how they can be integrated into more traditional revenue management frameworks, see Chun et al. (2017), Chun and Ovchinnikov (2018), as well as work on pricing with reference effects, where customers interacting with a seller repeatedly exhibit anchoring behavior which depends on past prices in a similar manner to our own budget dynamics, such as in Popescu and Wu (2007), Nasiry and Popescu (2011), and Hu et al. (2016). The idea of repeated customer interaction has also been explored in assortment selection: Ferreira and Goh (2018) show that selectively revealing a selling season's product assortment can extract more customer purchases.

3. Model

We consider a discrete time model which carries over T distinct periods. The system is endowed with a persistent population of m heterogeneous customers. These customers interact with the same platform at every period t. In each period, the platform has a supply of n different product types to allocate to the pool of customers. We assume this supply is replenished at the start of each period to some deterministic level s_i for product i.

Valuations and prices. Customer j has a constant valuation v_{ij} for one unit of a product of type i, and we denote by $\mathbf{v}_j$ customer j's vector of valuations for products $1 \le i \le n$. The platform allocates product type i to customer j at a time-invariant unit price p_{ij}; we denote by $\mathbf{p}_j$ the vector of prices customer j pays per unit of products $1 \le i \le n$. We assume that $v_{ij} \ge p_{ij}$ for every i and j.

[4] Our results can be naturally extended to the infinite horizon discounted criterion.

Customer state. At period t, each customer j is endowed with a budget $b_{j,t}$ to purchase products in that period. To simplify the exposition, we assume $b_{j,t} \in [0,1]$ for each customer j and period t. We further denote by $\mathbf{b}_t$ the vector of customer budgets at period t.

Control. At period t, the platform chooses the allocation $\mathbf{x}_t \in [0,s_1]^m \times \ldots \times [0,s_n]^m$ of products to customers from the feasible set $\mathbf{X}(\mathbf{b}_t)$:

$$\sum_{i=1}^{n} p_{ij} x_{ij,t} \le b_{j,t}, \quad \forall 1 \le j \le m \tag{1}$$

$$\sum_{j=1}^{m} x_{ij,t} \le s_i, \quad \forall 1 \le i \le n. \tag{2}$$

$$x_{ij,t} \ge 0, \quad \forall 1 \le i \le n, 1 \le j \le m.$$

We define the vector $\mathbf{x}_{j,t} = (x_{1j,t}, \ldots, x_{nj,t})$ to refer to the vector of allocations of products to customer j in period t, and the feasible set $\mathbf{X}_j(b_{j,t})$ to refer to the projection of $\mathbf{X}(\mathbf{b}_t)$ to customer j's allocations. Note that we allow fractional allocations of products. In applications such as online ad allocation where the volume of products transacted is large, it could be argued that the optimality gap versus the best integral solution is small. Such an assumption allows us to simplify the problem without losing any essential insights.

Platform revenues and customer utilities. In period t and for a feasible allocation $\mathbf{x}_t$, the platform garners revenues

$$R(\mathbf{x}_t) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} x_{ij,t}.$$

We assume that customer j's utility function is linear in her allocation. Specifically,

$$U_j(b_{j,t}, \mathbf{x}_{j,t}) = \begin{cases} \sum_{i=1}^{n} (v_{ij} - p_{ij}) x_{ij,t} & \text{if } \mathbf{x}_{j,t} \in \mathbf{X}_j(b_{j,t}) \\ 0 & \text{if } \mathbf{x}_{j,t} \notin \mathbf{X}_j(b_{j,t}), \end{cases}$$

Additionally, for a given budget $b_{j,t}$, we denote the maximum possible utility that customer j could attain by

$$U_j^*(b_{j,t}) = \max_{\mathbf{y}_t \in \mathbf{X}_j(b_{j,t})} U_j(b_{j,t}, \mathbf{y}_t).$$

We emphasize that if $\mathbf{y}_t^*$ is an optimal solution to this problem, i.e. it is feasible and attains $U_j^*(b_{j,t})$, then $\sum_i p_{ij} y_{i,t}^* = \min\left(b_{j,t}, \sum_{i=1}^{n} p_{ij} s_i\right)$. Namely, when computing the maximum possible utility for customer j in period t, either the budget constraint becomes tight or all the supply that the customer is interested in is exhausted.

Budget state dynamics. Our model of budget dynamics formalizes the concept of customer goodwill which we introduced in Section 1. We allow budgets to evolve over time as a deterministic process $\{\mathbf{b}_t\}_{1 \le t \le T}$ specified by budget update functions $\phi_j(\cdot,\cdot)$ which are potentially heterogeneous across customers. We allow for a broad range of memoryless functions $\phi_j(\cdot,\cdot)$ that depend on (i) customer j's current budget and (ii) the "service quality" provided to customer j in the current period. We assume that the latter is a function of that customer's allocation in the current period $\mathbf{x}_{j,t}$ and her available budget $b_{j,t}$. Specifically, we define the service quality provided to customer j in period t as

$$q_j(b_{j,t}, \mathbf{x}_{j,t}) \triangleq \begin{cases} \dfrac{U_j(b_{j,t}, \mathbf{x}_{j,t})}{U_j^*(b_{j,t})}, & \text{if } b_{j,t} > 0 \\ 1, & \text{if } b_{j,t} = 0. \end{cases} \tag{3}$$

Intuitively, $q_j(b_{j,t}, \mathbf{x}_{j,t})$ is the ratio of the utility customer j garnered from her allocation $\mathbf{x}_{j,t}$ and the maximum utility she could have garnered at state $b_{j,t}$, equal to $U_j^*(b_{j,t})$. Clearly, the definition implies that the range of $q_j(\cdot, \cdot)$ is the interval $[0,1]$ for any $\mathbf{x}_{j,t} \in \mathbf{X}_j(b_{j,t})$.

We define the service quality q_j to be 1 when the budget level is 0; this is a corner case that can be avoided by having $b_{j,t} \in [b_j^{\min}, b_j^{\max}]$ for each customer j and period t, where $b_j^{\min} > 0$. Although this results in a more general formulation, it also requires additional notation to project the budget back and forth between $[b_j^{\min}, b_j^{\max}]$ and $[0,1]$. For the sake of clarity, we assume that $b_{j,t} \in [0,1]$ for each customer j and period t, although our results extend to this more general case.

As a result, the domain and range of the budget update function $\phi_j(b_{j,t}, q_j(b_{j,t}, \mathbf{x}_{j,t}))$ are $\phi_j : [0,1]^2 \to [0,1]$. To denote the updated budget state vector induced by the budget update $\phi_j(b_{j,t}, q_j(b_{j,t}, \mathbf{x}_{j,t}))$, we use the vector notation $\boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t, \mathbf{x}_t))$. Thus, budgets over two periods are linked by the equation

$$\mathbf{b}_{t+1} = \boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t, \mathbf{x}_t)).$$

We note several connections between our choice of service quality metric and the the broader literature. In supply chains, the fraction of the customer's demand that is served with the available inventory, or fill rate, is a commonly monitored metric of supplier performance. For example, Gaur and Park (2007) and Adelman and Mersereau (2013) study supplier systems where customers place orders for a homogeneous good, and use fill rates as a measure of service quality in a way that is similar to our approach. One can think of our service quality metric $q_j(b_{j,t}, \mathbf{x}_{j,t})$ as a natural analogue (in the online advertising context) to the fill rate defined in Adelman and Mersereau (2013). Specifically, $U_j^*(b_{j,t})$ can be interpreted as the (best-case) utility that the advertiser demands, whereas $U_j(b_{j,t}, \mathbf{x}_{j,t})$ can be interpreted as the utility that is actually served to the advertiser under the allocation $\mathbf{x}_t$.

The comparison between $U_j(b_{j,t}, \mathbf{x}_{j,t})$ and $U_j^*(b_{j,t})$ is also related to the concept of envy-freeness (Foley 1967). Namely, $U_j^*(b_{j,t})$ is envy-free in the sense that it achieves the highest utility for customer j compared to any other allocation; thus, $q_j(b_{j,t}, \mathbf{x}_{j,t})$ can be interpreted as measuring how close $\mathbf{x}_{j,t}$ is to an envy-free allocation, in percentage terms.

A last remark is that in crafting such a model of customer goodwill one could ostensibly define a different service quality metric than the one used here. We believe ours is the right choice for a few reasons:

- First, it satisfies the condition that the metric $q_j(\cdot)$ should be increasing in the utility derived over the set of feasible allocations that customer j can receive at their current budget level.
- Second, our quality metric is scale-free in the sense that, if we change the scale of $\mathbf{v}$ and $\mathbf{p}$ by multiplying by a common constant (i.e., as if we converted the values of $\mathbf{v}$ and $\mathbf{p}$ into a different currency), we obtain the same quality as before; requiring scale-freeness invalidates other candidates, such as un-normalized metrics like $U_j(b_{j,t}, \mathbf{x}_{j,t})$. It can be shown that such an alternative budget update can suffer from ill-conditioning.[5]

Special Budget Update Case: Exponential Smoothing. While our results require some minimal structure on the budget update function of each customer $\phi_j(\cdot,\cdot)$, made precise in Assumption 1 in Section 4, our analysis accommodates a broad class of possible update functions. A natural example is an exponentially smoothed function of the form

$$\phi_j(b_j, q_j) = \alpha_j \cdot b_j + (1-\alpha_j)\cdot q_j, \tag{4}$$

for any $(b_j, q_j) \in [0,1]^2$ and some $\alpha_j \in [0,1]$. Note that this budget update function maintains a convex combination of the current budget level and service quality.

The exponentially smoothed update of the type (4) has been frequently considered in the literature on customer behavior as a model for customer state dynamics. For example, Gaur and Park (2007), Adelman and Mersereau (2013) and Aflaki and Popescu (2014) use exponential smoothing as an update function for customer goodwill evolution, while Nasiry and Popescu (2011) and Hu et al. (2016) use it for customer reference prices.

Dynamic programming formulation. Having stated the primitives of our model, we are now ready to formulate the platform's optimization, which is to find a sequence of allocation policies $\{\mathbf{x}_t\}_{1\le t\le T}$ that solve the following problem:

$$\begin{aligned} J_T^*(\mathbf{b}_1) = \max_{\mathbf{x}_1,\dots,\mathbf{x}_T} \; & \sum_{t=1}^{T} R(\mathbf{x}_t) \\ \text{s.t. } & \mathbf{x}_t \in \mathbf{X}(\mathbf{b}_t), \quad \forall t \\ & \mathbf{b}_{t+1} = \boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t,\mathbf{x}_t)), \quad \forall t. \end{aligned} \tag{5}$$

We note that the cost-to-go is indexed by the problem's remaining horizon length. Problem (5) is a deterministic dynamic program which, for any initial budget $\mathbf{b}$ and feasible allocation $\mathbf{x}$, can be solved by a Bellman recursion of the form:

$$J_\tau^*(\mathbf{b}) \triangleq \max_{\mathbf{x}\in\mathbf{X}(\mathbf{b})} \{R(\mathbf{x}) + J_{\tau-1}^*(\boldsymbol{\phi}(\mathbf{b},\mathbf{q}(\mathbf{b},\mathbf{x})))\}, \quad \forall\, 1\le\tau\le T, \tag{6}$$

[5] An example of ill-conditioning in the case of an un-normalized update is that a customer's budget can decrease to zero even though they keep receiving the maximum possible utility.

with the boundary condition $J_0^*(\mathbf{b}) = 0$.

Moreover, while we present the problem in a finite horizon setting, we emphasize that our results can be extended to a discounted infinite-horizon version of the problem.

Finally, for any policy π, specifying a sequence of feasible allocations $\mathbf{x}_1^\pi, \ldots, \mathbf{x}_T^\pi$, we define

$$J_T^\pi(\mathbf{b_1}) \triangleq \sum_{t=1}^{T} R(\mathbf{x}_t^\pi).$$

In particular, we now define the natural myopic policy for problem (5).

The myopic policy. As its name suggests, the myopic policy simply maximizes the platform's short-term revenue in each period, regardless of the customers' budget dynamics between periods. Thus in each period $t \in \{1, \ldots, T\}$, and given the available budgets $\mathbf{b}_t$, the myopic policy solves the following linear program:

$$\begin{aligned} \mathrm{MY}(\mathbf{b}_t) = \max_{\mathbf{x}_t} \; & R(\mathbf{x}_t) \\ \text{s.t. } & \mathbf{x}_t \in \mathbf{X}(\mathbf{b}_t). \end{aligned} \tag{7}$$

Throughout our paper, the myopic policy is denoted by the superscript MY, i.e. $\mathbf{x}_t^{MY} \in \mathbf{X}(\mathbf{b}_t)$ and $R(\mathbf{x}_t^{MY}) = \mathrm{MY}(\mathbf{b}_t)$. The myopic policy follows the natural budget update

$$\mathbf{b}_{t+1} = \boldsymbol{\phi}\left(\mathbf{b}_t, \mathbf{q}\left(\mathbf{b}_t, \mathbf{x}_t^{MY}\right)\right).$$

3.1. Model Discussion

The Bellman equation (6) emphasizes that our model captures the following trade-off: in each period the platform must find a balance between myopically maximizing its short-term revenue in that period, and providing a service quality to its customers that maximizes the revenue that can be garnered in the remaining horizon. Addressing this trade-off can be difficult due to the following model features:

1. The potential heterogeneity in the bang-per-buck $v_{ij}/p_{ij} - 1$ that different items i provide for each customer j and the revenue that the platform collects for their allocation p_{ij}, can create a conflict between the allocations the customers and the platform would prefer. For example, replacing a customer's preferred item by an alternative that garners more revenue in the current period can, in principle, have dramatically different effects on customer utility and, consequently, their future budget trajectories.
2. Customers can additionally be heterogeneous in their budget update functions, and in particular in the extent of their reaction to the service quality they receive, which is a function of their utility (cf. equation (3)). Some customers may be highly sensitive to receiving low service quality and drastically reduce their budget in the next period, e.g. the exponential

smoothing budget update function in equation (4) with $\alpha_j = 0$, while others may have a much more stable budget update function, e.g. the exponential smoothing budget update function in equation (4) with $\alpha_j = 1$.

Our main results in Section 4 will address the relative importance of these challenges as they pertain to the platform's ability to maximize its revenue using relatively simple policies.

3.2. Value Function Behavior

Although our problem is a shortest path DP, it is quite challenging to solve, both from a computational, as well as a structural perspective. In this section, we provide preliminary evidence of this by exhibiting the ill behavior of the value function of problem (5), as defined in equation (6). Specifically, in Proposition 1 below we show that the value function can be decreasing in customer budget levels, as well as not being quasi-concave nor quasi-convex.

First, we emphasize that in equation (6) computing J_1 requires solving a linear program, while computing J_t for $t \in \{2, \ldots, T\}$ may require solving a non-convex optimization problem due to the constraint $\mathbf{b}_{t+1} = \boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t, \mathbf{x}_t))$. In particular, this is already the case when computing J_3 even under the simple exponential smoothing budget update equation (4). Specifically, since

$$\begin{aligned} b_{j,3} &= \alpha_j b_{j,2} + (1-\alpha_j) \frac{U_j(b_{j,2}, \mathbf{x}_{j,2})}{U_j^*(b_{j,2})} \\ &= \alpha_j^2 b_{j,1} + \alpha_j (1-\alpha_j) \frac{U_j(b_{j,1}, \mathbf{x}_{j,1})}{U_j^*(b_{j,1})} + (1-\alpha_j)^2 \frac{U_j(b_{j,2}, \mathbf{x}_{j,2})}{U_j^*\left(\alpha_j b_{j,1} + (1-\alpha_j) \frac{U_j(b_{j,1}, \mathbf{x}_{j,1})}{U_j^*(b_{j,1})}\right)}, \end{aligned}$$

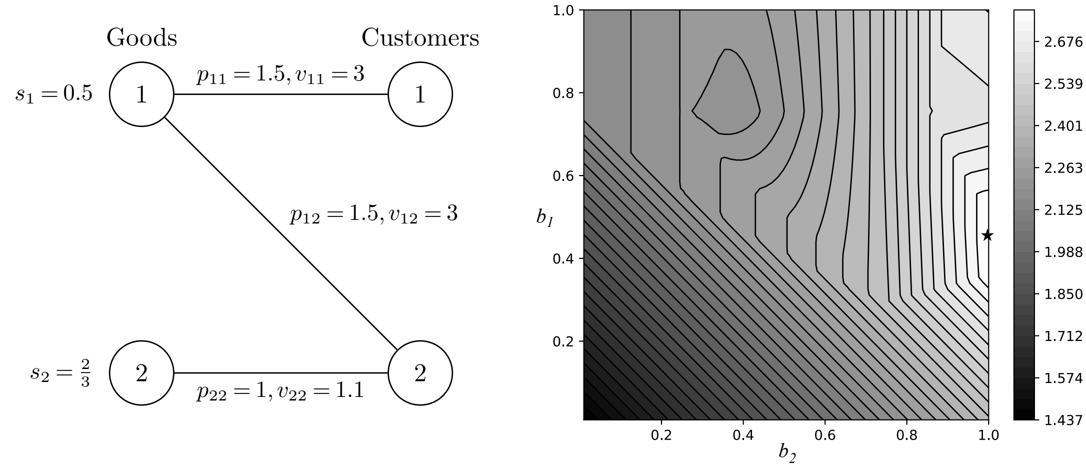
J_3 is non-convex in the first period allocation $\mathbf{x}_{j,1}$, see Proposition 1 for a numerical example. Thus, even though our problem is a shortest-path DP, it is computationally hard to find its optimum even assuming that the platform has full information about customer characteristics.

One question that arises with respect to our model is whether having customers with larger initial budgets is always beneficial for the platform. Another is whether there exist modified prices $\tilde{\mathbf{p}}$ that can incorporate the consumer's valuations into the platform's objective function such that acting myopically with respect to the modified prices $\tilde{\mathbf{p}}$ attains the optimal revenue for the platform. Specifically, it may be tempting to hope that there exist adjusted prices $\tilde{\mathbf{p}}$ which can replace the actual prices $\mathbf{p}$ in the myopic linear program, such that the policy obtained by solving

$$\begin{aligned} \max_{\mathbf{x}_t} & \sum_i \sum_j \tilde{p}_{ij} x_{ij,t} \\ \text{s.t. } & \mathbf{x}_t \in \mathbf{X}(\mathbf{b}_t), \end{aligned} \tag{8}$$

at each period t is optimal. Proposition 1, stated below, answers both questions in the negative by providing an instance of problem (5) that does not satisfy either of these statements.

Proposition 1. *There exist instances of problem* (5) *such that*



(a) Model parameters. (b) Contour plot of $J_2(\mathbf{b})$.

Figure 1 Instance that showcases problem ill behavior in Proposition 1.
Budget updates are given by equation (4). We assume $\alpha_1 = \alpha_2 = 0.2$. The initial budget that maximizes J_2 is (0.45, 1.0) and is highlighted is in the picture with a star.

(i) Increasing the customer's initial budgets decreases the platform's optimal revenue.

(ii) The revenue-to-go function is neither quasi-concave nor quasi-convex.

(iii) For any modified prices $\tilde{\mathbf{p}} \neq \mathbf{0}$, the myopic policy with respect to $\tilde{\mathbf{p}}$ is guaranteed to be strictly sub-optimal.

Proof. Consider the instance of problem (5) defined in Figure 1. It consists of two products and two customers, where the customers update their budgets according to equation (4). Moreover, Figure 1b displays the contour plots of $J_2(\mathbf{b})$, which satisfy the first two statements in the lemma.

For the last statement in the lemma, it can be verified that for $T = 3$ the unique optimal allocation of products can be such that all the supply and budget constraints are non-binding. For instance, if the initial budgets are $\mathbf{b} = (0.15, 0.3)$, the unique optimal allocation in the first period is $x^*_{11,1} = 0.063$, $x^*_{12,1} = 0.01$, and $x^*_{22,1} = 0$, and it is optimal not to spend all of the customers' budgets even if there is supply available. Note that since the unique optimal solution is in the interior of the polyhedron $\mathbf{X}(\mathbf{b}_1)$, then the myopic policy will be sub-optimal for any linear objective, i.e. for any modified prices $\tilde{\mathbf{p}} \neq \mathbf{0}$. $\square$

In short, in the instance in Figure 1 it is optimal for the platform to *withhold feasible allocations*; even if supply and demand are available, it may be optimal not to match them. The intuition behind this observation comes from the temporal dynamics captured in problem (5) and the definition of the service quality in equation (3). Specifically, it may be counterproductive to let customers' budgets grow too large when the system is supply constrained, since it may then be impossible to satisfy all their demands simultaneously. Moreover, this induced low service quality may lead to disappointed customers significantly reducing their budget in future interactions with the platform.

The complexity of this problem is compounded by the heterogeneity in the customers' budget update functions.

Importantly, Proposition 1(iii) states that, in general, the optimal policy for the RNRM problem cannot be obtained by solving any (modified) linear program at every stage as in equation (8). The unique interior-point solution from the proof of Proposition 1 cannot be the solution of an LP since it would never leave supply and demand unmatched. This shows that no linear-programming based approach can be optimal for this instance. We emphasize that the parameters of the instance in Figure 1 are reasonable for the model and do not correspond to a corner case.

4. Parametric Guarantees for the Myopic Policy

As discussed in the introduction, the desired features of a good heuristic policy for our problem are computational simplicity and as little use of customer-specific data as possible, since this data may be difficult to acquire in practice. However, it seems unlikely that such policies would perform well in the full generality of our setting. If this were the case, it would imply that, across all instances permitted by our model, customer dynamics can be ignored no matter how drastically they can alter customer budget trajectories, suggesting that our model is degenerate. Thus, we focus on injecting realistic regularity assumptions into the model, which will allow us to show performance guarantees for simple policies.

In this section, we show that the myopic policy is, in fact, a good heuristic policy under such relatively unrestrictive regularity assumptions on the problem structure. Specifically, we derive parametric worst-case performance guarantees for the myopic policy versus the optimum of problem (5), where the parametrization is in terms of the heterogeneity in the bang-per-buck of each item that a customer is interested in.

In our analysis, we make use of the budget trajectory induced by providing a fixed level of service quality, $q \in [0,1]$, to a customer in each period. We thus use for convenience a shorthand notation for the composition of the budget update function with itself under a fixed service quality.

Definition 1. *For each customer j and fixed service quality $q \in [0,1]$, let ϕ_j^t be defined as*

$$\phi_j^1(b,q) \triangleq \phi_j(b,q),$$
$$\phi_j^t(b,q) \triangleq \phi_j(\phi_j^{t-1}(b,q),q), \quad \forall t > 1.$$

To denote the updated budget state vector induced by the budget update $\phi_j^t(b,q)$, we use the vector notation $\boldsymbol{\phi}^t(\mathbf{b},\mathbf{q})$.

In order to derive a worst-case performance guarantee for the myopic policy, we first introduce two assumptions on the dynamics of customer budgets:

Assumption 1. *For any customer j, the budget update function $\phi_j(b,q)$ is such that:*

(i) It is non-decreasing in each component. Namely,

(a) $\phi_j(b_h, q) \geq \phi_j(b_l, q)$ *for each* $q \in [0,1]$ $b_h, b_l \in [0,1]$, $b_h \geq b_l$

(b) $\phi_j(b, q_h) \geq \phi_j(b, q_l)$ *for each* $b \in [0,1]$ $q_h, q_l \in [0,1]$, $q_h \geq q_l$.

(ii) For any $(b,q) \in [0,1]^2$, $\phi_j(b,q) \geq \min\{b,q\}$.

(iii) For any $b \in [0,1]$ *and* $t \in \{1,2,\ldots\}$, $\phi_j^t(b,(1-\vartheta)) \geq (1-\vartheta)\cdot\phi_j^t(b,1)$, *for any* $\vartheta \in [0,1]$.

Assumption 1(i) imposes the natural condition that, for each customer j, having a larger budget or receiving a higher service quality cannot lead to a smaller budget in the next state. In particular, Assumption 1(i) implies that $\phi_j^{t-1}(b_j, 1)$ is an upper bound on the budget state of customer j in period t, when starting from the initial budget b_j.

Assumption 1(ii) imposes some smoothness on the customer budget update. Specifically, the current budget can be interpreted as a summary statistic of the history of past service quality provided to the customer. Then, Assumption 1(ii) requires that the updated summary statistic cannot be lower than both its initial value and the new observation that it is being updated with. Any averaging rule will satisfy this assumption, e.g. the exponential smoothing budget update discussed in Section 3.

Finally, Assumption 1(iii) states that, for each customer, the budget induced by providing a consistent service quality of $(1-\vartheta)$ is no worse than scaling by $(1-\vartheta)$ the budget induced by consistently providing perfect service quality (i.e. $q = 1$). Note that the exponential smoothing budget update in equation (4) satisfies Assumption 1(iii). To build additional insight, we give examples of budget update functions which violate these assumptions in Appendix EC.1 of the e-companion.

Importantly, Assumption 1(i) allows us to derive the following relaxation of the platform's problem which will be useful in the proof of this section's main result: find a sequence of allocations $\{\mathbf{x}_t\}_{1\leq t\leq T}$ that maximizes the total revenue collected by the platform, assuming the largest possible budget update in each period t starting from the initial budgets $\mathbf{b}_1$, i.e. assuming $b_{j,t} = \phi_j^{t-1}(b_{j,1}, 1)$ for each customer j and period $t \in \{2,\ldots,T\}$. This natural relaxation can be cast in terms of the following linear program:

$$
\begin{aligned}
J_T^{\text{relax}}(\mathbf{b}_1) = \max_{\mathbf{x}_1,\ldots,\mathbf{x}_T} \; & \sum_{t=1}^{T} R(\mathbf{x}_t) \\
\text{s.t. } & \mathbf{x}_t \in \mathbf{X}\left(\boldsymbol{\phi}^{t-1}(\mathbf{b}_1, \mathbf{1})\right), \quad \forall t.
\end{aligned} \tag{9}
$$

We emphasize that $J_T^{\text{relax}}(\mathbf{b}_1) = \text{MY}(\mathbf{b}_1) + \sum_{t=2}^{T} \text{MY}(\boldsymbol{\phi}^{t-1}(\mathbf{b}_1, \mathbf{1}))$; in other words, it is an optimistic revenue upper bound which assumes all customer goodwill can be maximized at all times. The following proposition formalizes the relationship between the optimal objective value of problems (5) and (9). Its proof is presented in Appendix EC.2 in the e-companion to this paper.

Proposition 2. *Under Assumption 1(i), $J_T^{relax}(\mathbf{b}_1) \geq J_T^*(\mathbf{b}_1)$ for any horizon $T \geq 1$ and initial budget state $\mathbf{b}_1$.*

Note that Assumption 1 is *not sufficient* to ensure good performance of the myopic policy. In fact, the instance in the proof of Proposition 1, where the myopic policy is sub-optimal, satisfies this assumption.

We now move on to discussing our second assumption, which refers to the products' bang-per-buck ratios for each customer. This assumption yields the parametrization of our performance bound. It is motivated by the intuition that, in practice, while there may be significant heterogeneity in the products that a given customer desires, one would expect that the bang-per-buck

$$\frac{v_{ij} - p_{ij}}{p_{ij}} = \frac{v_{ij}}{p_{ij}} - 1$$

among a customer's desired goods to be similar. We state this precisely below:

Assumption 2. *We assume that each customer j is endowed with a characteristic set $A_j \subseteq [n]$ of products such that*

(i) $\min_{i \in A_j} \left\{ \frac{v_{ij}}{p_{ij}} \right\} - 1 \geq (1-\gamma) \cdot \left(\max_{i \in A_j} \left\{ \frac{v_{ij}}{p_{ij}} \right\} - 1 \right)$ *for some $\gamma \in [0,1]$,*

(ii) For all $i \notin A_j, v_{ij} = p_{ij} = 0$.

Assumption 2 imposes some smoothness on the customers' preferences, which propagate to the customers' budget update through the service quality they perceive. Eventually, since the customer budgets provide an upper bound on the revenues the platform can collect, Assumption 2 rules out extremely non-smooth behavior of the platform's revenues.

Although Assumptions 1 and 2 impose structure on the general RNRM problem, a priori it is not clear this should help bound the performance of the myopic policy. Specifically,

- Even in the case that $\gamma = 0$, namely when for each customer the bang-per-buck ratio is the same across items, supply scarcity might still make it optimal to ration this supply to customers with different potentials for future revenues in a non-myopic fashion.
- Although restricting heterogeneity via Assumption 2 could, in principle, restrict the amount of sub-optimality myopic accrues over *one* period, it is not clear why this sub-optimality would not compound over time, eventually leading to a large gap versus the optimal policy.
- Moreover, the structure on the customers' budget update functions imposed by Assumption 1 still allows for high heterogeneity, which remains as an important challenge that the platform needs to address when making its allocation decisions.

The following section shows that, surprisingly, *Assumptions 1 and 2 are sufficient for the myopic policy to be $(1-\gamma)$-optimal in the RNRM problem.*

4.1. A Parametric Worst-Case Guarantee for the Myopic Policy

Our main result shows that simple, myopic policies admit parametric guarantees under Assumptions 1 and 2, as explained below.

Theorem 1. *For any horizon T and initial budget state $\mathbf{b}_1$, let $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ be the myopic policy defined in Section 3.*

Then, under Assumptions 1 and 2, $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ is $(1-\gamma)$-optimal for problem (5), i.e.

$$J_T^{MY}(\mathbf{b}_1) \ge (1-\gamma)\cdot J_T^*(\mathbf{b}_1).$$

The proof of this result, which is presented in Appendix EC.2 in the e-companion to this paper, depends on auxiliary results which are presented in detail in the next subsection.

Theorem 1 shows that the level of heterogeneity in the bang-per-buck that each customer derives from the different products she is interested in, measured by the parameter $(1-\gamma)$ in Assumption 2, directly defines a worst-case performance guarantee for the myopic policy in maximizing the revenue collected by the platform. We emphasize that the $(1-\gamma)$ guarantee is independent of the length of the horizon T, or the number of customers m and products n. In other words, the myopic policy does not "compound" sub-optimality, with the $(1-\gamma)$ gap remaining invariant at every stage of the time horizon.

Moreover, the myopic policy has additional features that make it very attractive in practice. First, it is easy to compute since it only requires solving a linear program in each period. In fact, our result shows that continuing to use current technology built around the deterministic LP-based approaches that are common in the one-shot NRM literature is approximately optimal. In contrast, computing the optimal policy, or even a heuristic like look-ahead policies (see Section 5), may require solving a dynamic program where the revenue-to-go function is not quasi-concave nor quasi-convex, and it cannot be solved to optimality by a linear programmming based heuristic (cf. Proposition 1 and Figure 1).

Second, the myopic policy only requires knowledge of the prices, the supply of products, and the budgets available from each customer in the current period, but is agnostic to the customers' budget dynamics. In contrast, computing the optimal policy additionally requires knowledge of the customer valuations for each product and the customers' budget update functions. Although for simplicity we assumed a full-information setup where this information is available to the platform, in practice this may not be the case. In a private information setting, the platform may need to design a truthful mechanism or alternatively face the consequences of misspecified valuations.

Returning to the instance defined in Figure 1 with a horizon of $T=3$ periods, which was our example of bad behavior of the value function in Section 3.2, we computed the actual worst-case performance of the myopic policy by full enumeration. This is attained for an initial budget of 0.01

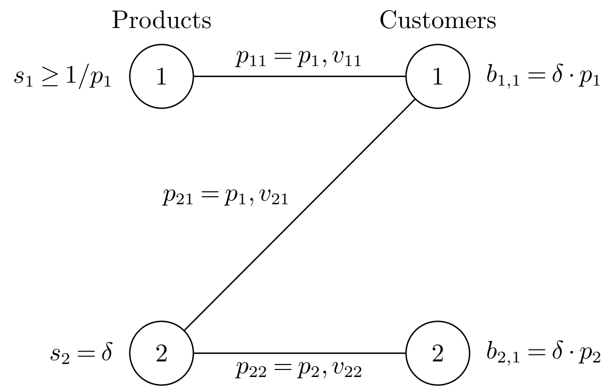


Figure 2 Instance that achieves the bound in Theorem 1. We assume δ is small and $v_{11} \leq v_{21}$.

for both customers, where the myopic policy collects about 66% of the optimal revenue. We note that in the instance in Figure 1 we have $\gamma = 0.9$.

The performance of the myopic policy in the instance in Figure 1 may suggest that the $(1-\gamma)$ bound provided in Theorem 1 is loose. This is, however, not the case and the bound in Theorem 1 is tight in γ. We show this fact by introducing an instance that achieves the bound.

Example 1. *Consider the instance with two products and two customers depicted in Figure 2 and assume $T = 2$. In this instance, there is ample supply of the first product ($s_1 \geq 1/p_1$), while there is limited supply of the second product ($s_2 = \delta$). The first customer desires both products, pays $p_{11} = p_{21} = p_1$ for each product, and values the second product more than the first ($v_{11} > v_{21}$). Specifically, $\frac{v_{11}/p_1 - 1}{v_{21}/p_1 - 1} = \frac{v_{11} - p_1}{v_{21} - p_1} = 1 - \gamma$. The second customer only desires the second product at a price $p_{22} = p_2$. We assume small initial budgets, $b_{1,1} = \delta \cdot p_1$ and $b_{2,1} = \delta \cdot p_2$, for some parameter $\delta > 0$. For both customers, we assume that the budget update function is an exponentially smoothed function as in (4).*

The next proposition, which we prove in Appendix EC.2, formalizes that the above instance achieves the bound in Theorem 1.

Proposition 3. *The instance described in Example 1 achieves the bound in Theorem 1 as $\delta \to 0$.*

We explore the performance of the myopic policy computationally in Section 4.3.

4.2. Optimality of the Myopic Policy when $\gamma = 0$

The main purpose of this section is to present a special case of Theorem 1, namely Theorem 2 which imposes the additional condition that $\gamma = 0$, i.e. that all goods within a customer's preferred basket have constant bang-per-buck. We specifically focus on this case for two reasons (a) its additional assumptions are motivated by online advertising practice and are sufficient for establishing the optimality of myopic policies, and (b) it provides an easier to analyze "base case" for Theorem 1. Thus, in proving the optimality result for $\gamma = 0$, we will build important technical machinery that we use for proving Theorem 1, combined with some other additional, non-trivial steps.

Note that the $\gamma = 0$ assumption is practically motivated by the structure of campaign contracts in certain online advertising systems. In particular, many such systems allow advertisers to target specific features, such as geographical location, age group, or household income, that their delivered impressions must satisfy. The space of features characterizing these impression types is often quite rich, and as such it is a daunting task for an advertiser to value each particular feature combination. To avoid these complexities, campaign contracts allow an advertiser to specify a subset of the feature space, such as $(\ldots, \mathsf{age\_group} = 18-29 \text{ or } 30-39, \mathsf{state} = \mathsf{MA} \text{ or } \mathsf{NH} \text{ or } \mathsf{VT}, \ldots)$, which identifies the range of acceptable impressions for the advertiser's campaign, akin to our definition of the characteristic set A_j. It is implicitly understood that the advertiser values all the impressions in this set equally at some value $v_{ij} = v_j$, and the campaign contract sets a single price $p_{ij} = p_j$ for any impression in this set. In fact, this campaign structure is slightly more restrictive than Assumption 2 with $\gamma = 0$, since it enforces that valuations and prices are constant across one advertiser's acceptable goods basket, rather than requiring this just for their bang-per-buck ratios.

A consequence of Assumption 2 with $\gamma = 0$ is that the budget update function is simplified. Namely, denoting customer j's constant valuation to price ratio by $\rho_j \triangleq \frac{v_{ij}}{p_{ij}}$ for any $i \in A_j$, the service quality experienced by customer j in period t becomes:

$$q_j(b_j, \mathbf{x}_j) = \frac{\sum_{i \in A_j} (v_{ij} - p_{ij}) x_{ij}}{U_j^*(b_j)} = \frac{(\rho_j - 1) \sum_{i \in A_j} p_{ij} x_{ij}}{(\rho_j - 1) \sum_{i \in A_j} p_{ij} y_i^*} = \frac{\sum_{i \in A_j} p_{ij} x_{ij}}{\min\left(b_j, \sum_{i \in A_j} p_{ij} s_i\right)}, \tag{10}$$

where $\mathbf{y}^*$ is an optimal solution to $U_j^*(b_j)$, i.e., the service quality provided to customer j becomes the fraction of customer j's budget that was used by the platform, or her fill rate. In particular, $q_j(b_j, \mathbf{x}_j)$ becomes independent of the customer's product valuations.

Having said that, we emphasize that the issues of supply scarcity and heterogeneity in customer update functions still remain. In particular, we would still expect to see the trade-off between short-term platform revenues and the long-term goodwill of the customers. For example if a customer is very sensitive to service quality, it could be better to prioritize this customer even if this contradicts the myopic policy. As a result, it is natural to expect that the myopic policy will remain sub-optimal, even in the presence of this structure that appears to simplify the problem. On the contrary, the theorem which we prove next shows that $\gamma = 0$ is sufficient for myopic optimality.

Before stating this theorem, we introduce two additional pieces of notation which we use in the theorem's proof, together with a key proposition that enables the result. We use $\mu_{j,t}$ to denote an optimal dual variable associated to the budget constraint (1) of customer j, and $\lambda_{i,t}$ to denote an optimal dual variable associated to the supply constraint (2) of item i in the myopic optimization problem (7) solved in period t. We define $\boldsymbol{\mu}_t = (\mu_{1,t}, \ldots, \mu_{m,t})$ and $\boldsymbol{\lambda}_t = (\lambda_{1,t}, \ldots, \lambda_{n,t})$. The key proposition is:

Proposition 4 (Dynamic Complementary Slackness). *Under Assumptions 1 and 2 with $\gamma = 0$, let $\mathbf{x}_1^{MY}$ be a myopic allocation with budgets $\mathbf{b}_1$. Similarly, let $\mathbf{x}_2^{MY}$ be a myopic allocation with the updated budgets $\mathbf{b}_2 = \boldsymbol{\phi}(\mathbf{b}_1, \mathbf{q}(\mathbf{b}_1, \mathbf{x}_1^{MY}))$. Then, if $\sum_{i\in A_j} p_{ij} x_{ij,1}^{MY} < b_{j,1}$ there exists a dual optimal solution $(\boldsymbol{\mu}_2, \boldsymbol{\lambda}_2)$ associated to $\mathbf{x}_2^{MY}$ such that $\mu_{j,2} = 0$ for each customer j.*

Informally, this proposition states that, given our assumptions and $\gamma = 0$, if the myopic policy does not exhaust a customer's budget in some time period, then the marginal value of increasing that customer's budget *in the next period* is zero. In other words, Proposition 4 shows that our model satisfies what one could call "dynamic complementary slackness". This is formally proved in Appendix EC.2 in the e-companion to this paper.

The dynamic complementary slackness proposition is crucial to the next theorem which states that if $\gamma = 0$ the myopic policy is optimal. To develop some intuition for this, consider the case when $T = 2$ and the platform performs a myopic allocation at $t = 1$. By equation (10), $b_{j,2}$ expands at the maximum possible rate for customers whose budgets were fully extracted, i.e. $b_{j,1} = \sum_{i\in A_j} p_{ij} x_{ij,1}^{MY}$. At the same time, for customers with $b_{j,1} > \sum_{i\in A_j} p_{ij} x_{ij,1}^{MY}$ for which $b_{j,2}$ possibly contracted (i.e. $b_{j,2} < b_{j,1}$), Proposition 4 guarantees that $\mu_{j,2} = 0$. Thus we can, without loss of generality, assume $b_{j,2} = \phi_j(b_{j,1}, 1)$ without affecting the revenues in the second period. Hence, myopic revenues are equal to the revenues obtained had service quality been 1 for all customers at period $t = 1$, and must therefore be optimal. The formal theorem is stated and proved next.

Theorem 2. *For any horizon T and initial budget state $\mathbf{b}_1$, let $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ be the myopic policy defined in Section 3. Then, under Assumptions 1 and 2 with $\gamma = 0$,*

(i) $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ is an optimal solution to problem (9), i.e. $J_T^{MY}(\mathbf{b}_1) = J_T^{relax}(\mathbf{b}_1)$.

(ii) $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ is an optimal solution to problem (5), i.e. $J_T^{MY}(\mathbf{b}_1) = J_T^{*}(\mathbf{b}_1)$.

Proof. We show that $J_T^{MY}(\mathbf{b}_1) = J_T^{\text{relax}}(\mathbf{b}_1)$. Since $\{\mathbf{x}_t^{MY}\}_{1\le t\le T}$ is feasible for problem (5), then Proposition 2 implies $J_T^{MY}(\mathbf{b}_1) = J_T^{*}(\mathbf{b}_1) = J_T^{\text{relax}}(\mathbf{b}_1)$. The proof is by induction on the horizon length T.

Base Case: If $T = 1$, then the RNRM problem collapses to the one-shot NRM problem and $J_1^{MY}(\mathbf{b}_1) = J_1^{*}(\mathbf{b}_1) = J_1^{\text{relax}}(\mathbf{b}_1)$.

Induction Step: Assume the statement of the Theorem holds for any problem with horizon length $(T-1)$, for some $T \ge 2$. Let $\{\mathbf{b}_t\}_{2\le t\le T}$ be the budget states induced by $\{\mathbf{x}_t^{MY}\}_{1\le t\le T-1}$. Hence, for any budget state $\mathbf{b}_2$, $\{\mathbf{x}_t^{MY}(\mathbf{b}_2)\}_{t\in\{2,\dots,T\}} \in \arg\max J_{T-1}^{*}(\mathbf{b}_2) = J_{T-1}^{\text{relax}}(\mathbf{b}_2)$, where we are abusing the notation to emphasize the dependence of the allocations $\mathbf{x}_t^{MY}$ on the budget state $\mathbf{b}_2$. In particular, since $J_{T-1}^{*}(\mathbf{b}_2) = J_{T-1}^{\text{relax}}(\mathbf{b}_2)$, it follows that $J_{T-1}^{*}(\mathbf{b}_2)$ can be computed by solving the associated linear program (9). Let $\left\{(\boldsymbol{\lambda}_t^{\text{relax}}, \boldsymbol{\mu}_t^{\text{relax}})\right\}_{2\le t\le T}$ be associated optimal dual solutions.

For each customer j there are two possible cases:

(a) If $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} = b_{j,1}$, then it follows from Assumption 2 with $\gamma = 0$ that $q_j\left(b_{j,1}, \mathbf{x}_{j,1}^{MY}\right) = 1$ (cf. equation (10)) and $b_{j,2} = \phi_j\left(b_{j,1}, 1\right)$.

(b) If $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} < b_{j,1}$, then from Proposition 4 it follows that $\mu_{j,2} = 0$. Moreover, from Assumption 1(ii) we have $\phi_j^{T-2}(b_{j,2}, 1) \geq \ldots \geq b_{j,2}$. Since the optimal objective value of a maximization linear program is concave on the right hand side of the constraints, then $\mu_{j,T}^{\text{relax}} \leq \ldots \leq \mu_{j,2}^{\text{relax}} = \mu_{j,2} = 0$, see Bertsimas and Tsitsiklis (1997). From dual feasibility ($\mu_{j,t}^{\text{relax}} \geq 0$ for each $t \in \{2, \ldots, T\}$) we conclude that $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} < b_{j,1}$ implies $\mu_{j,t}^{\text{relax}} = 0$ for each period $t \in \{2, \ldots, T\}$. Namely, increasing the budget $b_{j,2}$ to its upper bound $\phi_j\left(b_{j,1}, 1\right)$ (hence $\phi_j^{t-2}(b_{j,2}, 1)$ to $\phi_j^{t-1}(b_{j,1}, 1)$ for each $t \in \{2, \ldots, T\}$) does not impact the feasibility or optimality of the allocations $\{\mathbf{x}_t^{MY}\left(\mathbf{b}_2\right)\}_{2\leq t\leq T}$ in the subproblem with horizon $T-1$ starting in period $t = 2$.

Therefore, we conclude that

$$J_T^{MY}(\mathbf{b}_1) = \text{MY}\left(\mathbf{b}_1\right) + \sum_{t=2}^{T} \text{MY}\left(\boldsymbol{\phi}^{t-2}\left(\mathbf{b}_2, \mathbf{1}\right)\right) = \text{MY}\left(\mathbf{b}_1\right) + \sum_{t=2}^{T} \text{MY}\left(\boldsymbol{\phi}^{t-1}\left(\mathbf{b}_1, \mathbf{1}\right)\right) = J_T^{\text{relax}}(\mathbf{b}_1).$$

From Propositon 2, $J_T^{MY}(\mathbf{b}_1) = J_T^{*}(\mathbf{b}_1) = J_T^{\text{relax}}(\mathbf{b}_1)$. This concludes the proof. □

Before discussing the result in Theorem 2, we remark that its proof does not require Assumption 1(*iii*) since $\gamma = 0$. The latter is required in the proof of Theorem 1, which allows for $\gamma \neq 0$.

Theorem 2(*ii*) implies that, under Assumptions 1 and 2, when $\gamma = 0$ an optimal policy for problem (5) with horizon T can be computed by sequentially solving T linear programs, using the myopic allocation in one period to update the customers' budget state in the next period. Moreover, Theorem 2(*i*) shows that under these conditions the optimal objective function of problem (5) can be computed by solving one larger linear program for the whole horizon T, where the largest possible budget update is assumed for each customer in each period, i.e. problem (9). Note that then the budget state in one period is independent of the allocation in previous periods.

Additionally, Theorem 2 shows that under Assumptions 1 and 2 some of the problematic features of the general problem discussed in Proposition 1 are no longer present when $\gamma = 0$. In particular, the platform always benefits from customers with higher budgets. This argument is formalized in the following corollary, which is proved in the e-companion to this paper.

Corollary 1. *Under Assumptions 1 and 2 with* $\gamma = 0$, $J_T^{*}\left(\mathbf{b}_{1,h}\right) \geq J_T^{*}\left(\mathbf{b}_{1,l}\right)$ *for any horizon* $T \geq 2$ *and budget states* $\mathbf{b}_{1,h}$, $\mathbf{b}_{1,l}$ *such that* $\mathbf{b}_{1,h} \geq \mathbf{b}_{1,l}$ *component-wise.*

4.3. Numerical Performance of the Myopic Policy

We now proceed to investigate the numerical performance of the myopic policy. Since computing the optimal solution to problem (5) is, in general, hard, we will use $J_T^{\text{relax}}\left(\mathbf{b}_1\right)$ as a benchmark, i.e. the revenue obtained by the platform when the budgets increase at the highest possible rate, which can be computed by solving the linear program (9). Recall that Proposition 2 shows that this is

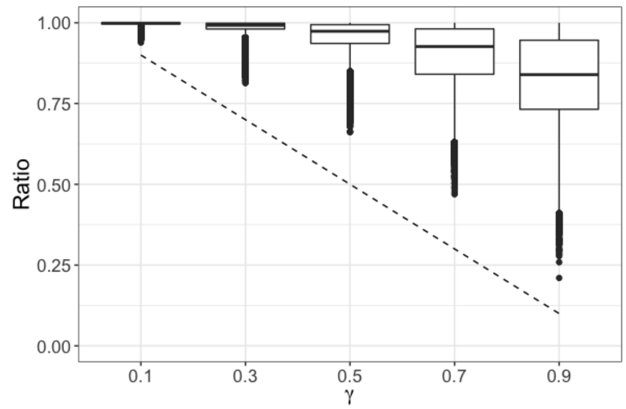


Figure 3 Boxplot of the ratio between the revenue of the myopic policy and the optimal revenue when budgets always increase at the largest possible rate. The box plot for each value of γ represents the distribution of ratios sampled from 250,000 randomly generated problem instances. The dotted line is the $1-\gamma$ bound. For these simulations, $n \in [3,10]$, $T \in [3,15]$, and $n = m$.

a valid upper bound on the optimal revenue. More specifically, assuming that customers have an exponential smoothing budget update function given by (4), we compute the performance ratio

$$r\left(n, m, T, \alpha, \mathbf{b}_1, \mathbf{s}, \{p_{ij}\}, \{v_{ij}\}\right) = \frac{J_T^{MY}\left(\mathbf{b}_1\right)}{J_T^{\text{relax}}\left(\mathbf{b}_1\right)} \tag{11}$$

for randomly generated instances of problem (5) with parameters $(n, m, T, \alpha, \mathbf{b}_1, \mathbf{s}, \{p_{ij}\}, \{v_{ij}\})$. Note that calculating $J_T^{MY}\left(\mathbf{b}_1\right)$ involves solving a sequence of linear programs. For different values of γ, we sample 250,000 problem instances and calculate the performance ratio (11) for each instance. The sampling procedure is described in detail in Appendix EC.5.1 in the e-companion to this paper.

The distribution of ratios r for different values of the parameter γ is depicted through box plots in Figure 3. The dotted line represents the $(1-\gamma)$ bound from Theorem 1. Note that for $\gamma \leq 0.7$, the revenue collected by the myopic policy is at least 85% of the revenue obtained when budgets increase at the highest possible rate (and therefore at least 85% of the optimal revenue) for over 75% of the randomly generated instances. However, there are many outliers in the simulation with a performance close to the $(1-\gamma)$ bound from Theorem 1, which is tight (cf. Proposition 3).

5. Problem Instances Where Simple Policies Fail

Having determined that under certain regularity assumptions myopic policies can perform well, we now examine the performance of myopic and other natural policies if we do not impose any regularity assumption on our baseline model.

First, we define another candidate class of heuristic policies for our problem, namely finite look-ahead policies. The reason we study this class of policies is that they have been successfully used in a large number of dynamic programming applications where myopic policy performance is unsatisfactory (see for example the survey of Bertsekas 2005). Moreover, unlike myopic, finite look-ahead policies explicitly account for customer goodwill effects, and is it reasonable to expect that they would perform well. We will show in this section that, in fact, *look-ahead policies, together with the myopic policy, perform arbitrarily badly when we remove all our previous assumptions.*

We begin with the definition of a look-ahead policy.

The L-step Look-ahead Policy $\pi^{L\text{-}\mathbf{LA}}$. For any horizon T, in each period $t \leq T$ the L-step look-ahead policy $\left\{\mathbf{x}_t^{L-LA}\right\}_{1\leq t\leq T}$ implements the first period allocation of the policy that maximizes the revenue garnered by the platform in the next $\min(L+1, T-t+1)$ periods, i.e. in whatever is shorter between the current period plus the following L periods and the remaining horizon. To simplify the notation let us define $\hat{L} = \min(L+1, T-t+1)$. We note that the myopic policy defined in Section 3 is a degenerate example of an L-step look-ahead policy with $L=0$.

Specifically, for any parameter $L \leq T-1$, period $t \leq T$, and budget state $\mathbf{b}_t$, let $\left\{\mathbf{y}_k^{\hat{L}}\right\}_{1\leq k\leq \hat{L}} \in \arg\max J_{\hat{L}}^*(\mathbf{b}_t)$. Then, $\mathbf{x}_t^{L-LA} = \mathbf{y}_1^{\hat{L}}$. We emphasize that the L-step look-ahead policy is a rolling horizon policy that requires solving a dynamic program in each period. However, it is oblivious to the customers' budget dynamics beyond the horizon $\hat{L} = \min(L+1, T-t+1)$. The L-step look-ahead policy follows the natural budget update

$$\mathbf{b}_{t+1} = \boldsymbol{\phi}\left(\mathbf{b}_t, \mathbf{q}\left(\mathbf{b}_t, \mathbf{x}_t^{L-LA}\right)\right).$$

Arbitrary Sub-optimality Gap for Myopic and L-step Look-ahead Policies. In the following, we provide a family of instances of problem (5) where finite look-ahead policies, as well as myopic which is a degenerate case with $L=0$, perform arbitrarily badly in an asymptotic regime as the number of customers of the platform grows. These results illustrate that problem (5) is hard to solve in general.

Example 2. *Consider a T period instance with $m+1$ customers. The first m customers have the same exponentially smoothed update as defined in* (4)*, with common parameter $\alpha_j = \alpha$:*

$$\phi(b_j, q_j(b_j, \mathbf{x}_j)) = \alpha \cdot b_j + (1-\alpha) q_j(b_j, \mathbf{x}_j), \quad \forall 1 \leq j \leq m.$$

We set α such that $\alpha^L = 1/2$. The $(m+1)$-th customer does not update her budget, i.e. she has an exponentially smoothed update with $\alpha_{m+1} = 1$.

There are a total of $m+1$ products. The first m are indexed by $1^1, \ldots, 1^j, \ldots, 1^m$, while the last is indexed by 2. The supply of product 1^j is $s_{1^j} = \frac{1}{m^2}$, while the supply of product 2 is $s_2 = m$. We set $T = 3\log_\alpha(e^{-1})\log m$. Furthermore:

1. *For customers $1 \le j \le m$ and some scale parameter $\eta > 2$, the product valuations are*

$$v_{1^l j} = \begin{cases} \eta m + 1, & \text{if } l = j, \\ m + 1 - \frac{1}{m}, & \text{if } l \neq j, \end{cases} \qquad p_{1^l j} = \begin{cases} 1, & \text{if } l = j, \\ 1 - \frac{1}{m}, & \text{if } l \neq j \end{cases}$$
$$v_{2j} = 1, \quad \forall j \qquad p_{2j} = 1, \quad \forall j.$$

2. *For customer $m+1$, $v_{1^l(m+1)} = p_{1^l(m+1)} = 0$ for $1 \le l \le m$ and $v_{2(m+1)} = p_{2(m+1)} = \frac{1}{m^2}$.*
3. *For all customers $1 \le j \le m$, the starting budget is $b_{1,j} = \frac{\eta}{\eta + m - 1}$, while for customer $(m+1)$, the starting (and constant) budget is $b_{(m+1),1} = 1$.*

Proposition 5. *On the instance described in Example 2, with $\eta > 2$ and $m \ge \max\left\{\frac{1}{\alpha^2}, \frac{2(\eta-1)}{\eta-2}\right\}$, both the myopic policy and the L-step look-ahead policy, for any $L \ge 1$, produce revenues that are an $o\left(\frac{\log m}{m}\right)$ fraction of the optimum.*

The proof of proposition 5 is provided in the e-companion to this paper.

To conclude this section, we make some observations about why look-ahead policies can perform so poorly in this context. In our instance defined in Example 2, products 1^l, which are the ones with positive utility for the customers, are scarce enough that it is not feasible to simultaneously satisfy all customers $1 \le j \le m$ when their budgets are non-negligible. Thus, the feasible strategy that we exploit is to fully deplete the budgets of customers $2 \le j \le m$ to essentially 0, when it becomes possible to fully satisfy them with the scarce supply of positive utility products 1^l. This allocation induces a one-time jump in their budget from 0 to approximately $(1-\alpha)$ between periods $T-1$ and T, which is then monetized in the last period T. As a consequence, the revenues garnered by this feasible policy over the time horizon are highly non-smooth, with essentially all its revenues coming in the last period. Such a strategy cannot be found by $\pi^{L\text{-LA}}$ due to its limited look-ahead $L < T$; indeed, in only L periods it is not possible to ration the 1^l products in such a way as to cause a similar coordinated one-time jump in budgets.

6. Extensions

Two important features of our model are the presence of the same pool of customers in each period, and the time-invariance of the available supply. We now introduce two extensions which incorporate uncertainty in each of these dimensions. We assume throughout this section that $\gamma = 0$, which simplifies the customers' utility as in equation (10).

6.1. Uncertain Customer Arrivals

We now consider the extension of our base model where customers do not necessarily show up in every single period. Specifically, we develop a worst-case model where customers can be absent from the system at arbitrary times. We naturally assume that customers only update their budgets in periods when they are present. Then, a customer's budget in period t depends on her budget and service level received in her last interaction with the platform (need not be at $t-1$).

We present the main results here, and defer the technical analysis of the results to Appendix EC.4.1 in the e-companion to this paper.

We first remark on the difficulty of establishing bounds when we allow for customers to skip periods of interaction with the platform. Recall that our myopic optimality result for $\gamma = 0$ crucially relies on a "dynamic complementary slackness" condition stated in Proposition 4. Informally, this condition states that, as long as all customers are always present, if a customer's budget constraint at time t is not tight, then this customer will never have a positive shadow price on their budget at any point in the future. Intuitively, the fact that the active set of customers changes over time now breaks dynamic complementary slackness: a customer whose budget constraint was not tight at time t may have a positive shadow price at $t+1$ in the case that other more profitable customers from period t are not present in $t+1$. Thus, the technical challenge to establishing lower bounds for the myopic policy in this model will be to replace dynamic complementary slackness with weaker conditions (see Proposition 8 and the discussion before it in the e-companion to this paper).

Our first result in this section shows a parametric lower bound on the performance of the myopic policy. Namely, in this model the revenue collected by the optimal policy must be at most T times the revenue collected by the myopic policy, where T is the horizon the RNRM problem.

Theorem 3. *Under Assumptions 1 and 2 with $\gamma = 0$, for any horizon T and initial budget state $\mathbf{b}_1$,*

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{T}.$$

We emphasize that this bound holds in a worst-case model, where nature has full power to choose when each customer is present in the system. A natural question is whether the bound in Theorem 3 can be improved in this adversarial setup. The following example and proposition answer it on the negative; interestingly enough, this is achieved via an instance with a single product type:

Example 3. *Consider any integer T and $\alpha \in [0,1)$ such that $1/(1-\alpha)$ is integer. Let there be $n = 1$ products, and $m = 1 + (T-1)/(1-\alpha)$ customers. For customer 1, we set $p_{11} = 1$, $v_{11} = 2$ and their starting budget to $b_{1,1} = 1$. Customer 1 is present at period $t = 1$ but never returns to the system. We index customers $2, \ldots, (T-1)/(1-\alpha)$ by $\{j^1, \ldots, j^{1/(1-\alpha)}\}$ for all $2 \leq j \leq T$. Customer j^l, for $1 \leq l \leq 1/(1-\alpha)$, is present in the system only at times $t = 1$ and $t = j$. Moreover, $p_{1j^l} = 1 - \delta$, for some $\delta > 0$, and $v_{1j^l} = 2$. Customer j^l starts with budget $b_{j^l,1} = \delta$. The budget update function of all customers is $\phi(b,q) = \alpha b + (1-\alpha)q$.*

The above instance attains the bound of $1/T$, as the following proposition shows.

Proposition 6. *The bound in Theorem 3 is tight. Namely, for the instance in Example 3, $\lim_{\delta \to 0} J_T^{MY}(\mathbf{b}_1)/J_T^*(\mathbf{b}_1) = 1/T$.*

Proposition 6 shows that if customers are not necessarily present in each period, then the performance of the myopic policy can be arbitrarily bad when compared to the optimal policy as the length of the horizon grows. This is in stark contrast to our results in Section 4, which showed that, for any horizon length, the myopic policy was either optimal or it had a parametric performance guarantee when all customers are present in each period.

We emphasize that the instance in Example 3 is fairly general with respect to the customer budget updates, in that the value of the parameter α in their exponential smoothing function is not very restrictive. Having said that, the instance in Example 3 is somewhat artificial in that the most profitable customer joins the platform in the first period and then never returns. Then, the myopic policy allocates the item to her at first, without taking into account that all the other customers will receive low service and hence will have a negligible budget when they come back to the platform. As a result, the platform collects minimal revenue in each period after the first.

It is thus interesting to ask whether slightly restricting nature's power to choose a worst-case sequence of customer arrivals can improve this lower bound, by removing such "corner case" instances. We explore this question in the remainder of this subsection.

Improved Bounds. Example 3 suggests that the myopic policy performs poorly on instances where it allocates many products to customers who at present look the most profitable, but who in the future will return infrequently, or even never return at all to the system, no matter how large the time horizon T. One realistic assumption that could avoid this difficulty is to make sure that the gap between customer arrivals cannot scale with T:

Assumption 3. *Assume that each customer must be present at least once every L periods. Namely, let t_j^k and t_j^{k+1} denote the periods in which any given customer j arrives for the k-th and $k+1$-th times, respectively. Then any inter-arrival time for this customer, $t_j^{k+1} - t_j^k$, is at most L.*

This assumption can be interpreted as saying that, while customers may exhibit some infrequency in their returns to the platform, the length of a "cycle" of inactivity cannot be arbitrarily large. We believe that this is a reasonable assumption in an online advertising context, since it is unlikely an advertiser would not schedule any campaigns for an extended amount of time.

Returning to Example 3, Assumption 3 no longer allows T to scale arbitrarily: to enforce the arrival pattern where customer one can be present only at time $t = 1$, we require that $T \leq L$. Enforcing Assumption 3 for this instance naturally gives us a $1/L$ bound. One could then hope that, even for instances where $T \gg L$, the performance of myopic remains independent of T.

We next show that this indeed turns out to be true: we provide bounds that only depend on L, and possibly on the shape of the customer budget update functions. We emphasize that these bounds are still the result of a worst-case analysis on the customers' arrival pattern, but up to every customer returning to the system at least every L periods as per Assumption 3.

We begin by precisely stating a result for a single product type, which matches the instance family from Example 3, and provides a stronger bound on the performance of the myopic policy.

Theorem 4. *Under Assumptions 1-3 with $\gamma = 0$, if $n = 1$ then for any horizon T and initial budget state $\mathbf{b}_1$,*

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{\min(L+1, T)}.$$

For $T \gg L$, Theorem 4 provides a $1/(L+1)$ bound that nearly matches the $1/L$ performance of the myopic policy in Example 3 under Assumption 3. We remark that, as discussed earlier, since customers are not present in the system in every period then the "dynamic complementary slackness" condition from Proposition 4 no longer holds. Therefore, showing Theorem 4 requires new proof techniques, which we explain in detail in Appendix EC.4.1 in the e-companion to this paper.

A similar analysis for the general case where the platform allocates multiple products leads to the following bound on the performance of the myopic policy:

Theorem 5. *Let $\bar{\alpha} \in [0,1]$ be defined as:*

$$\bar{\alpha} = \min_{j \in [m], b \in [0,1]} \{\phi_j(b, 1)\}.$$

Under Assumptions 1-3 with $\gamma = 0$, for any horizon T and initial budget state $\mathbf{b}_1$,

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{\min\left(L + \frac{1}{\bar{\alpha}}, T\right)}.$$

In Theorem 5, $\bar{\alpha}$ can be interpreted as a lower bound on the one-time increase in budget of a customer who receives full service. Note that, if ϕ_j is defined as an exponential smoothing update with parameter α_j (as in equation (4)), then $\bar{\alpha} = 1 - \max_j \alpha_j$.

The bound for multiple products from Theorem 5 depends on the customer budget update functions, and as a result can be weaker than the bound for one product from Theorem 4. Having said that, there are special cases where the bound in Theorem 5 recovers the full strength of the bound in Theorem 4. Specifically, this is the case if all customers follow an exponential smoothing budget update (as in equation (4)) with parameter $\alpha_j = 0$, i.e. if all customers are fully reactive and their updated budget always matches the service quality they received the last time they interacted with the platform (and hence $\bar{\alpha} = 1$).

Note that implementing the optimal policy in this model would require a perfect forecast of the sequence of customer arrivals over time. In practice, this requirement adds a new and significant difficulty to the learning problem that the platform would face in order to be able to collect a high revenue from customers. As a result, this feature puts an even higher premium on simple policies, such as myopic, having a guaranteed performance. Our results are important since they show that, as long as customers come back to the platform often enough, a platform acting myopically is still guaranteed to garner a significant fraction of the optimal revenues. For example, if customers come back to he platform at least for every other campaign, i.e. $L=2$, then the platform is guaranteed to collect at least one third of the optimal revenue.

To conclude this section, we examine the numerical performance of the myopic policy with uncertain customer arrivals for randomly generated systems in Appendix EC.5.2 in the e-companion to this paper. To summarize, we find that the myopic policy collects at least 90% of the optimal revenue for 98% of the sampled systems, suggesting that its average performance is significantly higher than the worst-case bounds we derived in this section.

6.2. Time-Varying Supply

We extend our original model by assuming that the supply of goods changes from one period to the next. We assume that, in each period t, the supply of good i, which we now denote by $s_{i,t}$, lies within the interval $[(1-\epsilon)\bar{s}_i, (1+\epsilon)\bar{s}_i]$, where ϵ is the same for all goods[6]. We denote $\mathbf{s}_t = (s_{1,t}, \ldots, s_{n,t})$ and $\bar{\mathbf{s}} = (\bar{s}_1, \ldots, \bar{s}_n)$. We also extend our notation and denote the feasible allocation set at time t by $\mathbf{X}(\mathbf{b}_t, \mathbf{s}_t)$, where $\mathbf{X}(\mathbf{b}_t, \mathbf{s}_t)$ is

$$\begin{aligned}
&\sum_{i=1}^{n} p_{ij} x_{ij,t} \le b_{j,t}, \quad \forall 1 \le j \le m \\
&\sum_{j=1}^{m} x_{ij,t} \le s_{i,t}, \quad \forall 1 \le i \le n. \\
&x_{ij,t} \ge 0, \quad \forall 1 \le i \le n, 1 \le j \le m.
\end{aligned}$$

Note the dependence of $\mathbf{X}$ on the random vector $\mathbf{s}_t$. Furthermore, the myopic allocation for budgets $\mathbf{b}_t$ and some supply vector $\mathbf{s}$ (for example, equal to $\mathbf{s}_t$ or $\bar{\mathbf{s}}$) becomes

$$\begin{aligned}
\mathbf{x}_t^{MY}(\mathbf{b}_t, \mathbf{s}) \in \arg\max_{\mathbf{x}_t} \; & R(\mathbf{x}_t) \\
\text{s.t. } & \mathbf{x}_t \in \mathbf{X}(\mathbf{b}_t, \mathbf{s}),
\end{aligned} \tag{12}$$

where we emphasize that this allocation is defined based on $\mathbf{s}$ and we make this explicit by parametrizing $\mathbf{x}_t^{MY}$ on current budget $\mathbf{b_t}$ and supply $\mathbf{s}$.

The scaled myopic allocation policy. We examine the performance of the *scaled myopic allocation* policy. In each period t, for customer budgets $\mathbf{b}_t$ and supply $\mathbf{s_t}$, the scaled myopic allocation denoted by $\mathbf{x}_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t)$, is constructed by calculating the myopic allocation with budgets $\mathbf{b}_t$ and supply $\bar{\mathbf{s}}$ (as defined in (12)) and scaling it by a factor $1-\epsilon$. Namely, set

$$\mathbf{x}_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t) \triangleq (1-\epsilon) \cdot \mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}). \tag{13}$$

[6] The result also holds if ϵ was the maximum possible deviation across all goods; for simplicity we assume common ϵ.

This allocation is guaranteed to be feasible since, if $\mathbf{s}_t \geq (1-\epsilon)\bar{\mathbf{s}}$, $\mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}) \in \mathbf{X}(\mathbf{b}_t, \bar{\mathbf{s}})$ implies $(1-\epsilon)\mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}) \in \mathbf{X}(\mathbf{b}_t, \mathbf{s}_t)$. The only additional information that the scaled myopic information uses compared to the myopic policy is the range supply variation ϵ. Similar to the myopic policy, it does not use information about consumers' valuations and budget update functions. We denote the total revenue of this policy for a given starting budget $\mathbf{b}_1$ and horizon T by $J_T^{S-MY}(\mathbf{b}_1)$.

Qualitatively, the scaled myopic allocation $\mathbf{x}_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t)$ differs from the myopic policy $\mathbf{x}_t^{MY}(\mathbf{b}_t, \mathbf{s}_t)$ in one important way: it guarantees a somewhat "linear" allocation across customers as $\mathbf{s}_t$ varies. Since the scaled myopic allocation is anchored around the constant supply level $\bar{\mathbf{s}}$, the service quality received by each customer will be at least a fraction of the optimal quality level if supply was equal to $\bar{\mathbf{s}}$. On the other hand, this is not the case if we were to use $\mathbf{x}_t^{MY}(\mathbf{b}_t, \mathbf{s}_t)$ instead of $\mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$: if $\mathbf{s}_t$ decreases, the new allocation might disproportionately reduce the amount of goods allocated to low-price customers when compared to the scaled myopic allocation. These low-price customers might end up with a low service quality level and, as a result, a low budget in the next period. Then, depending on the supply of goods in the next period, this might lead to loss of revenue compared to the optimal policy.

Formally, given the definition of $\mathbf{x}_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t)$, we have for any customer j and period t,

$$\sum_i p_{ij} x_{ij,t}^{S-MY}(\mathbf{b}_t, \mathbf{s}_t) \geq (1-\epsilon) \sum_i p_{ij} x_{ij,t}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}).$$

Also, given the bounded variability of supply, $\min\{b_j, \sum_i p_{ij} s_{i,t}\} \leq (1+\epsilon)\min\{b_{j,t}, \sum_i p_{ij}\bar{s}_i\}$. Thus, when starting with common budget $\mathbf{b}_t$, we can bound the quality level of the scaled myopic policy, which we denote $\mathbf{q}_t^{S-MY}$, by the quality level of the myopic policy with constant supply, which we denote $\mathbf{q}_t^{MY}$:

$$q_{j,t}^{S-MY}(b_{j,t}, \mathbf{x}_{j,t}^{S-MY}, \mathbf{s}_t) = \frac{\sum_i p_{ij} x_{ij,t}^{S-MY}(b_{j,t}, \mathbf{s_t})}{\min\{b_j, \sum_i p_{ij} s_{i,t}\}} \geq \frac{(1-\epsilon)\sum_i p_{ij} x_{ij,t}^{MY}(b_{j,t}, \bar{\mathbf{s}})}{(1+\epsilon)\min\{b_{j,t}, \sum_i p_{ij}\bar{s}_i\}} \geq (1-\epsilon)^2 q_{j,t}^{MY}(b_{j,t}, \mathbf{x}_{j,t}^{MY}, \bar{\mathbf{s}}), \tag{14}$$

where in the last inequality we use the definition of $\mathbf{q}_t^{MY}$ and the fact that $\frac{1-\epsilon}{1+\epsilon} \geq (1-\epsilon)^2$. Note that, to make the dependence of varying supply explicit, we have overloaded the notation for the quality level to include the supply vector as a parameter.

With the observation above in hand, we state a theorem which bounds the performance of the scaled myopic policy for every supply path bounded by $[(1-\epsilon)\bar{s}_i, (1+\epsilon)\bar{s}_i]$.

Theorem 6. *Let $J_T^*(\mathbf{b}_1)$ be the optimal revenue for the system with time-varying supply for a starting budget $\mathbf{b}_1$ and a horizon of T. If for each period t and good i we have $s_{i,t} \in [(1-\epsilon)\bar{s}_i, (1+\epsilon)\bar{s}_i]$, then under Assumptions 1 and 2 with $\gamma = 0$,*

$$\frac{J_T^{S-MY}(\mathbf{b}_1)}{J_T^*(\mathbf{b}_1)} \geq (1-\epsilon)^4.$$

We make a few remarks regarding Theorem 6 and the underlying model. First, as we detail in Appendix EC.4.2 in the e-companion to this paper, the proof of this result crucially depends on the fact that Proposition 7 functions in a more general, black-box fashion, than we used it in Theorem 1. That is, Proposition 7 allows us to compare the revenues of separate systems which follow the sequences of service qualities $\{\mathbf{q}_t^{S-MY}\}$ and, respectively, $\{(1-\epsilon)^2\mathbf{q}_t^{MY}\}$, given the fact that, as per equation (14), the former service quality is lower bounded by the latter for the same budget.

Secondly, one can compare our model to the fluid models typically used in the revenue management literature (Gallego and van Ryzin 1997, Talluri and Van Ryzin 1998, Jasin and Kumar 2012). There, one typically models varying supply as arrivals coming from a Poisson process with rate λ, together with a heuristic control policy that allocates each arriving supply unit in an online fashion. It is then possible to prove that, as the arrival rate is increased to $\theta\lambda$ and the other relevant dimensions of the problem are also scaled up by the same θ, the heuristic's loss in the scaled system with respect to the optimal policy vanishes with θ. It is in fact possible to rephrase our result as a similar fluid bound where ϵ vanishes with θ, but we present our result in its current form as we believe it makes the proof (particularly the use of Proposition 7) more transparent and simple.

We denote $J_T^{S-relax}$ as the revenue when budgets increase at the maximum possible rate (quality level is always one) and supply varies over time. We note that $J_T^{S-relax}$ is an upper bound to the optimal revenue. In Appendix EC.5.3 in the e-companion to this paper, we present numerical simulations where we sample randomly generated systems and observe that, in general, a modified version of the scaled myopic policy (which satisfies Theorem 6) collects at least 95% of the optimal revenue. Furthermore, even for large values of ϵ, the scaled myopic policy collects at least half of the optimal revenue. We also present simulations where we assume that supply evolves according to a discrete-time random walk and we find that the "regular" myopic policy captures at least 80% of the optimal revenue.

7. Conclusions

In this paper, we have considered a multi-period model of network revenue management where customers' behavior changes from one period to another as a function of the quality of the past interactions between the customers and the platform controlling the allocation of products. While the problem is hard to solve in general, we have shown that by imposing reasonable conditions on the problem structure, simple myopic policies which ignore future customer behavior work well. Additionally, we have provided guarantees in the case that there is uncertainty both on the side of customers and of supply.

We hope that our model and results inspire further efforts to understand whether classical operations management prescriptions are transferable to dynamic settings where customer interactions repeat over time. Specifically, two avenues are worth exploring: (i) understanding how prices, along with allocations should be set in such repeated settings and (ii) building a model where customers can explicitly switch between competing platforms, depending on their history of service quality.

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E-Companion to:
Revenue Management with Repeated Customer Interactions

EC.1. Examples of Budget Update Functions Which Violate Assumption 1

To build more insight into the budget update functions allowed by Assumption 1, we also discuss which types of functions would violate it, focusing specifically on assumptions (ii) and (iii). Broadly speaking, such examples either involve "large penalties" or "large rewards".

A type of update which violates both Assumptions 1(ii) and (iii) is one which has "large penalties", such as

$$\phi(b,q) = (\alpha b + (1-\alpha)q) \cdot \mathbf{I}\{q \geq q_0\},$$

for some $\alpha, q_0 \in (0,1)$. This rules out ultimatum like update functions, where the customer only brings positive budget in the next period if her service quality in the current period exceeds some minimum threshold.

Another example update, which this time satisfies Assumptions 1(i) and (ii), but not (iii), is

$$\phi(b,q) = \begin{cases} 1, & \text{if } q = 1 \\ \min\{b,q\}, & \text{otherwise.} \end{cases}$$

Then, for some δ small enough and $t = 1$,

$$\phi(\delta, 1-\vartheta) = \delta < (1-\vartheta) = (1-\vartheta)\phi(\delta, 1),$$

which violates Assumption 1 (iii), due to the disproportionately "large reward" when $q = 1$.

EC.2. Appendix for Section 4

We organize this appendix into three pieces. First, we prove that $J_T^{\text{relax}}(\mathbf{b}_1)$ is indeed an upper bound problem for $J_T^*(\mathbf{b}_1)$, together with a helper lemma. Then, in a separate subsection (Section EC.2.1), we provide the proofs pertaining to Theorem 2, which is the myopic optimality result for $\gamma = 0$. Lastly, the second subsection (Section EC.2.2) contains the proof of Theorem 1, which is the parametric performance guarantees result for general γ, and all its auxiliary results, finalizing with the proof of the tightness of the bound in Theorem 1.

As stated above, we begin by restating and proving a proposition from Section 4, which shows that the relaxed problem where customers always receive the maximum service quality is indeed an upper on the RNRM optimal objective value:

Proposition 2. *Under Assumption 1(i), $J_T^{relax}(\mathbf{b}_1) \geq J_T^*(\mathbf{b}_1)$ for any horizon $T \geq 1$ and initial budget state $\mathbf{b}_1$.*

Proof. Let $\{\mathbf{x}_t^*\}_{t\in\{1,\dots,T\}}$ be an optimal policy for problem (5) with initial budgets $\mathbf{b}_1$, and let $\mathbf{b}_t^*$ be the budget trajectory induced by it, i.e. $\mathbf{b}_{t+1}^* = \boldsymbol{\phi}(\mathbf{b}_t^*, \mathbf{q}(\mathbf{b}_t^*, \mathbf{x}_t^*))$ for each $t \in \{1,\dots,T-1\}$, where $\mathbf{b}_1^* = \mathbf{b}_1$. From Assumption 1(*i*), $b_{j,t}^* \leq \phi_j^{t-1}(b_{j,1}, 1)$ for each period t and customer j. Then,

$$J_T^{\text{relax}}(\mathbf{b}_1) = \text{MY}(\mathbf{b}_1) + \sum_{t=2}^{T} \text{MY}\left(\boldsymbol{\phi}^{t-1}(\mathbf{b}_1, \mathbf{1})\right) \geq \text{MY}(\mathbf{b}_1) + \sum_{t=2}^{T} \text{MY}(\mathbf{b}_t^*) \geq J_T^*(\mathbf{b}_1),$$

where the first inequality follows from Lemma 1 below. The second inequality follows from $\mathbf{x}_t^* \in \mathbf{X}(\mathbf{b}_t^*)$ for each period t, which implies that $R(\mathbf{x}_t^*) \leq \text{MY}(\mathbf{b}_t^*)$. □

It is not hard to see that $\text{MY}(\mathbf{b})$ is monotonically increasing in the budget state vector $\mathbf{b}$. The following lemma formalizes this observation.

Lemma 1. *Let $\mathbf{b}$, $\bar{\mathbf{b}}$, be such that $\bar{b}_j \geq b_j$ for each customer j. Then, $MY(\bar{\mathbf{b}}) \geq MY(\mathbf{b})$.*

Proof. Let $\mathbf{x}^{MY}$ be a myopic policy with budgets $\mathbf{b}$. Note that $\bar{b}_j \geq b_j$ implies $\mathbf{x}^{MY} \in \mathbf{X}(\bar{\mathbf{b}})$. Hence, $\text{MY}(\bar{\mathbf{b}}) \geq R(\mathbf{x}^{MY}) = \text{MY}(\mathbf{b})$. □

EC.2.1. Proofs for Theorem 2

The main technical proposition of this section now follows and is restated from Section 4; this is a key result in the proof of Theorem 2. As noted in Section 4, the proposition captures a notion of "dynamic complementary slackness". Before we restate it, we observe that the dual for problem (7), which is the linear program which can be solved to find the myopic policy, is

$$\begin{aligned}
\min_{\mu_t,\lambda_t} \quad & \sum_{i=1}^{n} s_i \lambda_{i,t} + \sum_{j=1}^{m} b_{j,t}\mu_{j,t} \\
\text{s.t.} \quad & \lambda_{i,t} + p_{ij}\mu_{j,t} \geq p_{ij}, \quad \forall 1 \leq i \leq n, 1 \leq j \leq m \\
& \lambda_{i,t} \geq 0 \quad \forall 1 \leq i \leq n \\
& \mu_{j,t} \geq 0 \quad \forall 1 \leq j \leq m,
\end{aligned}$$

where we remind that $\mu_{j,t}$ denotes an optimal dual variable associated to the budget constraint (1) of customer j, and $\lambda_{i,t}$ denotes an optimal dual variable associated to the supply constraint (2) of item i. We then have:

Proposition 4 (Dynamic Complementary Slackness)**.** *Under Assumptions 1 and 2 with $\gamma = 0$, let $\mathbf{x}_1^{MY}$ be a myopic allocation with budgets $\mathbf{b}_1$. Similarly, let $\mathbf{x}_2^{MY}$ be a myopic allocation with the updated budgets $\mathbf{b}_2 = \boldsymbol{\phi}(\mathbf{b}_1, \mathbf{q}(\mathbf{b}_1, \mathbf{x}_1^{MY}))$. Then, if $\sum_{i\in A_j} p_{ij} x_{ij,1}^{MY} < b_{j,1}$ there exists a dual optimal solution $(\boldsymbol{\mu}_2, \boldsymbol{\lambda}_2)$ associated to $\mathbf{x}_2^{MY}$ such that $\mu_{j,2} = 0$ for each customer j.*

The proof of Proposition 4 requires some auxiliary results, which we prove first. We begin with the following definition:

Definition 2. *Let $\mathbf{x}_1^{MY}$ be a myopic allocation with budgets $\mathbf{b}_1$ such that there exists a customer j with a strictly loose budget constraint, i.e. such that $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} < b_{j,1}$. Then, customer j defines a non-empty class of customers $C_j \subseteq \{1,\dots,m\}$ as follows.*

Start with $C_j = \{j\}$. In each iteration define the set of customers

$$D_j(C_j) = \left\{ k \in \{1\dots,n\} : x_{ik,1}^{MY} > 0 \text{ for some } i \in \bigcup_{l\in C_j} A_l \right\}.$$

While $D_j(C_j) \neq C_j$, update $C_j = D_j(C_j)$ and iterate.

Note that by construction the class of customers C_j has the property that all the items that the customers in C_j are interested in are allocated by $\mathbf{x}_1^{MY}$, if at all, to customers in C_j only. Namely, the class of customers C_j is such that $x_{ik,1}^{MY} = 0$ for any $i \in \bigcup_{l\in C_j} A_l$, $k \notin C_j$.

Furthermore, the following Lemma shows that since $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} < b_{j,1}$ it must be the case that all the items that customers in the class C_j are interested in are *fully* allocated by $\mathbf{x}_1^{MY}$ to customers in the class C_j. This property will be instrumental in the proof of Proposition 4.

Lemma 2. *Let $\mathbf{x}_1^{MY}$ be a myopic allocation with budgets $\mathbf{b}_1$ such that there exists a customer j with $\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY} < b_{j,1}$, and let C_j be the associated class of customers from Definition 2, then*

$$\sum_{l\in C_j} x_{il,1}^{MY} = s_i, \text{ for all } i \in \bigcup_{l\in C_j} A_l.$$

Proof. Assume for a contradiction that there exists an item $i \in \bigcup_{l\in C_j} A_l$ with a strictly loose supply constraint, i.e. such that $\sum_{l\in C_j} x_{il,1}^{MY} < s_i$. We show that then there must exist an feasible augmenting path in the network induced by $\mathbf{x}_1^{MY}$, contradicting its myopic optimality.

The construction of the class C_j in Definition 2 specifies a path from customer j to item i, through items $k \in \bigcup_{l\in C_j} A_l$ and customers $l \in C_j$. Let us denote this path by $\text{path}_j(i)$. Assume, without loss of generality, that $\text{path}_j(i)$ is composed of $r \geq 1$ items labeled $\{k_1,\dots,k_r\}$, and r customers labeled $\{l_1,\dots,l_r\}$, where $l_1 = j$ and $k_r = i$. Namely, $\text{path}_j(i) = \{l_1,k_1,\dots,l_r,k_r\}$. We show that $\text{path}_j(l)$ is an augmenting path.

Specifically, consider moving a flow $\eta > 0$ small enough from item i to customer j through $\text{path}_j(i)$ while keeping the spend of each customer in the path, except j, the same. Namely, denote $\eta_r = \eta$ and $\eta_0 = 0$, and for each $s \in \{1,\dots,r\}$ consider increasing the allocation $x_{k_s l_s,1}^{MY}$ by η_s, and decreasing the allocation $x_{k_{s-1} l_s,1}^{MY}$ by η_{s-1}, such that $p_{k_s l_s}\eta_s = p_{k_{s-1} l_s}\eta_{s-1}$ for each $s \in \{2,\dots,r\}$. Equivalently, $\eta_s = \prod_{q=s+1}^{r} \frac{p_{k_q l_q}}{p_{k_{q-1} l_q}}\eta$ for each $s \in \{1,\dots,r\}$. We now verify that this change to $\mathbf{x}_1^{MY}$ is feasible for $\eta > 0$ small enough and strictly improves the revenue collected by the platform, a contradiction.

The feasibility of the change to $\mathbf{x}_1^{MY}$ is guaranteed for any $\eta > 0$ such that

$$\eta \le \min\left(\min_{s\in\{1,\dots,r-1\}}\left(\prod_{q=s+1}^{r}\frac{p_{k_{q-1}l_q}}{p_{k_q l_q}}x^{MY}_{k_s l_{s+1},1}\right), \frac{b_{j,1}-\sum_{k\in A_j}p_{kj}x^{MY}_{kj,1}}{p_{k_1 j}}\prod_{q=2}^{r}\frac{p_{k_{q-1}l_q}}{p_{k_q l_q}}, s_i - \sum_{l\in C_j}x^{MY}_{il,1}\right),$$

where the first term in the outer min guarantees the non-negativity of the modified allocations (i.e. $\eta_s \le x^{MY}_{k_s l_{s+1},1}$ for each $s\in\{1,\dots,r-1\}$, where $x^{MY}_{k_s l_{s+1},1} > 0$ by the construction of the class C_j in Definition 2), the second term in the outer min guarantees the budget feasibility for customer j (i.e. $\sum_{i\in A_j}p_{ij}x^{MY}_{ij,1} + p_{k_1 j}\eta_1 \le b_{j,1}$), and the third term in the outer min guarantees the supply feasibility for item i.

Finally, since by construction the change to $\mathbf{x}_1^{MY}$ keeps the spend of each customer in $\text{path}_j(i)$, except j, the same, then it follows that the strict increase in the revenue collected by the platform is equal to $p_{k_1,j}\eta_1 = p_{k_1,j}\prod_{q=2}^{r}\frac{p_{k_q l_q}}{p_{k_{q-1}l_q}}\eta > 0$. This concludes the proof. $\square$

Before proving Proposition 4, we also require the next auxiliary lemma, which states that, if $\gamma = 0$, then the allocation to a customer j will remain budget-feasible in the next period. We note that the proof depends crucially on Assumption 1(ii), while the proof can be extended to different, including un-normalized, service quality metrics.

Lemma 3. *Let $\mathbf{x}_1^{MY}$ be a myopic allocation with budgets $\mathbf{b}_1$, and let $\mathbf{b}_2$ be the updated budgets, i.e. $\mathbf{b}_2 = \boldsymbol{\phi}(\mathbf{b}_1, \mathbf{q}(\mathbf{b}_1, \mathbf{x}_1^{MY}))$. Then, under Assumptions 1 and 2 with $\gamma = 0$, $\mathbf{x}_1^{MY} \in \mathbf{X}(\mathbf{b}_2)$, i.e.*

$$\sum_{i\in A_l}p_{il}x^{MY}_{il,1} \le b_{l,2} \text{ for each customer } l. \tag{EC.1}$$

Proof. First, assume $b_{j,1} \le q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1})$. Note that then

$$b_{j,2} = \phi_j(b_{j,1}, q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1})) \ge \min(b_{j,1}, q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1})) = b_{j,1} \ge \sum_{i\in A_j}p_{ij}x^{MY}_{ij,1},$$

where the first inequality follows from Assumption 1(ii).

Now assume $b_{j,1} > q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1})$. Then,

$$b_{j,2} = \phi_j(b_{j,1}, q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1})) \ge q_j(b_{j,1}, \mathbf{x}^{MY}_{j,1}) = \frac{\sum_{i\in A_j}p_{ij}x^{MY}_{ij,1}}{\min\left\{b_{j,1}, \sum_{i\in A_j}p_{ij}s_i\right\}} \ge \sum_{i\in A_j}p_{ij}x^{MY}_{ij,1}.$$

where the first inequality follows from Assumption 1(ii), the second equality follows from Assumption 2 with $\gamma = 0$ (cf. equation (10)), and the second inequality follows from $b_{j,1}\in[0,1]$. $\square$

Having established these auxiliary results, we are now ready to prove Proposition 4:

Proof of Proposition 4. Assume, for contradiction, that there exists a customer j such that $\sum_{i\in A_j} p_{ij} x^{MY}_{ij,1} < b_{j,1}$, *and* $\mu_{j,2} > 0$ for all dual optimal solutions associated to all myopic allocations with budgets $\mathbf{b}_2$.

First, note that $\mu_{j,2} > 0$ for *all* dual optimal solutions associated to *all* myopic allocations with budgets $\mathbf{b}_2$ implies that for a scalar $\delta > 0$ small enough, $\tilde{\mu}_{j,2} > 0$ for *all* dual optimal solutions associated to *all* myopic allocations with budgets $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 + \delta \mathbf{e}_j$. In particular, let $\tilde{\mathbf{x}}_2$ be an optimal myopic allocation with budgets $\tilde{\mathbf{b}}_2$, i.e. $\tilde{\mathbf{x}}_2 \in \mathbf{X}\left(\tilde{\mathbf{b}}_2\right)$ and $R(\tilde{\mathbf{x}}_2) = \mathrm{MY}(\tilde{\mathbf{b}}_2)$.

Since $\sum_{i\in A_j} p_{ij} x^{MY}_{ij,1} < b_{j,1}$, let C_j be the associated class of customers from Definition 2. We will show that assuming $\tilde{\mu}_{j,2} > 0$ for *all* dual optimal solutions associated to *all* myopic allocations with budgets $\tilde{\mathbf{b}}_2$ implies that all the customers in the class C_j spent at least as much under allocation $\tilde{\mathbf{x}}_2$ than under allocation $\mathbf{x}^{MY}_1$, and customer j spent strictly more. Namely, that the class of customers C_j is such that

$$\sum_{i\in A_k} p_{ik} x^{MY}_{ik,1} \le \sum_{i\in A_k} p_{ik} \tilde{x}_{ik,2} \text{ for each } k \in C_j, \text{ and}$$
$$\sum_{i\in A_j} p_{ij} x^{MY}_{ij,1} < \sum_{i\in A_j} p_{ij} \tilde{x}_{ij,2} \text{ for } j. \tag{EC.2}$$

Before proving that (EC.2) holds, we argue that (EC.2) leads to a contradiction. Specifically, since customer j spent strictly more under allocation $\tilde{\mathbf{x}}_2$ than under allocation $\mathbf{x}^{MY}_1$, i.e. $\sum_{i\in A_j} p_{ij} x^{MY}_{ij,1} < \sum_{i\in A_j} p_{ij} \tilde{x}_{ij,2}$, and from Lemma 2 all the items that customers in the class C_j are interested in are *fully* allocated by $\mathbf{x}^{MY}_1$ to customers in the class C_j, i.e. $\sum_{l\in C_j} x^{MY}_{il,1} = s_i$ for all $i \in \bigcup_{l\in C_j} A_l$, then there must exist a customer $k_1 \in C_j \setminus \{j\}$ and item $i_1 \in A_j \cap A_{k_1}$ such that some fraction of item i_1 that was assigned by $\mathbf{x}^{MY}_1$ to customer k_1 is now assigned by $\tilde{\mathbf{x}}_2$ to customer j, i.e. $x^{MY}_{i_1k_1,1} > \tilde{x}_{i_1k_1,2} \ge 0$ and $0 \le x^{MY}_{i_1j,1} < \tilde{x}_{i_1j,2}$. Similarly, since all the customers in the class C_j spent at least as much under allocation $\tilde{\mathbf{x}}_2$ than under allocation $\mathbf{x}^{MY}_1$, i.e. $\sum_{i\in A_k} p_{ik} x^{MY}_{ik,1} \le \sum_{i\in A_k} p_{ik} \tilde{x}_{ik,2}$ for each $k \in C_j$, then there must be a customer $k_2 \in C_j \setminus \{j, k_1\}$ and item $i_2 \in A_{k_1} \cap A_{k_2}$ such that some fraction of item i_2 that was assigned by $\mathbf{x}^{MY}_1$ to customer k_2 is now assigned by $\tilde{\mathbf{x}}_2$ to customer k_1, i.e. $x^{MY}_{i_2k_2,1} > \tilde{x}_{i_2k_2,2} \ge 0$ and $0 \le x^{MY}_{i_2k_1,1} < \tilde{x}_{i_2k_1,2}$. Note that we can assume $k_2 \ne j$ without loss of generality, otherwise we would obtain a feasible augmenting path to improve $\mathbf{x}^{MY}_1$, contradicting its myopic optimality. By iterating this argument $|C_j|$ times, we conclude that either there must exist a feasible augmenting path to improve $\mathbf{x}^{MY}_1$, or there must exist an advertiser $k \in C_j$ such that $\sum_{i\in A_k} p_{ik} x^{MY}_{ik,1} > \sum_{i\in A_k} p_{ik} \tilde{x}_{ik,2}$, a contradiction with (EC.2).

What is left to be shown, is that $\tilde{\mu}_{j,2} > 0$ for *all* dual optimal solutions associated to *all* myopic allocations with budgets $\tilde{\mathbf{b}}_2$ implies equation (EC.2). We *reconstruct* the class C_j as follows. We iteratively construct a class of customers $E_j \subseteq \{1, \ldots, m\}$ such that in finitely many iterations $E_j = C_j$ and (EC.2) is satisfied, which leads to a contradiction as previously discussed. Specifically, we start with $E_j = \{j\} \subseteq C_j$, and note that $\tilde{\mu}_{j,2} > \mu_{j,1} = 0$. Then, we add a customer $l \in C_j \setminus E_j$ to the class E_j in each iteration such that the updated class preserves the properties that $\tilde{\mu}_{k,2} > \mu_{k,1} \geq 0$ for all $k \in E_j$, and $E_j \subseteq C_j$. In more details, each iteration follows the next three steps.

1. We show that since, by construction, $\tilde{\mu}_{k,2} > \mu_{k,1} \geq 0$ for all $k \in E_j$, all the customers in the class E_j spent at least as much under allocation $\tilde{\mathbf{x}}_2$ than under allocation $\mathbf{x}_1^{MY}$, and customer j spent strictly more. Namely, that the class of customers E_j is such that

$$\sum_{i \in A_k} p_{ik} x_{ik,1}^{MY} \leq b_{k,2} \leq \tilde{b}_{k,2} = \sum_{i \in A_k} p_{ik} \tilde{x}_{ik,2} \text{ for each } k \in E_j, \text{ and}$$
$$\sum_{i \in A_j} p_{ij} x_{ij,1}^{MY} \leq b_{j,2} < \tilde{b}_{j,2} = \sum_{i \in A_j} p_{ij} \tilde{x}_{ij,2} \text{ for } j \in E_j. \tag{EC.3}$$

   In both cases in (EC.3) the first inequality follows from Lemma 3 (where Assumptions 1 and 2 with $\gamma = 0$ are used), the second inequality follows from $\tilde{\mathbf{b}}_2 = \mathbf{b}_2 + \delta \mathbf{e}_j$ and $j \in E_j$, and the last equality follows from $\tilde{\mu}_{k,2} > \mu_{k,1} \geq 0$ for all $k \in E_j$.
2. Since, by construction, $E_j \subseteq C_j$, and from Lemma 2 all the items that customers in the class C_j are interested in are *fully* allocated by $\mathbf{x}_1^{MY}$ to customers in the class C_j, i.e. $\sum_{l \in C_j} x_{il,1}^{MY} = s_i$ for all $i \in \bigcup_{l \in C_j} A_l$, then note that (EC.3) implies that there exists a customer $l \in C_j \setminus E_j$ and item $i \in \bigcup_{k \in E_j} A_k \cap A_l$ such that some fraction of item i that was assigned by $\mathbf{x}_1^{MY}$ to customer l is now assigned by $\tilde{\mathbf{x}}_2$ to some customer $k \in E_j$, i.e. $x_{il,1}^{MY} > \tilde{x}_{il,2} \geq 0$ and $0 \leq x_{ik,1}^{MY} < \tilde{x}_{ik,2}$ for some $l \in C_j \setminus E_j$, $k \in E_j$, and $i \in A_k \cap A_l$.
3. Consider the customers $l \in C_j \setminus E_j$ and $k \in E_j$, and the item $i \in A_k \cap A_l$, characterized in step 2. We now show that $\tilde{\mu}_{l,2} > \mu_{l,1} \geq 0$. Specifically,

$$(1 - \mu_{l,1}) p_{il} = \lambda_{i,1} \geq (1 - \mu_{k,1}) p_{ik} > (1 - \tilde{\mu}_{k,2}) p_{ik} = \lambda_{i,2} \geq (1 - \tilde{\mu}_{l,2}) p_{il}, \tag{EC.4}$$

   which implies $\tilde{\mu}_{l,2} > \mu_{l,1} \geq 0$. In (EC.4), the first equality follows from $x_{il,1}^{MY} > 0$ and complementary slackness, the first and last inequalities follow from dual feasibility, the second inequality follows from $\tilde{\mu}_{k,2} > \mu_{k,1}$ since $k \in E_j$, the second equality follows from $\tilde{x}_{ik,2} > 0$ and complementary slackness.

   Finally, update the class of customers E_j by adding customer $l \in C_j \setminus E_j$ to it, i.e. let $E_j = E_j \cup \{l\}$. Hence, the updated class E_j is such that $\tilde{\mu}_{k,2} > \mu_{k,1} \geq 0$ for all $k \in E_j$, and $E_j \subseteq C_j$. Iterate by going to step 1.

To summarize, we start with $E_j = \{j\} \subseteq C_j$ and in each iteration we add a customer $l \in C_j \setminus E_j$ to the class E_j, such that (EC.3) is preserved. Since C_j has finitely many members, in finitely many iterations $E_j = C_j$ and (EC.2) is satisfied. Specifically, (EC.2) is equivalent to (EC.3) in step 1 of the iteration for $E_j = C_j$. Finally, (EC.2) leads to a contradiction as previously discussed. This concludes the proof. $\square$

Having completed the technical steps used in the proof of Theorem 2, we observe that we only require the following consequence of Assumption 2 with $\gamma = 0$ for the result to go through: that the service quality provided to each customer satisfies equation (10). This observation will be formalized in the next subsection.

We conclude this subsection by restating and proving Corollary 1.

Corollary 1. *Under Assumptions 1 and 2 with $\gamma = 0$, $J_T^*(\mathbf{b}_{1,h}) \geq J_T^*(\mathbf{b}_{1,l})$ for any horizon $T \geq 2$ and budget states $\mathbf{b}_{1,h}$, $\mathbf{b}_{1,l}$ such that $\mathbf{b}_{1,h} \geq \mathbf{b}_{1,l}$ component-wise.*

Proof. From Assumption 1(i) we have $\boldsymbol{\phi}^{t-1}(\mathbf{b}_{1,h}, 1) \geq \boldsymbol{\phi}^{t-1}(\mathbf{b}_{1,l}, 1)$ component-wise, for each $t \in \{2, \ldots, T\}$. Hence,

$$J_T^*(\mathbf{b}_{1,h}) = J_T^{\text{relax}}(\mathbf{b}_{1,h}) = \text{MY}(\mathbf{b}_{1,h}) + \sum_{t=2}^{T} \text{MY}\left(\boldsymbol{\phi}^{t-1}(\mathbf{b}_{1,h}, \mathbf{1})\right)$$

$$\geq \text{MY}(\mathbf{b}_{1,l}) + \sum_{t=2}^{T} \text{MY}\left(\boldsymbol{\phi}^{t-1}(\mathbf{b}_{1,l}, \mathbf{1})\right) = J_T^{\text{relax}}(\mathbf{b}_{1,l}) = J_T^*(\mathbf{b}_{1,l})$$

where the first and last equalities follow from Theorem 2, and the inequality follows from Lemma 1 at the start of Appendix EC.2. □

EC.2.2. Proofs for Theorem 1

We begin this section by providing a roadmap of the proof. Observe that, for any budget b_j and allocation $\mathbf{x}_j$, the service quality perceived by customer j can be bounded in the following way using Assumption 2:

$$\begin{aligned} q_j(b_j, \mathbf{x}_j) &= \frac{\sum_{i \in A_j}(v_{ij} - p_{ij})x_{ij}}{\sum_{i \in A_j}(v_{ij} - p_{ij})y_i^*(b_{j,1})} \\ &\geq \frac{\left(\min_{i \in A_j}\left\{\frac{v_{ij}}{p_{ij}}\right\} - 1\right)\sum_{i \in A_j} p_{ij}x_{ij}}{\left(\max_{i \in A_j}\left\{\frac{v_{ij}}{p_{ij}}\right\} - 1\right)\sum_{i \in A_j} p_{ij}y_i^*(b_j)} \\ &\geq (1-\gamma) \cdot \frac{\sum_{i \in A_j} p_{ij}x_{ij}}{\min\left(b_j, \sum_{i \in A_j} p_{ij}s_i\right)}. \end{aligned} \tag{EC.5}$$

Thus, the service quality can be lower bounded by a modified service quality, which essentially assumes $\gamma = 0$ and scales down the result by a factor of $1 - \gamma$.

Now, recall from Theorem 2, that *if* $\gamma = 0$, then $J_T^{MY}(\mathbf{b}_1)$ is equal to the revenues of the myopic policy in a system where service quality is always equal to 1 (which we denoted by $J_T^{relax}(\mathbf{b}_1)$). This result, coupled with equation (EC.5), suggests lower bounding $J_T^{MY}(\mathbf{b}_1)$ by a system where service quality is always equal to $1 - \gamma$ instead. Formally, we define this system as:

$$\begin{aligned} J_T^{LBMY}(\mathbf{b}_1) = \max_{\mathbf{x}_1, \ldots, \mathbf{x}_T} \; & \sum_{t=1}^{T} R(\mathbf{x}_t) \\ \text{s.t. } & \mathbf{x}_t \in \mathbf{X}\left(\boldsymbol{\phi}^{t-1}(\mathbf{b}_1, \mathbf{1} - \boldsymbol{\gamma})\right), \quad \forall t. \end{aligned} \tag{EC.6}$$

Analogous to problem (9), we emphasize that $J_T^{LBMY}(\mathbf{b}_1) = \text{MY}(\mathbf{b}_1) + \sum_{t=2}^{T} \text{MY}(\boldsymbol{\phi}^{t-1}(\mathbf{b}_1, \mathbf{1} - \boldsymbol{\gamma}))$. In order to prove Theorem 1, we will then show that, indeed,

$$J_T^{MY}(\mathbf{b}_1) \geq J_T^{LBMY}(\mathbf{b}_1).$$

We do so in Proposition 7 via a coupling argument. Having established this lower bound on the performance of the myopic policy, the bound of $(1-\gamma)$ with respect to the optimal revenue follows naturally, as concluded in the proof of Theorem 1.

We begin the proofs in this section by showing a corollary which proves an extension of Theorem 2 and Corollary 1 that is important in the derivation for the lower bound of the myopic policy performance.

Corollary 2. *Under Assumption 1, for any horizon T, initial budget state $\mathbf{b}_1$, and parameter $\gamma \in [0,1)$, Theorem 2 and Corollary 1 also hold for any service quality that satisfies equation (10), and such that it is scaled in the budget update function as $\phi_j(b_j, (1-\gamma)q_j)$ for each customer j.*

Proof. It is straightforward to replicate the proofs of Theorem 2 and Corollary 1 under these assumptions. We omit the details for the sake of brevity. □

The next proposition establishes that the simpler policy defined by (EC.6), where the service quality is set to $1-\gamma$ in each period, is indeed a valid lower bound on the performance of myopic.

Proposition 7. *Let $\{\mathbf{x}_t^{MY}\}_{1 \leq t \leq T}$ be the myopic policy defined in Section 3. Under Assumptions 1 and 2, $J_T^{MY}(\mathbf{b}_1) \geq J_T^{LBMY}(\mathbf{b}_1)$ for any horizon $T \geq 2$ and initial budget state $\mathbf{b}_1$.*

Proof. The proof structure is as follows. We consider the revenues collected by the myopic policy under different budget updates, parametrized by an index $k \in \{2, \dots, T\}$. Specifically, we assume that the true budget update $\mathbf{b}_{t+1} = \boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t, \mathbf{x}_t^{MY}))$ is followed between the first k periods, while a modified budget update is followed in the rest of the horizon. The modified budget update is specified below, but informally it is characterized by an alternative service quality measure $(1-\gamma)\tilde{\mathbf{q}}(\mathbf{b}_t, \mathbf{x}_t^{MY})$, where $\tilde{\mathbf{q}}(\mathbf{b}_t, \mathbf{x}_t^{MY})$ satisfies equation (10). One interpretation of this alternative service quality measure is that it assumes a constant bang-per-buck ratio for each customer and then it scales the result by $(1-\gamma)$.

Although the net effect of using the modified budget update on the total revenue collected by the myopic policy is a priori unclear, we show that it is actually nondecreasing in the index k (cf. equation (EC.7)), i.e. the larger the number of periods where the modified budget update is used the smaller the total revenue collected by the myopic policy. This result is useful since it implies the statement in the proposition as a special case (cf. equation (EC.8)).

More precisely, let $\{\mathbf{x}_t^{MY}\}_{t \in \{1,\dots,k\}}$ be the myopic policy starting from the initial budget state $\mathbf{b}_1$ and following the true budget update $\mathbf{b}_{t+1} = \boldsymbol{\phi}(\mathbf{b}_t, \mathbf{q}(\mathbf{b}_t, \mathbf{x}_t^{MY}))$ for each period $t \in \{1, \dots, k-1\}$. Additionally, let $\left\{\tilde{\mathbf{x}}_{k,t}^{MY}\right\}_{t \in \{k+1,\dots,T\}}$ be the myopic policy starting from the budget state $\mathbf{b}_k$ and following the modified budget update $\tilde{\mathbf{b}}_{k+1} = \boldsymbol{\phi}\left(\mathbf{b}_k, (1-\gamma)\tilde{\mathbf{q}}\left(\mathbf{b}_k, \mathbf{x}_k^{MY}\right)\right)$, and $\tilde{\mathbf{b}}_{t+1} = \boldsymbol{\phi}\left(\tilde{\mathbf{b}}_t, (1-\gamma)\tilde{\mathbf{q}}\left(\tilde{\mathbf{b}}_t, \tilde{\mathbf{x}}_{k,t}^{MY}\right)\right)$ for each period $t \in \{k+2, \ldots, T-1\}$, where the alternative service quality function $\tilde{q}_j\left(\tilde{b}_{j,t}, \tilde{\mathbf{x}}_{j,k,t}^{MY}\right) \triangleq \frac{\sum_{i\in A_j} p_{ij}\tilde{x}_{ij,k,t}^{MY}}{\min\left(\tilde{b}_{j,t}, \sum_{i\in A_j} p_{ij}s_i\right)}$ satisfies equation (10), and it is scaled by $(1-\gamma)$.

Then, we show that the total revenue collected by the myopic policy $\left\{\{\mathbf{x}_t^{MY}\}_{t\in\{1,\ldots,k\}}, \{\tilde{\mathbf{x}}_{k,t}^{MY}\}_{t\in\{k+1,\ldots,T\}}\right\}$ is nondecreasing in the index k. Specifically, for each $k \in \{2, \ldots, T\}$,

$$\sum_{t=1}^{k} R\left(\mathbf{x}_t^{MY}\right) + \sum_{t=k+1}^{T} R\left(\tilde{\mathbf{x}}_{k,t}^{MY}\right) \geq \sum_{t=1}^{k-1} R\left(\mathbf{x}_t^{MY}\right) + \sum_{t=k}^{T} R\left(\tilde{\mathbf{x}}_{k-1,t}^{MY}\right). \tag{EC.7}$$

Therefore, in particular,

$$J_T^{MY}(\mathbf{b}_1) = \sum_{t=1}^{T} R\left(\mathbf{x}_t^{MY}\right) \geq R\left(\mathbf{x}_1^{MY}\right) + \sum_{t=2}^{T} R\left(\tilde{\mathbf{x}}_{1,t}^{MY}\right) = J_T^{LBMY}(\mathbf{b}_1), \tag{EC.8}$$

where the last equality follows from Theorem 2 and Corollary 2.

We prove equation (EC.7). The proof is by induction on the length of the horizon T.

Base Case: Consider $T = 2$. Namely, we prove equation (EC.7) for $k = 2$, i.e. $R\left(\mathbf{x}_1^{MY}\right) + R\left(\mathbf{x}_2^{MY}\right) \geq R\left(\mathbf{x}_1^{MY}\right) + R\left(\tilde{\mathbf{x}}_{2,2}^{MY}\right)$. We show $\tilde{\mathbf{x}}_{2,2}^{MY} \in \mathbf{X}(\mathbf{b}_2)$, hence $R\left(\tilde{\mathbf{x}}_{2,2}^{MY}\right) \leq \mathrm{MY}(\mathbf{b}_2) = R\left(\mathbf{x}_2^{MY}\right)$. Specifically,

$$\begin{aligned}
b_{j,2} &= \phi_j\left(b_{j,1}, q_j\left(b_{j,1}, \mathbf{x}_{j,1}^{MY}\right)\right) \\
&= \phi_j\left(b_{j,1}, \frac{\sum_{i\in A_j}(v_{ij}-p_{ij})x_{ij,1}^{MY}}{\sum_{i\in A_j}(v_{ij}-p_{ij})y_i^*(b_{j,1})}\right) \\
&\geq \phi_j\left(b_{j,1}, \frac{\left(\min_{i\in A_j}\left\{\frac{v_{ij}}{p_{ij}}\right\}-1\right)\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY}}{\left(\max_{i\in A_j}\left\{\frac{v_{ij}}{p_{ij}}\right\}-1\right)\sum_{i\in A_j} p_{ij}y_i^*(b_{j,1})}\right) \\
&\geq \phi_j\left(b_{j,1}, (1-\gamma)\frac{\sum_{i\in A_j} p_{ij}x_{ij,1}^{MY}}{\min\left(b_{j,1}, \sum_{i\in A_j} p_{ij}s_i\right)}\right) \\
&= \phi_j\left(b_{j,1}, (1-\gamma)\tilde{q}_j\left(b_{j,1}, \mathbf{x}_{j,1}^{MY}\right)\right) \\
&= \tilde{b}_j^2 \\
&\geq \sum_{i\in A_j} p_{ij}\tilde{x}_{ij,2,2}^{MY},
\end{aligned}$$

where the first inequality follows from Assumption 1(i), the second inequality follows from Assumptions 1(i) and 2(i), and the third inequality follows from $\tilde{\mathbf{x}}_{2,2}^{MY} \in \mathbf{X}\left(\tilde{\mathbf{b}}_2\right)$.

Induction Step: For any $T \geq 3$ assume that equation (EC.7) holds for any problem with horizon $T-1$, and for each policy defined by a parameter $k \in \{2, \ldots, T-1\}$.

First, note that this implies that equation (EC.7) holds for any problem with horizon T, and for each policy defined by a parameter $k \in \{3, \ldots, T\}$. Specifically, equation (EC.7) for a problem with horizon $(T-1)\geq 2$ and a policy defined by a parameter $k\in\{2,\dots,T-1\}$ implies equation (EC.7) for a problem with horizon T and a policy defined by a parameter $(k+1)$. Namely,

$$\begin{aligned}\sum_{t=1}^{k+1} R\left(\mathbf{x}_t^{MY}\right)+\sum_{t=k+2}^{T} R\left(\tilde{\mathbf{x}}_{(k+1),t}^{MY}\right) &= R\left(\mathbf{x}_1^{MY}\right)+\sum_{t=2}^{k+1} R\left(\mathbf{x}_t^{MY}\right)+\sum_{t=k+2}^{T} R\left(\tilde{\mathbf{x}}_{(k+1),t}^{MY}\right)\\ &\geq R\left(\mathbf{x}_1^{MY}\right)+\sum_{t=2}^{k} R\left(\mathbf{x}_t^{MY}\right)+\sum_{t=k+1}^{T} R\left(\tilde{\mathbf{x}}_{k,t}^{MY}\right)\\ &= \sum_{t=1}^{k} R\left(\mathbf{x}_t^{MY}\right)+\sum_{t=k+1}^{T} R\left(\tilde{\mathbf{x}}_{k,t}^{MY}\right).\end{aligned}$$

It remains to show that equation (EC.7) holds for $k=2$. Namely, $\sum_{t=1}^{2} R\left(\mathbf{x}_t^{MY}\right)+\sum_{t=3}^{T} R\left(\tilde{\mathbf{x}}_{2,t}^{MY}\right)\geq R\left(\mathbf{x}_1^{MY}\right)+\sum_{t=2}^{T} R\left(\tilde{\mathbf{x}}_{1,t}^{MY}\right)$. Or equivalently,

$$R\left(\mathbf{x}_2^{MY}\right)+\sum_{t=3}^{T} R\left(\tilde{\mathbf{x}}_{2,t}^{MY}\right)\geq \sum_{t=2}^{T} R\left(\tilde{\mathbf{x}}_{1,t}^{MY}\right). \tag{EC.9}$$

Equation (EC.9) follows from Corollaries 1 and 2. Specifically, inequality (EC.9) compares the revenue collected by two myopic allocations with the same scaled budget update function $\tilde{\mathbf{b}}_{t+1}=\boldsymbol{\phi}\left(\tilde{\mathbf{b}}_t,(1-\gamma)\tilde{\mathbf{q}}\left(\tilde{\mathbf{b}}_t,\tilde{\mathbf{x}}_{k,t}^{MY}\right)\right)$ with initial budgets $\mathbf{b}_2$ and $\tilde{\mathbf{b}}_2$, respectively, such that $\mathbf{b}_2\geq\tilde{\mathbf{b}}_2$ component-wise. This concludes the proof of equation (EC.7).

Finally, equation (EC.8) follows from equation (EC.7) by comparing the largest left hand side $(k=T)$ with the smallest right hand side $(k=2)$. This concludes the proof. □

We are now ready to complete the proof of the main result in Section 4.

Theorem 1. *For any horizon T and initial budget state $\mathbf{b}_1$, let $\{\mathbf{x}_t^{MY}\}_{1\leq t\leq T}$ be the myopic policy defined in Section 3.*

Then, under Assumptions 1 and 2, $\{\mathbf{x}_t^{MY}\}_{1\leq t\leq T}$ is $(1-\gamma)$-optimal for problem (5), i.e.

$$J_T^{MY}\left(\mathbf{b}_1\right)\geq(1-\gamma)\cdot J_T^{*}\left(\mathbf{b}_1\right).$$

Proof. We have the following chain of inequalities,

$$J_T^{MY}\left(\mathbf{b}_1\right)\geq J_T^{LBMY}\left(\mathbf{b}_1\right)\geq(1-\gamma)J_T^{\mathrm{relax}}\left(\mathbf{b}_1\right)\geq(1-\gamma)J_T^{*}\left(\mathbf{b}_1\right),$$

where the first and last inequalities follow from Propositions 7 and 2, respectively. The second inequality follows from Assumption 1(iii), since then any solution to problem (9) scaled by $(1-\gamma)$ is feasible in problem (EC.6). Specifically, $(1-\gamma)\mathbf{X}\left(\boldsymbol{\phi}^{t-1}\left(\mathbf{b}_1,\mathbf{1}\right)\right)\subseteq\mathbf{X}\left(\boldsymbol{\phi}^{t-1}\left(\mathbf{b}_1,\mathbf{1}-\boldsymbol{\gamma}\right)\right)$ for each $t\in\{1,\dots,T\}$. This completes the proof. □

We conclude this subsection by proving the tightness of the bound in Theorem 1.

Proposition 3. *The instance described in Example 1 achieves the bound in Theorem 1 as $\delta\to 0$.*

Proof. We explicitly construct the allocation under the myopic policy and a feasible policy whose revenues lower bound those of the optimal policy.

Myopic Policy Allocation: In $t=1$, the myopic policy sets $x_{11,1}^{MY}=\delta$, $x_{22,1}^{MY}=\delta$, and $x_{12,1}^{MY}=x_{21,1}^{MY}=0$. The first-period revenue is $\delta\cdot(p_1+p_2)$. Note that $x_{22,1}^{MY}$ supply exhausts the supply of product two, and this is the unique myopic solution.

The first customer's utility in the first period is $U_1(b_{1,1},\mathbf{x}_{1,1}^{MY})=(v_{11}-p_1)\delta$ while $U_1^*(b_{1,1})=(v_{12}-p_1)\delta$. Thus, $q_{1,1}=\frac{(v_{11}-p_1)\delta}{(v_{12}-p_1)\delta}=1-\gamma$ and the first customer's second period initial budget is $b_{1,2}=\alpha_1 b_{1,1}+(1-\alpha_1)(1-\gamma)$. The second customer's utility in the first period is $U_2(b_{2,1},\mathbf{x}_{2,1}^{MY})=(v_{22}-p_2)\delta$ while $U_2^*(b_{2,1})=(v_{22}-p_2)\delta$. Thus, $q_{2,1}=\frac{(v_{22}-p_2)\delta}{(v_{22}-p_2)\delta}=1$ and the second customer's second period initial budget is $b_{2,2}=\alpha_2 b_{2,1}+(1-\alpha_2)\geq b_{2,1}$.

In $t=2$, the myopic policy sets $x_{11,2}^{MY}=\frac{b_{1,2}}{p_1}=\frac{\alpha_1 b_{1,1}+(1-\alpha_1)(1-\gamma)}{p_1}$, $x_{22,2}^{MY}=\delta$, and $x_{12,2}^{MY}=x_{21,2}^{MY}=0$. Note that we are able to allocate the whole supply of the second product to the second customer since $b_{2,2}\geq b_{2,1}$. We are also able to spend all the budget of the first customer since the supply of good one is large.

The second period myopic revenue is

$$b_{1,2}+\delta p_2=\alpha_1 b_{1,1}+(1-\alpha_1)(1-\gamma)+\delta p_2=\delta(\alpha_1 p_1+p_2)+(1-\alpha_1)(1-\gamma).$$

The total profit of the myopic policy over the two periods is

$$J_2^{MY}(\mathbf{b}_1)=\delta((1+\alpha_1)p_1+2p_2)+(1-\alpha_1)(1-\gamma).$$

We now examine the allocation policy which lower bounds the optimal.

Optimal Policy Lower Bound Allocation: We denote this policy π. In $t=1$, the optimal policy sets $x_{21,1}^{\pi}=\delta$, and $x_{12,1}^{\pi}=x_{21,1}^{\pi}=x_{22,1}^{\pi}=0$. The first-period revenue is $\delta\cdot p_1$, which is strictly less than the myopic policy.

The first customer's utility in the first period is $U_1(b_{1,1},\mathbf{x}_{1,1}^{\pi})=U_1^*(b_{1,1})=(v_{12}-p_1)\delta$. Thus, $q_{1,1}=1$ and the first customer's second period initial budget is $b_{1,2}=\alpha_1 b_{1,1}+(1-\alpha_1)$.

The second customer's utility in the first period is $U_2(b_{2,1},\mathbf{x}_{2,1}^{\pi})=0$ while $U_2^*(b_{2,1})=(v_{22}-p_2)\delta$. Thus, $q_{2,1}=0$ and the second customer's second period initial budget is $b_{2,2}=\alpha_2 b_{2,1}\leq b_{2,1}$.

In $t=2$, the lower bound policy sets $x_{11,2}^{\pi}=b_{1,2}/p_1, x_{22,2}^{\pi}=b_{2,2}/p_2$, and $x_{12,2}^{\pi}=x_{21,2}^{*}=0$. As before, we are able to spend all the budget of the first customer since the supply of the first good is large. The lower policy's revenue in the second period is

$$b_{1,2}+b_{2,2}=\alpha_1 b_{1,1}+(1-\alpha_1)+\alpha_2 b_{2,1}=\delta(\alpha_1 p_1+\alpha_2 p_2)+(1-\alpha_1).$$

The total revenue of the lower bound policy over the two periods is

$$J_2^{\pi}(\mathbf{b}_1)=\delta((1+\alpha_1)p_1+\alpha_2 p_2)+(1-\alpha_1).$$

The ratio between the two revenues is

$$\frac{J_2^{MY}(\mathbf{b}_1)}{J_2^*(\mathbf{b}_1)} \leq \frac{J_2^{MY}(\mathbf{b}_1)}{J_2^{\pi}(\mathbf{b}_1)} = \frac{\delta((1+\alpha_1)p_1+2p_2)+(1-\alpha_1)(1-\gamma)}{\delta((1+\alpha_1)p_1+\alpha_2 p_2)+(1-\alpha_1)}. \tag{EC.10}$$

Thus, if the initial budgets are very small, i.e. $\delta \to 0$, we have

$$\lim_{\delta\to 0}\frac{J_2^{MY}(\mathbf{b}_1)}{J_2^*(\mathbf{b}_1)} = 1-\gamma,$$

achieving the bound from Theorem 1. □

EC.3. Proofs from Section 5

In this section, we prove our result regarding the sub-optimality of L look-ahead policies.

Proposition 5. *On the instance described in Example 2, with $\eta > 2$ and $m \geq \max\left\{\frac{1}{\alpha^2}, \frac{2(\eta-1)}{\eta-2}\right\}$, both the myopic policy and the L-step look-ahead policy, for any $L \geq 1$, produce revenues that are an $o\left(\frac{\log m}{m}\right)$ fraction of the optimum.*

Proof. We first prove this statement for the myopic policy MY. The proof proceeds in two parts. First, in part 1 we characterize the allocation that the myopic policy induces and upper bound the revenues it can garner over the T time periods. In the second part, we construct a sub-optimal policy and lower bound the revenues it garners. The result then follows by comparing these upper and lower bounds.

Part 1. Assume that at some time t all customers $1 \leq j \leq m$ have their current budget $b_{j,t} = \frac{\eta}{\eta+m-1}$. For this budget level, consider an allocation $\mathbf{x}^{\text{uniform}}$ which, for all $j \in \{1,\dots,m\}$, allocates product 1^j to customer j and then fills their remaining budgets with product 2. Finally, $\mathbf{x}^{\text{uniform}}$ fills customer $(m+1)$'s budget with the remaining supply of product 2. Namely,

$$x_{1^l j,t}^{\text{uniform}} = \begin{cases} \frac{1}{m^2}, & \text{if } l=j \text{ and } l,j\in\{1,\dots,m\},\\ 0, & \text{if } l\neq j \text{ and } l,j\in\{1,\dots,m\},\end{cases}$$
$$x_{2j}^{\text{uniform}} = \begin{cases} \frac{\eta}{\eta+m-1}-\frac{1}{m^2}, & \text{if } j\in\{1,\dots,m\},\\ m-\frac{\eta m}{\eta+m-1}+\frac{1}{m}, & \text{if } j=m+1.\end{cases} \tag{EC.11}$$

It is easy to verify that $\mathbf{x}^{\text{uniform}}$ is budget feasible for any $\eta > 2$ and $m \geq 1$.

Applying Lemma 4, where the assumptions that $\eta > 2$ and $m \geq \left\{\frac{1}{\alpha^2}, \frac{2(\eta-1)}{\eta-2}\right\}$ are used, we know that at $t=1$ with a budget $\mathbf{b}_1$ as defined in Example 2, the myopic policy implements the allocation $\mathbf{x}^{\text{uniform}}$. It is easy to verify that $\phi(b_{j,1}, q(b_{j,1}, \mathbf{x}_j^{\text{uniform}})) = b_{j,1} = \frac{\eta}{\eta+m-1}$ for all customers $j \in \{1,\dots,m\}$. Thus, $\mathbf{b}_1$ is a fixed-point under the myopic policy; this implies that MY satisfies $b_{j,t} = \frac{\eta}{\eta+m-1}$ for all customers $j \in \{1,\dots,m\}$ and periods $t = 1,\dots,T$. We can then construct an upper bound on J^{MY} by assuming that the myopic policy completely exhausts customer budgets in each period over the horizon T:

$$J^{\text{MY}}(\mathbf{b}_1) \leq T\left(\frac{\eta m}{\eta+m-1}+1\right) = 3\log_\alpha(e^{-1})\log m\left(\frac{\eta m}{\eta+m-1}+1\right) = \Theta(\log m), \tag{EC.12}$$

where the term $\frac{\eta m}{\eta+m-1}$ corresponds to the budgets of customers $1,\ldots,m$, and the 1 corresponds to the budget of customer $(m+1)$.

Part 2. Now, let us consider an alternative policy π^{deplete}, which concentrates all the items on customer 1 for each period t such that $1 \le t \le T-2$. Namely, for $1 \le t \le T-2$, π^{deplete} sets:

$$x^{\text{deplete}}_{1^l j,t} = \begin{cases} \frac{1}{m^2}, \text{ if } j=1 \\ 0, \text{ otherwise,} \end{cases}$$

and fills the remaining customer budgets with product 2. Clearly, for all customers $2 \le j \le m$ and periods $t \le T-2$, $q_{j,t}(b_{j,t}, \mathbf{x}^{\text{deplete}}_{j,t}) = 0$ so that $b_{j,t+1} = \alpha b_{j,t}$. Thus, under the π^{deplete} policy, the budget of each customer $2 \le j \le m$ at time $T-1$ will be small enough such that the customer can be fully satisfied by only allocating a fraction of item 1^j to her. Namely,

$$b_{j,T-1} = \alpha^{T-2}\frac{\eta}{\eta+m-1} = \alpha^{\log_\alpha(m^{-3})-2}\frac{\eta}{\eta+m-1} = \frac{1}{\alpha^2 m^3}\frac{\eta}{\eta+m-1} \le \frac{1}{\alpha^2 m^3} \le \frac{1}{m^2},$$

where the first inequality follows from $\frac{\eta}{\eta+m-1} \le 1$ for any $m \ge 1$, and the second inequality follows from the assumption $m \ge \frac{1}{\alpha^2}$. Then, at period $T-1$ the policy π^{deplete} sets:

$$x^{\text{deplete}}_{1^l j,T-1} = \begin{cases} 0, \text{ if } j=1 \\ \alpha^{T-2}\frac{\eta}{\eta+m-1}, \text{ if } j \ne 1, l=j \\ 0, \text{ otherwise,} \end{cases}$$

thus obtaining $q_{j,T-1}(b_{j,T-1}, \mathbf{x}^{\text{deplete}}_{j,T-1}) = 1$ and $b_{j,T} = \alpha b_{j,T-1} + (1-\alpha) \ge 1-\alpha$ for all customers $2 \le j \le m$. Lastly, in period T, π^{deplete} exhausts the budgets of all customers $2 \le j \le m$ with products of type 2. Therefore,

$$J^{\pi^{\text{deplete}}}(\mathbf{b}) \ge (1-\alpha)(m-1) = \Theta(m). \tag{EC.13}$$

Combining equations (EC.12) and (EC.13), we obtain that:

$$\frac{J^{\text{MY}}(\mathbf{b}_1)}{J^*(\mathbf{b}_1)} \le \frac{J^{\text{MY}}(\mathbf{b}_1)}{J^{\pi^{\text{deplete}}}(\mathbf{b}_1)} = o\left(\frac{\log m}{m}\right).$$

Where the inequality holds since $J^{\pi^{\text{deplete}}}(\mathbf{b}_1)$ is a lower bound on $J^*(\mathbf{b}_1)$.

The proof that

$$\frac{J^{\pi^{L\text{-LA}}}(\mathbf{b}_1)}{J^*(\mathbf{b}_1)} = o\left(\frac{\log m}{m}\right),$$

proceeds similarly and is omitted. The main difference is that we use a different argument to show that starting at $\mathbf{b}_1$, $\pi^{L\text{-LA}}$ allocates uniformly products 1^j to j, thus implying that $\mathbf{b}_1$ remains a steady state budget. This argument is precisely described in Lemma 4. □

Lemma 4. *On the instance described in Example 2, and for $\eta > 2$ and $m \ge \left\{\frac{1}{\alpha^2}, \frac{2(\eta-1)}{\eta-2}\right\}$, both the myopic and L-step look-ahead policies produce $\mathbf{x}^{uniform}$ as the optimal allocation for $t=1$.*

Proof. We prove, separately for the case of the myopic policy and the L-step look-ahead policy, that $\mathbf{x}^{\text{uniform}}$ is yielded as the optimal allocation on the instance described in Example 2.

Part 1: Myopic policy. Note that at period $t=1$ all customers $1 \leq j \leq m$ start with equal budget levels. Therefore, by symmetry, the optimal allocation can be either one which allocates each good $\{1^j\}_{1\leq j\leq m}$ to the corresponding customers $1 \leq j \leq m$, which is $\mathbf{x}^{\text{uniform}}$, or one which concentrates all these goods to, without loss of generality, customer 1 only. By examination, we can see that at this budget level, $\mathbf{x}^{\text{uniform}}_{2j} = \frac{\eta}{\eta+m-1} - \frac{1}{m^2}$ for customers $1 \leq j \leq m$, and thus $\mathbf{x}^{\text{uniform}}_{2(m+1)} = m - \frac{m\eta}{\eta+m-1} - \frac{1}{m} > 0$ for customer $(m+1)$, while all customers $1 \leq j \leq m$ have their budgets exhausted. Since the price garnered by allocating good 1^j to customer j is higher than the price of allocating some other good 1^l, $l \neq j$, to customer j, then under the concentrated allocation there is less leftover of good 2 to allocate to customer $(m+1)$, and as such $\mathbf{x}^{\text{uniform}}$ garners more revenues in this period than the concentrated allocation. This highlights the purpose of customer $(m+1)$ in Example 2: without this customer, the optimal myopic solution at $t=1$ would be non-unique, with both concentrated and uniform allocations yielding the same objective value, whereas the presence of customer $(m+1)$ solves this issue and guarantees that $\mathbf{x}^{\text{uniform}}$ is the unique myopic allocation.

Part 2: L-step look-ahead. Consider the L-step look-ahead policy with a starting budget $b_{j,1} = \frac{\eta}{\eta+m-1}$ for $1 \leq j \leq m$. Note that the structure of the problem is such that it is always optimal to fully exhaust the budget of a customer $1 \leq j \leq m$; moreover, since the supply of good 2 is m, it is always possible to exhaust these budgets. Thus, the optimal sequence of allocations produced by $\pi^{L\text{-LA}}$ is the one that maximizes $\sum_{t=1}^{L+1}\sum_{j=1}^{m} b_{j,t}$.

First, note that under any sequence of allocations $\mathbf{z}^1, \ldots, \mathbf{z}^{L+1}$, the budget trajectories in the $L+1$ periods, namely $\mathbf{b}_1, \ldots, \mathbf{b}_{L+1}$, are such that for any j, $b_{j,t} \geq \alpha^L \frac{\eta}{\eta+m-1} \geq \frac{\eta}{2(\eta+m-1)}$. This implies $\frac{1}{m^2} \leq \frac{\eta}{2(\eta+m-1)} \leq b_{j,t}$, where the first inequality holds for any $m \geq 2$ and $\eta \geq 2$, and thus $\mathbf{x}^{\text{uniform}}$ is feasible for any $\mathbf{b}_t$ along the budget path. Moreover,

$$U_j^*(b_{j,t}) = \eta m \frac{1}{m^2} + m \min\left\{ \frac{b_{j,t} - \frac{1}{m^2}}{1 - \frac{1}{m}}, \frac{m-1}{m^2} \right\} = \frac{\eta}{m} + \min\left\{ \frac{m^2 b_{j,t} - 1}{m-1}, \frac{m-1}{m} \right\}, \tag{EC.14}$$

and, since $b_{j,t} \geq \frac{\eta}{2(\eta+m-1)}$, then $\frac{m^2 b_{j,t}-1}{m-1} \geq \frac{\frac{\eta m^2}{2(\eta+m-1)}-1}{m-1} \geq 1 \geq \frac{m-1}{m}$ as long as $\eta > 2$ and $m \geq \frac{2(\eta-1)}{\eta-2}$. Thus, equation (EC.14) becomes

$$U_j^*(b_{j,t}) = \frac{\eta - 1}{m} + 1, \quad \text{for } t \leq L+1. \tag{EC.15}$$

Now observe that given any budget level $\mathbf{b}_t$ such that $b_{j,t} \geq \frac{\eta}{2(\eta+m-1)}$ and for any feasible allocation $\mathbf{x}$,

$$\sum_j q_{j,t}(b_{j,t}, \mathbf{x}_j) = \sum_j \frac{\eta m x_{1^j j} + m \sum_{l\neq j} x_{1^l j}}{U_j^*(b_{j,t})} = \frac{\eta m \sum_j x_{1^j j} + m \sum_j \sum_{l \neq j} x_{1^l j}}{\frac{\eta-1}{m} + 1}.$$

We can cast optimizing $\sum_j q_{j,t}(b_{j,t}, \mathbf{x}_j)$ as the fractional knapsack problem:

$$\max_{\mathbf{x}\geq 0} \frac{\eta m \sum_j x_{1^j j} + m \sum_j \sum_{l\neq j} x_{1^l j}}{\frac{\eta-1}{m}+1}$$
$$\text{s.t.} \sum_j x_{1^l j} \leq \frac{1}{m^2}, \quad \forall l.$$

Since we have set $\eta > 2$, the solution to the knapsack is the one that allocates all the supply of good 1^j to customer j, for each $1 \leq j \leq m$, i.e. sets $x_{1^j j}$ to its maximum feasible value for all j, which is precisely the allocation $\mathbf{x}^{\text{uniform}}$. This implies that, starting at $t = 1$, choosing $\mathbf{x}^{\text{uniform}}$ at $t = 1, \ldots, L+1$ maximizes each $\sum_j q_{j,t}(b_{j,t}, \mathbf{x}_j)$, and consequently $\sum_{t=1}^{L+1} \sum_{j=1}^{m} b_{j,t}$. Thus $\mathbf{x}^{\text{uniform}}$ is the optimal first period allocation for $\pi^{L\text{-LA}}$. □

EC.4. Proofs from Section 6

In this section, we provide the proofs from two extensions to our model described in Sections 6.1 and 6.2, respectively.

EC.4.1. Proofs from Section 6.1

In this subsection, we start by defining additional notation that is relevant in the extension with uncertain customer arrivals. Then, we prove several bounds on the performance of the myopic policy in this context, including all the necessary auxiliary results.

Let P^t be the set of customers that are present in period $t \in \{1, \ldots, T\}$. Let I^t be the set of customers that are present in period t and do not get full service by the myopic policy, i.e. $I^t = \{j : q_{j,t}^{\text{MY}} < 1\}$.

Let $b_{j,t}^*$ and $b_{j,t}^{\text{MY}}$ be customer's j budget in period t if the platform follows the optimal or myopic policy in each period $\tau \in \{1, \ldots, t-1\}$, respectively.

Consider an optimal dual solution associated to the myopic allocation in period t. Let $\mu_{j,t}^{MY}$ be the dual variable of customer's j budget constraint, and $\lambda_{i,t}^{MY}$ be the dual variable of product's i supply constraint.

First, we prove the result that if we allow a completely arbitrary arrival pattern, the performance of myopic is lower bounded by $1/T$ times the performance of the optimal policy.

Theorem 3. *Under Assumptions 1 and 2 with $\gamma = 0$, for any horizon T and initial budget state $\mathbf{b}_1$,*

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{T}.$$

Proof. We prove

$$J_T^*(\mathbf{b}_1) \leq \sum_{t=1}^{T} \sum_{\tau=1}^{t} R\left(\mathbf{x}_\tau^{\text{MY}}\right) \leq T \cdot J_T^{\text{MY}}(\mathbf{b}_1). \tag{EC.16}$$

The second inequality in (EC.16) is trivial. We show that

$$R\left(\mathbf{x}_t^*\right) \le \sum_{\tau=1}^{t} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) \text{ for each } t \in \{1,\dots,T\}. \tag{EC.17}$$

The first inequality in (EC.16) then follows by adding up (EC.17) over $t \in \{1,\dots,T\}$.

First, note that $R(\mathbf{x}_1^*) \le R(\mathbf{x}_1^{\mathrm{MY}})$. Hence, (EC.17) holds for $t = 1$.

For any $T \ge 2$, consider an arbitrary time $t \in \{2,\dots,T\}$. Let G^t be the set of customers that either are present in period t for the first time, or get full service by the myopic policy in each period $\tau \in \{1,\dots,t-1\}$ that they are present in. Namely,

$$G^t = \left\{ j: \ j \notin \bigcup_{\tau=1}^{t-1} P^\tau \text{ or } q_{j,\tau}^{\mathrm{MY}} = 1 \ \forall \tau \in \{1,\dots,t-1\} \text{ such that } j \in P^\tau \right\}.$$

Assumption 1 (i) then implies $b_{j,t}^* \le b_{j,t}^{\mathrm{MY}}$ for all $j \in G^t$, hence

$$\sum_{j\in G^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^* \le R\left(\mathbf{x}_t^{\mathrm{MY}}\right). \tag{EC.18}$$

On the other hand, by definition of the sets G^t and I^t, for any customer $j \notin G^t$ there must exist a period $\tau < t$ such that customer j did not get full service, i.e. such that $j \in I^\tau$. Moreover, note that from Assumption 2 with $\gamma = 0$ it follows that for each $\tau \in \{1,\dots,t-1\}$

$$\sum_{j\in I^\tau} \sum_{i\in A_j} p_{ij} x_{ij,t}^* \le \sum_{i \in \bigcup_{j\in I^\tau} A_j} \max_{j\in I^\tau}\{p_{ij}\} S_i \le \sum_{i \in \bigcup_{j\in I^\tau} A_j} \lambda_{i,\tau}^{\mathrm{MY}} S_i \le \sum_j b_{j,\tau}^{\mathrm{MY}} \mu_{j,\tau}^{\mathrm{MY}} + \sum_i S_i \lambda_{i,\tau}^{\mathrm{MY}} = R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right), \tag{EC.19}$$

where the first inequality follows since the revenue collected from customers $j \in I^\tau$ for each product in any period is upper bounded by allocating all the supply of each item they are interested in, $\bigcup_{j\in I^\tau} A_j$, to the highest paying customer among them, $\arg\max_{j\in I^\tau}\{p_{ij}\}$. The second inequality holds since, from Assumption 2 with $\gamma = 0$, $j \in I^\tau$ implies that customer j's budget constraint is loose in period τ (cf. equation 10), hence by complementary slackness $\mu_{j,\tau}^{MY} = 0$ for all $j \in I^\tau$, and then by dual feasibility $\lambda_{i,\tau}^{MY} \ge p_{ij}$ for all $i \in A_j$ and $j \in I^\tau$. The third inequality holds since $\mu_{j,\tau}^{MY} \ge 0$ for all j, and $\lambda_{i,\tau}^{MY} \ge 0$ for all i. Finally, the equality follows from strong duality of the linear program that computes the myopic allocation.

Therefore,

$$R\left(\mathbf{x}_t^*\right) \le \sum_{\tau=1}^{t-1} \sum_{j\in I^\tau} \sum_{i\in A_j} p_{ij} x_{ij,t}^* + \sum_{j\in G^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^* \le \sum_{\tau=1}^{t} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right),$$

where the first inequality follows since each customer $j \in P^t$ either belongs to G^t or to at least one set I^τ, $\tau \in \{1,\dots,t-1\}$. The second inequality follows from equations (EC.18) and (EC.19). This completes the proof of (EC.17). The result in the theorem follows from adding up (EC.17) over $t \in \{1,\dots,T\}$. □

We now restate and prove the proposition showing the tightness of the bound from Theorem 3.

Proposition 6. *The bound in Theorem 3 is tight. Namely, for the instance in Example 3,* $\lim_{\delta\to 0} J_T^{MY}(\mathbf{b}_1)/J_T^*(\mathbf{b}_1) = 1/T$.

Proof. We compare the revenues of the myopic and those of another feasible policy whose revenues lower bound those of the optimal policy. We assume that δ is such that $\delta/(1-\delta) < 1$, as well as $\delta \le (1-\delta)(1-\alpha)/(T-1)$.

Myopic policy: At $t=1$, this policy allocates the entirety of the good to the first customer, for a revenue of 1. The other customers receive nothing and they transition to a budget of $\alpha\delta$. Then, when customer j^l arrives at time j, they are allocated $\alpha\delta/(1-\delta)$ of the good; this is feasible since over all customers present at $2 \le j \le T$, we allocate a total of $\delta/(1-\delta) \le 1$ by construction.

Thus,

$$J_T^{MY}(\mathbf{b}_1) \le 1 + \frac{T-1}{1-\alpha}\alpha\delta = 1 + (T-1)\frac{\alpha}{1-\alpha}\delta.$$

Lower bound policy to the optimal: Let us denote this policy by π. At $t=1$ this policy first allocates the good equally to all customers j^l, over all $2 \le j \le T$ and $1 \le l \le 1/\alpha$, and then allocates any leftover to customer 1. Since by assumption, $\delta \le (1-\delta)(1-\alpha)/(T-1)$, the budget constraint of these customers will be binding, and the allocation customer j^l will be set to $\delta/(1-\delta)$. This implies that the service quality of customer j^l is equal to 1, and the budget in the next period in which they are present is then $\alpha\delta + (1-\alpha)$. The revenues accrued in the first period are $(T-1)/(1-\alpha)\delta$ from customers j^l, plus a revenue of $1-(T-1)/(1-\alpha)\delta/(1-\delta)$ from customer 1.

Consequently, in any period $t = j \in \{2,\ldots,T\}$, customers j^l for $l \in \{1,\ldots,1/(1-\alpha)\}$ will collectively have $(\alpha\delta + (1-\alpha))/(1-\alpha) = \alpha\delta/(1-\alpha)+1$ in budget, and as such the platform can allocate all of the product to them for an aggregate spend of $1-\delta$. The revenues in periods $2,\ldots,T$ are thus $(T-1)(1-\delta)$. We then have

$$J_T^{\pi}(\mathbf{b}_1) = 1 - \frac{T-1}{1-\alpha}\frac{\delta}{1-\delta} + \frac{T-1}{1-\alpha}\delta + (T-1)(1-\delta) = 1 - \frac{T-1}{1-\alpha}\frac{\delta^2}{1-\delta} + (T-1)(1-\delta).$$

Summing up, we have as δ becomes small,

$$\lim_{\delta\to 0}\frac{J_T^{MY}(\mathbf{b}_1)}{J_T^*(\mathbf{b}_1)} \le \lim_{\delta\to 0}\frac{J_T^{MY}(\mathbf{b}_1)}{J_T^{\pi}(\mathbf{b}_1)} = \frac{1}{T}.$$

□

We now prove the main result in this subsection, which shows that if customers must be present at the platform at least once every L periods, see Assumption 3, then the worst-case performance of the myopic policy, when the platform allocates a single product, depends on L and does not scale as the problem's horizon T grows.

Theorem 4. *Under Assumptions 1-3 with $\gamma = 0$, if $n = 1$ then for any horizon T and initial budget state $\mathbf{b}_1$,*

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{\min(L+1,T)}.$$

Proof. For any $T \leq L+1$ the result follows from Theorem 3. Therefore, assume $T > L+1$. We prove

$$J_T^*(\mathbf{b}_1) \leq \sum_{t=1}^{T} \sum_{\tau=\max(1,t-L)}^{t} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) \leq (L+1) \cdot J_T^{\mathrm{MY}}(\mathbf{b}_1). \tag{EC.20}$$

The second inequality in (EC.20) is trivial. We show that

$$R\left(\mathbf{x}_t^*\right) \leq \sum_{\tau=\max(1,t-L)}^{t} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) \text{ for each } t \in \{1,\ldots,T\}. \tag{EC.21}$$

The first inequality in (EC.20) then follows by adding up (EC.21) over $t \in \{1,\ldots,T\}$.

For any $t \leq L+1$, (EC.21) follows from (EC.17). Therefore, assume $T \geq t > L+1$.

Let C^t be the set of customers that cannot afford their optimal allocation in period t with the budget induced by following the myopic policy, i.e.

$$C^t = \left\{ j:\ p_{ij} x_{ij,t}^* > b_{j,t}^{\mathrm{MY}} \right\}.$$

First, note that $p_{ij} x_{ij,t}^* \leq b_{j,t}^{\mathrm{MY}}$ for any customer $j \notin C^t$. Hence, the solution $x_{ij} = 0$ for all $j \in C^t$, and $x_{ij} = x_{ij,t}^*$ for all $j \notin C^t$, is myopic feasible in period t. Therefore,

$$\sum_{j \notin C^t} p_{ij} x_{ij,t}^* \leq R\left(\mathbf{x}_t^{\mathrm{MY}}\right). \tag{EC.22}$$

Additionally, each customer $j \in C^t$ must have gotten less than full service in some period $\tau \in \{1,\ldots,t-1\}$, otherwise Assumption 1 (i) would imply $p_{ij} x_{ij,t}^* \leq b_{j,t}^* \leq b_{j,t}^{\mathrm{MY}}$ and $j \notin C^t$. Moreover, we can partition the set C^t into subsets C_τ^t, $\tau \in \{1,\ldots,t-1\}$, according to the latest period that each customer in C^t got less than full service. Namely, let

$$C_\tau^t = \left\{ j:\ p_{ij} x_{ij,t}^* > b_{j,t}^{\mathrm{MY}},\ q_{j,\tau}^{\mathrm{MY}} < 1,\ q_{j,l}^{\mathrm{MY}} = 1\ \forall l \in \{\tau+1,\ldots,t-1\} \text{ such that } j \in P^l \right\}.$$

Then, the sets C_τ^t, $\tau \in \{1,\ldots,t-1\}$, are disjoint and $C^t = \bigcup_{\tau=1}^{t-1} C_\tau^t$. Note that $\tau < t-L$ is possible, i.e. is possible that τ could not be counted in the right hand side of (EC.21).

Let $\hat{j}$ be the customer in C^t willing to pay the highest price for the item, i.e. $\hat{j} = \arg\max_{j \in C^t} p_{ij}$. Let $\tau \in \{1,\ldots,t-1\}$ be such that $\hat{j} \in C_\tau^t$. Then, from Assumption 2 with $\gamma = 0$ it follows that

$$\sum_{j \in C^t} p_{ij} x_{ij,t}^* \leq p_{i\hat{j}} S_i \leq \lambda_{i,\tau}^{\mathrm{MY}} S_i \leq \sum_j b_{j,\tau}^{\mathrm{MY}} \mu_{j,\tau}^{\mathrm{MY}} + S_i \lambda_{i,\tau}^{\mathrm{MY}} = R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right), \tag{EC.23}$$

where the first inequality follows since the revenue collected from customers $j \in C^t$ in any period is upper bounded by allocating all the supply of the item to the highest paying customer among them, $\hat{j}$. The second inequality holds since, from Assumption 2 with $\gamma = 0$, $j \in C^t_\tau$ implies that customer j's budget constraint is loose in period τ (cf. equation 10), hence by complementary slackness $\mu^{MY}_{j,\tau} = 0$ for all $j \in C^t_\tau$, and then by dual feasibility $\lambda^{MY}_{i,\tau} \geq p_{ij}$ for all $j \in C^t_\tau$, and in particular $\lambda^{MY}_{i,\tau} \geq p_{i\hat{j}}$. The third inequality holds since $\mu^{MY}_{j,\tau} \geq 0$ for all j, and $\lambda^{MY}_{i,\tau} \geq 0$. Finally, the equality follows from strong duality of the linear program that computes the myopic allocation.

Therefore, adding up (EC.22) and (EC.23) we conclude

$$R\left(\mathbf{x}^*_t\right) = \sum_{j \in C^t} p_{ij} x^*_{ij,t} + \sum_{j \notin C^t} p_{ij} x^*_{ij,t} \leq R\left(\mathbf{x}^{\mathrm{MY}}_\tau\right) + R\left(\mathbf{x}^{\mathrm{MY}}_t\right). \tag{EC.24}$$

If $\tau \geq t - L$ then (EC.24) implies (EC.21). Hence, assume $1 \leq \tau < t - L$. Then,

$$R\left(\mathbf{x}^*_t\right) \leq R\left(\mathbf{x}^{\mathrm{MY}}_\tau\right) + R\left(\mathbf{x}^{\mathrm{MY}}_t\right) \leq \sum_{l=t-L}^{t-1} R\left(\mathbf{x}^{\mathrm{MY}}_l\right) + R\left(\mathbf{x}^{\mathrm{MY}}_t\right) = \sum_{l=t-L}^{t} R\left(\mathbf{x}^{\mathrm{MY}}_l\right),$$

where the first inequality follows from (EC.24), and the second inequality follows from Proposition 8 below. This completes the proof. □

The following key proposition, which we used to conclude our proof of Theorem 4, shows that, if we consider some time period τ and then a "cycle" of L time periods $\{t, \ldots, t+L-1\}$ for arbitrary $\tau < t \leq T - L + 1$, then the myopic revenues in period τ are no more than the myopic revenues over the entire cycle $\{t, \ldots, t+L-1\}$:

$$R\left(\mathbf{x}^{\mathrm{MY}}_\tau\right) \leq \sum_{r=t}^{t+L-1} R\left(\mathbf{x}^{\mathrm{MY}}_r\right).$$

We note that, while it is plausible that this inequality is true since by Assumption 3 all present customers at time τ are guaranteed to also be present at some time during the cycle, this statement is not a priori obvious. For example, τ could occur long before t, the start of the cycle; moreover, at this τ, customers could start in a highly profitable budget state that the system never returns to, and thus the platform could extract more revenue at τ than over the cycle.

From this perspective, one can also interpret this inequality as generalizing key elements in the proof of Theorem 2, to the uncertain customers case. Namely, if $L = 1$ then all customers are always present in the system, and we can apply Lemma 3, which we use for the "dynamic complementary slackness" argument. This lemma says that it is always feasible to repeat the myopic allocation from some time t at the next period $t+1$, implying that myopic per period revenues are non-decreasing. Thus, for $L = 1$, our old analysis proves that

$$R\left(\mathbf{x}^{\mathrm{MY}}_\tau\right) \leq R\left(\mathbf{x}^{\mathrm{MY}}_t\right), \qquad \text{for all } t > \tau,$$

no matter what the starting budget at time τ was. The inequality which we present in the next proposition generalizes this for any $L \geq 1$, at the expense of weakening the RHS to be revenues over an entire cycle of length L:

Proposition 8. *Under Assumptions 1-3 with $\gamma = 0$, for any horizon T and periods $1 \leq \tau \leq t-1$ such that $t-1 \leq T-L$,*

$$R\left(\mathbf{x}_\tau^{MY}\right) \leq \sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^{MY}\right).$$

Proof. For arbitrary periods $1 \leq \tau \leq t-1$ such that $t-1 \leq T-L$, we construct allocations $\mathbf{x}_r^t$, for periods r within the "cycle" $\{t, \ldots, t+L-1\}$, such that

$$R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) \leq \sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^t\right), \tag{EC.25}$$

and

$$\sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^t\right) \leq \sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^{\mathrm{MY}}\right). \tag{EC.26}$$

We prove (EC.25) in Lemma 5 and (EC.26) in Lemma 6. The proofs of both Lemmas, shown below, depend on $\mathbf{x}_r^t$ which we construct next.

In order to construct $\mathbf{x}_r^t$, $r \in \{t, \ldots, t+L-1\}$, we iteratively define a restricted-supply system for each period before the cycle $u \in \{\tau, \ldots, t-1\}$. The allocations $\mathbf{x}_r^t$, which happen for time periods inside the cycle $\{t, \ldots, t+L-1\}$, will then depend on the restricted-supply system defined for periods outside the cycle, i.e. in $\{\tau, \ldots, t-1\}$. The available supply in the restricted-supply system will be less than $\mathbf{s}$; while this may seem counterproductive, reducing supply will help in establishing the lower bound in equation (EC.26).

Construction of Restricted-Supply System (Before the Cycle): To construct the restricted-supply system in each period $u \in \{\tau, \ldots, t-1\}$, we first specify the budget of each customer and then the supply of each product. These are conceptually defined as follows.

The budgets are set equal to the amounts induced by the myopic allocation in the original system with full supply, $b_{j,u}^{\mathrm{MY}}$. In other words, in the restricted-supply system budgets do not follow our usual dynamics; instead, the budget path in this system is "hard-coded" to be $b_{j,u}^{\mathrm{MY}}$, regardless of the allocation implemented in each period in the restricted-supply system.

The definition of the supply available in each period is more intricate. Initially, in period τ, a supply $x_{ij,\tau}^{\mathrm{MY}}$ of product i is "tied" to each customer j. The supply "tied" to customer j in period τ will then only be available to the platform the next time customer j is present. If customer j returns to the platform before period t, the platform can then potentially reallocate the supply associated to customer j to some higher-paying customers and the supply then becomes "tied" to these customers. The items will then only become available again to the platform when the higher-paying customers that became "tied" to the items return. This reallocation process guarantees that at any period $u \in \{\tau+1, ..., t-1\}$ there will be a supply of item i of at most $\sum_j x^{\text{MY}}_{ij,\tau}$ available to the platform. In this restricted-supply system, we assume that the platform allocates items myopically in each period based on the available supply. We will argue that this reallocation process guarantees that the items originally allocated by the myopic policy in period τ are progressively reallocated in a manner that garners higher revenue.

Formally, we define the initial restricted-supply allocation for period τ as

$$x^{\text{RS}}_{ij,\tau} = \begin{cases} x^{\text{MY}}_{ij,\tau} \text{ if } j \in P^\tau, i \in A_j, \\ 0 \text{ otherwise.} \end{cases}$$

We emphasize that, by definition, $x^{\text{RS}}_{ij,\tau} = 0$ for all $j \notin P^\tau$. Then, the restricted-supply system for each period $u \in \{\tau+1, ..., t-1\}$ is defined iteratively as follows.

(i) Let each customer's budget be equal to the budget induced by using the myopic policy in the model with full supply in each period, i.e. $b_{j,u} = b^{\text{MY}}_{j,u}$ for each customer j.

(ii) Let $l_j(u)$ be the last period, before period u, that customer j was present in the platform, where we define $l_j(u) = -1$ if period u is the first time customer j is present. Similarly, let $l^\tau_j(u) \in \{\tau, \dots, u-1\}$ be the last period, before period u and truncated by τ, that customer j was present in the platform, i.e. $l^\tau_j(u) = \max(l_j(u), \tau)$. Then, define the restricted supply in period u as

$$s^{\text{RS}}_{i,u} = \sum_{j \in P^u} x^{\text{RS}}_{ij, l^\tau_j(u)},$$

where $\mathbf{x}^{\text{RS}}_u$ is a myopic allocation in the restricted-supply system in period u. For example, at $u = \tau+1$ the definition of the restricted supply evaluates to $s^{\text{RS}}_{i,\tau+1} = \sum_{j \in P^{\tau+1}} x^{\text{RS}}_{ij, l^\tau_j(\tau+1)} = \sum_{j \in P^{\tau+1}} x^{\text{RS}}_{ij,\tau}$. Recall that for any $j \notin P^\tau$, $x^{\text{RS}}_{ij,\tau} = 0$ by definition.

Construction of $\mathbf{x}^t_r$ (Within the Cycle): Let $f^t_j \in \{t, \dots, t+L-1\}$ be the first period such that customer j joins the platform during the cycle $\{t, \dots, t+L-1\}$, which is well defined by Assumption 3. Additionally, let F^t_r be the set of customers that show up for the first time in the cycle $\{t, \dots, t+L-1\}$ in period $r \in \{t, \dots, t+L-1\}$, i.e.

$$F^t_r = \{j : f^t_j = r\}.$$

Recall that from Assumption 3 all customers must appear at least once in any interval of L periods. Therefore, the sets $\{F^t_r\}_{r \in \{t, \dots, t+L-1\}}$ are a partition of all the customers in the system, which will be useful in the proofs of Lemmas 5 and 6 below.

Then, the allocation $\mathbf{x}^t_r$ for each period $r \in \{t, \dots, t+L-1\}$ is defined as follows,

$$x^t_{ij,r} = \begin{cases} x^{\text{RS}}_{ij, l^\tau_j(f^t_j)} \text{ if } r = f^t_j, \\ 0 \text{ otherwise.} \end{cases}$$

This concludes the construction of the allocation $\mathbf{x}^t_r$, $r \in \{t, \dots, t+L-1\}$.

Using $\mathbf{x}^t_r$ as constructed, Lemma 5 shows (EC.25), and Lemma 6 shows (EC.26). The statement in the proposition then follows from (EC.25) and (EC.26). $\square$

Before proving Lemmas 5 and 6, which complete the proof of Theorem 4, we pause for a moment to discuss why we restrict the proof of Theorem 4 to a single product. The key equations that illustrate the reason are (EC.23) and (EC.24). Specifically, if we allow for multiple products then we would have to consider one equation (EC.23) per product. As a result, we would end up with up to one term $R(\mathbf{x}_\tau^{\mathrm{MY}})$ per product in equation (EC.24). Consequently, it would be necessary to modify Proposition 8 to include several expressions $R(\mathbf{x}_\tau^{\mathrm{MY}})$ on the left hand side, or alternatively, the corresponding left hand side terms from the appropriate equation (EC.23) related to each product. In either case, it is unclear how to modify the restricted-supply system from the proof of Proposition 8, which define the allocation $\{\mathbf{x}_r^t\}_{r\in\{t,\dots,t+L-1\}}$, to accommodate these additional terms, while simultaneously preserving myopic feasibility and the revenue guarantee. In fact, even doing so for a single product requires a non-trivial analysis that relies on heavy machinery, as illustrated by the proofs of Lemmas 5 and 6 that we complete next.

Lemma 5. *Under Assumptions 1-3 with $\gamma = 0$, for any horizon T and periods $1 \le \tau \le t-1$ such that $t-1 \le T-L$. Let $\mathbf{x}_r^t$, $r \in \{t,\dots,t+L-1\}$, be the allocation constructed in the proof of Proposition 8. Then,*

$$R\left(\mathbf{x}_\tau^{MY}\right) \le \sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^t\right).$$

Proof. For any period $u \in \{\tau,\dots,t-1\}$, let $\mathbf{x}_u^{\mathrm{RS}}$ be a myopic solution of the restricted-supply system in period u defined in the proof of Proposition 8.

The proof is by induction on t, holding τ fixed; essentially we are inducting on the length of the interval from τ to $t-1$.

<u>Base Case:</u> Assume $t=\tau+1$, then

$$\begin{aligned}
R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) &= \sum_{j\in P^\tau}\sum_{i\in A_j} p_{ij} x_{ij,\tau}^{\mathrm{MY}} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}}\sum_{i\in A_j} p_{ij} x_{ij,\tau}^{\mathrm{MY}} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}}\sum_{i\in A_j} p_{ij} x_{ij,\tau}^{\mathrm{RS}} \\
&= \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}}\sum_{i\in A_j} p_{ij} x_{ij,l_j^\tau(r)}^{\mathrm{RS}} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}}\sum_{i\in A_j} p_{ij} x_{ij,r}^{\tau+1} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in P^r}\sum_{i\in A_j} p_{ij} x_{ij,r}^{\tau+1} \\
&= \sum_{r=\tau+1}^{\tau+L} R\left(\mathbf{x}_r^{\tau+1}\right),
\end{aligned}$$

where the second equality follows since from Assumption 3 $F_r^{\tau+1}$, $r \in \{\tau+1,\dots,\tau+L\}$, is a partition of all the customers in the system. The third equality is by definition of $\mathbf{x}_\tau^{\mathrm{RS}}$. The fourth equality follows since $l_j^\tau(r) = \tau$ for all customers $j \in F_r^{\tau+1}$ and periods $r \in \{\tau+1,\dots,\tau+L\}$ (see the definitions of $l_j^\tau(r)$ and $F_r^{\tau+1}$ in the proof of Proposition 8). The fifth equality is by definition of $\mathbf{x}_r^{\tau+1}$, $r \in \{\tau+1,\dots,\tau+L\}$. The sixth equality follows since, by definition, $x_{ij,r}^{\tau+1} = 0$ for all $j \in P^r \setminus F_r^{\tau+1}$, $r \in \{\tau+1,\dots,\tau+L\}$.

Induction Step: Assume the statement of the lemma holds for t, then

$$
\begin{aligned}
R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) &\leq \sum_{r=t}^{t+L-1} R\left(\mathbf{x}_r^t\right) = \sum_{r=t}^{t+L-1} \sum_{j\in P^r} \sum_{i\in A_j} p_{ij} x_{ij,r}^t \\
&= \sum_{r=t}^{t+L-1} \sum_{j\in F_r^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^t \\
&= \sum_{j\in F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^t + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^t \\
&= \sum_{j\in F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^t + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^{t+1}\setminus F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&= \sum_{r=t+1}^{t+L} \sum_{j\in F_r^{t+1}\cap F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^t + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^{t+1}\setminus F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&= \sum_{r=t+1}^{t+L} \sum_{j\in F_r^{t+1}\cap F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,l_j^\tau(t)}^{\mathrm{RS}} + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^{t+1}\setminus F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&\leq \sum_{r=t+1}^{t+L} \sum_{j\in F_r^{t+1}\cap F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,t}^{\mathrm{RS}} + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^{t+1}\setminus F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&= \sum_{r=t+1}^{t+L} \sum_{j\in F_r^{t+1}\cap F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} + \sum_{r=t+1}^{t+L-1} \sum_{j\in F_r^{t+1}\setminus F_t^t} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&= \sum_{r=t+1}^{t+L} \sum_{j\in F_r^{t+1}} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} \\
&= \sum_{r=t+1}^{t+L} \sum_{j\in P^r} \sum_{i\in A_j} p_{ij} x_{ij,r}^{t+1} = \sum_{r=t+1}^{t+L} R\left(\mathbf{x}_r^{t+1}\right).
\end{aligned}
$$

The first inequality follows from the induction hypothesis. The second equality follows since, by definition, $x_{ij,r}^t = 0$ for all $j \in P^r \setminus F_r^t$, $r \in \{t, \dots, t+L-1\}$. The fourth equality holds because, by definition, $x_{ij,r}^{t+1} = x_{ij,r}^t = x_{ij,l_j^\tau(r)}^{\mathrm{RS}}$ for all customers $j \in F_r^t = F_r^{t+1} \setminus F_t^t$ for all periods $r \in \{t+1, \dots, t+L\}$. The fifth equality follows since from Assumption 3 F_r^{t+1}, $r \in \{t+1, \dots, t+L\}$, is a partition of all the customers in the system. The sixth equality is by definition of $\mathbf{x}_t^t$. The seventh equality is by definition of $\mathbf{x}_r^{t+1}$, for each customer $j \in F_t^t$ and period $r \in \{t+1, \dots, t+L\}$. The ninth equality follows since, by definition, $x_{ij,r}^{t+1} = 0$ for all $j \in P^r \setminus F_r^{t+1}$, $r \in \{t+1, \dots, t+L\}$.

To conclude, the second inequality holds since from Assumption 3 $\{F_r^{t+1}\}_{r=t+1}^{t+L}$ is a partition of all the customers and the allocation $x_{i,j,l_j^\tau(t)}^{\mathrm{RS}}$ is feasible in the restricted-supply system in period t. In fact, by definition $\sum_{j\in P^r} x_{ij,l_j^\tau(t)}^{\mathrm{RS}} = s_{i,t}^{\mathrm{RS}}$, hence $x_{i,j,l_j^\tau(t)}^{\mathrm{RS}}$ is supply feasible. Moreover, $x_{i,j,l_j^\tau(t)}^{\mathrm{RS}}$ is also budget feasible since

$$\sum_{i\in A_j} p_{ij} x_{ij,l_j^\tau(t)}^{\mathrm{RS}} \leq \sum_{i\in A_j} p_{ij} x_{ij,l_j^\tau(t)}^{\mathrm{MY}} \leq b_{j,t}^{\mathrm{MY}}.$$

In the latter chain of inequalities, the first inequality follows since by definition[7] $x^{\text{RS}}_{ij,\tau} = x^{\text{MY}}_{ij,\tau}$, and, for each $u \in \{\tau+1,\ldots,t-1\}$, $\mathbf{x}^{\text{MY}}_u$ is defined on a problem with the same budgets and uniformly larger supply than $\mathbf{x}^{\text{RS}}_u$. The last inequality follows from Lemma 3, where Assumptions 1 and 2 with $\gamma = 0$ are used. This completes the proof of the lemma. □

Lemma 6. *Under Assumptions 1-3 with $\gamma = 0$, for any horizon T and periods $1 \le \tau \le t-1$ such that $t-1 \le T-L$. Let $\mathbf{x}^t_r$, $r \in \{t,\ldots,t+L-1\}$, be the allocation constructed in the proof of Proposition 8. Then,*

$$\sum_{r=t}^{t+L-1} R\left(\mathbf{x}^t_r\right) \le \sum_{r=t}^{t+L-1} R\left(\mathbf{x}^{MY}_r\right).$$

Proof. For any period $u \in \{\tau,\ldots,t-1\}$, let $\mathbf{x}^{\text{RS}}_u$ be a myopic solution of the restricted-supply system in period u defined in the proof of Proposition 8. We prove that the allocation $\mathbf{x}^t_r$ is myopic feasible in each period $r \in \{t,\ldots,t+L-1\}$.

First, $\mathbf{x}^t_r$ is budget feasible in each period $r \in \{t,\ldots,t+L-1\}$ since

$$\sum_{i\in A_j} p_{ij} x^t_{ij,r} = 0 \le b^{\text{MY}}_{j,r} \ \forall j, \ \forall r \in \{t,\ldots,t+L-1\} \setminus \{f^t_j\},$$

where the equality follows from the definition of the allocation $\mathbf{x}^t_r$ for $r \ne f^t_j$. Additionally, for each customer j and period $r = f^t_j$,

$$\sum_{i\in A_j} p_{ij} x^t_{ij,r} = \sum_{i\in A_j} p_{ij} x^t_{ij,f^t_j} = \sum_{i\in A_j} p_{ij} x^{\text{RS}}_{ij,l^\tau_j(f^t_j)} \le \sum_{i\in A_j} p_{ij} x^{\text{MY}}_{ij,l^\tau_j(f^t_j)} \le b^{\text{MY}}_{j,f^t_j},$$

where the equality follows from the definition of the allocation $\mathbf{x}^t_r$ for $r = f^t_j$. The first inequality follows since by definition $x^{\text{RS}}_{ij,\tau} = x^{\text{MY}}_{ij,\tau}$, and for each $u \in \{\tau+1,\ldots,t-1\}$, $\mathbf{x}^{\text{MY}}_u$ is defined on a problem with the same budgets and uniformly larger supply than $\mathbf{x}^{\text{RS}}_u$. The last inequality follows from Lemma 3, where Assumptions 1 and 2 with $\gamma = 0$ are used.

Second, we show that $\mathbf{x}^t_r$ is supply feasible since

$$\sum_{r=t}^{t+L-1} \sum_{j\in P^r} x^t_{ij,r} = \sum_{r=t}^{t+L-1} \sum_{j\in F^t_r} x^{\text{RS}}_{ij,l_j(r)} \le \sum_{j\in P^\tau} x^{\text{MY}}_{ij,\tau} \le S_i. \tag{EC.27}$$

In equation (EC.27), the equality follows from the definition of $\mathbf{x}^t_r$, $r \in \{t,\ldots,t+L-1\}$, and we show that the first inequality holds via an induction argument in the following.

The proof that $\sum_{r=t}^{t+L-1} \sum_{j\in F^t_r} x^{\text{RS}}_{ij,l_j(r)} \le \sum_{j\in P^\tau} x^{\text{MY}}_{ij,\tau}$ is by induction on t, holding τ fixed as in Lemma 5.

[7] By convention we define $x^{\text{MY}}_{ij,\tau} = 0$ for all customers $j \notin P^\tau$.

<u>Base Case:</u> Assume $t=\tau+1$, then

$$\sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}} x^{\mathrm{RS}}_{ij,\tau} = \sum_{r=\tau+1}^{\tau+L}\sum_{j\in F_r^{\tau+1}} x^{\mathrm{MY}}_{ij,\tau} = \sum_{j\in P^{\tau}} x^{\mathrm{MY}}_{ij,\tau}.$$

where the first equality follows since $l_j^{\tau}(r)=\tau$ for all $j\in F_r^{\tau+1}$, $r\in\{\tau+1,\ldots,\tau+L\}$, the second equality is by definition of $\mathbf{x}^{\mathrm{RS}}_{\tau}$, and the last equality follows since from Assumption 3 F_r^t, $r\in\{t,\ldots,t+L-1\}$, is a partition of all the customers in the system, for any t.

<u>Induction Step:</u> Assume the statement holds for t, then

$$\begin{aligned}
\sum_{r=t+1}^{t+L}\sum_{j\in F_r^{t+1}} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)} &= \sum_{r=t+1}^{t+L}\sum_{j\in F_r^{t+1}\setminus F_t^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)} + \sum_{r=t+1}^{t+L}\sum_{j\in F_r^{t+1}\cap F_t^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)}\\
&= \sum_{r=t+1}^{t+L}\sum_{j\in F_r^{t+1}\setminus F_t^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)} + \sum_{j\in F_t^t} x^{\mathrm{RS}}_{ij,t}\\
&= \sum_{r=t+1}^{t+L-1}\sum_{j\in F_r^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)} + \sum_{j\in F_t^t} x^{\mathrm{RS}}_{ij,t}\\
&\le \sum_{r=t}^{t+L-1}\sum_{j\in F_r^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(r)}\\
&\le \sum_{j\in P^{\tau}} x^{\mathrm{MY}}_{ij,\tau},
\end{aligned}$$

where the second equality follows since $l_j^{\tau}(r)=t$ for all customers $j\in F_r^{t+1}\cap F_t^t$ and periods $r\in\{t+1,\ldots,t+L\}$, and from Assumption 3 F_r^{t+1}, $r\in\{t+1,\ldots,t+L\}$, is a partition of all the customers in the system. The third equality holds because $F_r^t=F_r^{t+1}\setminus F_t^t$ for all $r\in\{t+1,\ldots,t+L\}$. The first inequality follows since $\mathbf{x}^{\mathrm{RS}}_t$ is feasible in the restricted supply-system, hence for each product i, $\sum_{j\in P^t} x^{\mathrm{RS}}_{ij,t}\le s^{\mathrm{RS}}_{i,t}=\sum_{j\in P^t} x^{\mathrm{RS}}_{ij,l_j^{\tau}(t)}$ and $P^t=F_t^t$, while the second inequality follows from the induction hypothesis. This completes the proof of the second inequality in equation (EC.27).

In order to conclude, we return to equation (EC.27): since the allocation $\mathbf{x}_r^t$ is myopic feasible in each period $r\in\{t,\ldots,t+L-1\}$, it follows that any myopic solution can only improve the revenues over $\mathbf{x}_r^t$, i.e. $R(\mathbf{x}_r^t)\le R(\mathbf{x}_r^{\mathrm{MY}})$, completing the proof of the lemma. $\square$

We conclude this subsection with the proof of Theorem 5, which provides a parametric lower bound on the performance of the myopic policy when the platform allocates multiple products, i.e. for $n\ge 1$. This bound depends both on the parameter L and on the shape of the customers' budget update functions.

Theorem 5. *Let $\bar{\alpha}\in[0,1]$ be defined as:*

$$\bar{\alpha}=\min_{j\in[m],b\in[0,1]}\{\phi_j(b,1)\}.$$

Under Assumptions 1-3 with $\gamma = 0$, for any horizon T and initial budget state $\mathbf{b}_1$,

$$J_T^{MY}(\mathbf{b}_1) \geq \frac{J_T^*(\mathbf{b}_1)}{\min\left(L + \frac{1}{\bar{\alpha}}, T\right)}.$$

Proof. Fix some $t \leq T$. We assume that $t > L+1$, or otherwise we can directly use Theorem 3 where we have shown that

$$R(\mathbf{x}_t^*) \leq \sum_{\tau=1}^{t} R(\mathbf{x}_\tau^{\mathrm{MY}}).$$

Similarly to the proof of Theorem 3, define G^t to be the set of customers who have received full service in each period up to time t (note that by Assumption 3, such customers must have been present in the system before $t > L+1$). For customers $j \in G^t$,

$$b_{j,t}^{\mathrm{MY}} \geq b_{j,t}^* \geq \bar{\alpha} b_{j,t}^*.$$

Furthermore, following the reasoning from equation (EC.24), for $\tau \in \{t-L, \ldots, t-1\}$,

$$\sum_{j \in I^\tau} \sum_{i \in A_j} p_{ij} x_{ij,t}^* \leq R(\mathbf{x}_\tau^{\mathrm{MY}}).$$

It remains to handle the customers who are present at t but belong to neither G^t, nor $\cup_{\tau \in \{t-L,\ldots,t-1\}} I^\tau$; let us call this set Q^t. These are customers who, via the myopic policy, have received full service at every time greater than or equal to $t-L$ up to $t-1$, but which did not receive full service in some period before $t-L-1$. Consider one such customer $j \in Q^t$. There must be at least one time between $t-L$ and $t-1$ when j was present in the system, by Assumption 3, and when j received full service, by construction. Thus, the myopic policy budget that this customer will have at time t is at least:

$$b_{j,t}^{\mathrm{MY}} \geq \bar{\alpha} = \min_{b \in [0,1]} \{\phi_j(b, 1)\} \geq \bar{\alpha} b_{j,t}^*.$$

Recalling that for $j \in G^t$, $b_{j,t}^{\mathrm{MY}} \geq b_{j,t}^* \geq \bar{\alpha} b_{j,t}^*$, and since the objective value of the LP that the myopic policy solves in period t is homogeneous in the RHS of the budget constraints,

$$\sum_{j \in G^t \cup Q^t} \sum_{i \in A_j} p_{ij} x_{ij,t}^* \leq \frac{1}{\bar{\alpha}} R(\mathbf{x}_t^{\mathrm{MY}}),$$

Putting everything together, we have:

$$\begin{aligned} R\left(\mathbf{x}_t^*\right) &\leq \sum_{\tau=t-L}^{t-1} \sum_{j \in I^\tau} \sum_{i \in A_j} p_{ij} x_{ij,t}^* + \sum_{j \in G^t} \sum_{i \in A_j} p_{ij} x_{ij,t}^* + \sum_{j \in Q^t} \sum_{i \in A_j} p_{ij} x_{ij,t}^* \\ &\leq \sum_{\tau=t-L}^{t-1} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right) + \frac{1}{\bar{\alpha}} R\left(\mathbf{x}_t^{\mathrm{MY}}\right), \end{aligned}$$

Thus, we can sum over all $t \in \{1,\dots,T\}$ to obtain:

$$\sum_{t=1}^{T} R\left(\mathbf{x}_t^*\right) \leq \sum_{t=1}^{T}\left(\sum_{\tau=\max(t-L,1)}^{t-1} R\left(\mathbf{x}_\tau^{\mathrm{MY}}\right)+\frac{1}{\bar{\alpha}} R\left(\mathbf{x}_t^{\mathrm{MY}}\right)\right) \leq\left(L+\frac{1}{\bar{\alpha}}\right) \sum_{t=1}^{T} R\left(\mathbf{x}_t^{\mathrm{MY}}\right).$$

□

EC.4.2. Proofs for Section 6.2

In this section, we give a proof of our main result for the varying supply case. We begin by restating this result:

Theorem 6. *Let $J_T^*(\mathbf{b}_1)$ be the optimal revenue for the system with time-varying supply for a starting budget $\mathbf{b}_1$ and a horizon of T. If for each period t and good i we have $s_{i,t} \in [(1-\epsilon)\bar{s}_i, (1+\epsilon)\bar{s}_i]$, then under Assumptions 1 and 2 with $\gamma = 0$,*

$$\frac{J_T^{S-MY}(\mathbf{b}_1)}{J_T^*(\mathbf{b}_1)} \geq (1-\epsilon)^4.$$

At a high level, the proof of the above theorem compares the scaled myopic revenues $J_T^{S-MY}(\mathbf{b}_1)$ in the varying supply system with supply $\mathbf{s}_t$ with a system in which the supply level is constantly at $\bar{\mathbf{s}}$, and in which the service quality is set to $(1-\epsilon)^2$. To give some intuition for the $(1-\epsilon)^2$ factor, recall that by Equation 14,

$$q_{j,t}^{S-MY}(b_{j,t}, x_{j,t}^{S-MY}, \mathbf{s}_t) \geq (1-\epsilon)^2 q_{j,t}^{MY}(b_{j,t}, x_{j,t}^{MY}, \bar{\mathbf{s}}),$$

where on the right hand side, we have the quality of service achieved by the myopic policy if supply were constant. In a way similar to that of Theorem 1, we leverage this inequality to compare the trajectories of two systems following the distinct budget updates implied by these service qualities. We observe that, here, the gap of $(1-\epsilon)^2$ comes from the variation of $\mathbf{s}_t$, whereas in Theorem 1 the $1-\gamma$ gap comes from the heterogeneity in bang-per-buck ratios.

We start with a lemma which compares the revenues in two auxiliary systems, which we refer to by H and S-LBMY. Both have initial budget $\mathbf{b}_1$, and the goods supply is equal to $\bar{\mathbf{s}}$ in every period. Furthermore,

(i) In system H, the quality of service at time t and budget $\mathbf{b}_t$ is defined as $q_{j,t}^{S-MY}(b_{j,t}, x_{j,t}^{S-MY}, \mathbf{s}_t)$. We denote by $J_T^H(\mathbf{b}_1)$ the revenues of this system under the myopic policy $\mathbf{x}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$.

(ii) In system S-LBMY, the quality of service at time t and budget $\mathbf{b}_t$ is always equal to $(1-\epsilon)^2$. Denote by $J_T^{S-LBMY}(\mathbf{b}_1)$ the myopic policy revenues of this system. This can be phrased as the following optimization:

$$\begin{aligned} J_T^{S-LBMY}\left(\mathbf{b}_1\right) = \max_{\mathbf{x}_1,\dots,\mathbf{x}_T} & \sum_{t=1}^{T} R\left(\mathbf{x}_t\right) \\ \text{s.t. } & \mathbf{x}_t \in \mathbf{X}\left(\boldsymbol{\phi}^{t-1}\left(\mathbf{b}_1, (\mathbf{1}-\boldsymbol{\epsilon})^2\right)\right), \quad \forall t, \end{aligned} \tag{EC.28}$$

and we have $J_T^{S-LBMY}(\mathbf{b}_1) = \mathrm{MY}(\mathbf{b}_1) + \sum_{t=2}^{T} \mathrm{MY}\left(\boldsymbol{\phi}^{t-1}\left(\mathbf{b}_1, (\mathbf{1}-\boldsymbol{\epsilon})^2\right)\right)$ We note that this system is similar to the system defined in (EC.6), except that we set the fixed quality of service to $(1-\epsilon)^2$ instead of $1-\gamma$.

We then have:

Lemma 7. *For any horizon $T \geq 2$ and budget $\mathbf{b}_1$, $J_T^H(\mathbf{b}_1) \geq J_T^{S-LBMY}(\mathbf{b}_1)$.*

Proof. Note that, as per equation (14),

$$q_{j,t}^{S-MY}(b_{j,t}, \mathbf{x}_{j,t}^{S-MY}, \mathbf{s}_t) \geq (1-\epsilon)^2 q_{j,t}^{MY}(b_{j,t}, \mathbf{x}_{j,t}^{MY}, \bar{\mathbf{s}})$$

We can then use this inequality to repeat the proof of Proposition 4 by setting $\tilde{\mathbf{q}}_t(\mathbf{b}_t, \mathbf{x}_t^{MY}) = \mathbf{q}_t^{MY}(\mathbf{b}_t, \mathbf{x}_t^{MY})$, to obtain obtain that: $J_T^H(\mathbf{b}_1) \geq J_T^{S-LBMY}(\mathbf{b}_1)$. The only change in the proof is that the chain of inequalities in the base case compares the budget update functions of systems H and LBMY. □

We note that in system H, both the quality of service and budget paths exactly follow those of the original varying supply system under the policy $\mathbf{x}_t^{S-MY}$. Also, while Lemma 7 is not sufficient to bound $J_T^*(\mathbf{b}_1)$ itself, it provides a crucial intermediate inequality towards this goal. In particular, Lemma 7 compares, for a system where supply is always equal to $\bar{\mathbf{s}}$ instead of varying, the effect of following the same budget update as the varying supply system, $\mathbf{q}_t^{S-MY}(\mathbf{b}_t, \mathbf{x}_t^{S-MY})$, versus following the budget update $\mathbf{q}_t^{MY}(\mathbf{b}_t, \mathbf{x}_t^{MY})$, which would transpire for fixed supply $\bar{\mathbf{s}}$.

Having established this, it remains to (a) bound the performance of $J_T^{S-MY}(\mathbf{b}_1)$ versus $J_T^H(\mathbf{b}_1)$, and then (b) bound the performance of $J_T^{S-LBMY}(\mathbf{b}_1)$ versus $J_T^*(\mathbf{b}_1)$. We do this in the next lemma:

Lemma 8. *For any $T \geq 2$ and starting budget $\mathbf{b}_1$, we have*

(i) $J_T^{S-MY}(\mathbf{b}_1) \geq (1-\epsilon) \cdot J_T^H(\mathbf{b}_1)$.

(ii) $J_T^{S-LBMY}(\mathbf{b}_1) \geq (1-\epsilon)^3 \cdot J_T^*(\mathbf{b}_1)$.

Proof. For part 1, recall that system H is designed in such a way that it shares the same budget path as the real stochastic system. Since in system H we are using the myopic policy, and using the definition of the scaled myopic policy, we have that $(1-\epsilon)x_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}) \leq x_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t)$; thus, at every t, $\sum_{ij} p_{ij} x_{ij,t}^{S-MY}(\mathbf{b}_t, \mathbf{s}_t) \geq (1-\epsilon) \sum_{ij} p_{ij} x_{ij,t}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$. The inequality then directly follows.

For part 2, note that $J_T^{S-LBMY}(\mathbf{b}_1) \geq (1-\epsilon)^2 J_T^{relax}(\mathbf{b}_1)$, where $J_T^{relax}(\mathbf{b}_1)$ is defined as in equation (9) to set the service quality to 1 at every period and receive constant supply $\bar{\mathbf{s}}$. Since $(1+\epsilon)\bar{\mathbf{s}} \geq \mathbf{s}_t$, it immediately follows that $J_T^{relax}(\mathbf{b}_1) \geq (1+\epsilon)^{-1} J_T^*(\mathbf{b}_1) \geq (1-\epsilon) J_T^*(\mathbf{b}_1)$. Thus, $J_T^{S-LBMY}(\mathbf{b}_1) \geq (1-\epsilon)^3 \cdot J_T^*(\mathbf{b}_1)$. □

Armed with the above lemmas, we can now complete the proof of Theorem 6.

Proof of Theorem 6. We have,

$$J_T^{S-MY}(\mathbf{b}_1) \geq (1-\epsilon) J_T^H(\mathbf{b}_1) \geq (1-\epsilon) J_T^{S-LBMY}(\mathbf{b}_1) \geq (1-\epsilon)^4 J_T^*(\mathbf{b}_1),$$

where the first inequality follows from part 1 of Lemma 8, the second from Lemma 7, and the third from part 2 of Lemma 8. □

EC.5. Numerical Experiments

In this section, we describe the sampling procedure for the numerical experiments in Section 4.3 and present a set of numerical experiments that illustrate the performance of the myopic policy in the extensions to our model detailed in Section 6.

EC.5.1. Sampling Procedure for the Numerical Experiments from Section 4.3

For each value of γ the parameters of each problem instance in the simulation in Section 4.3 were sampled as follows:

- n is sampled form a discrete uniform with support $\{3,\dots,10\}$ and we set $m=n$;
- T is sampled from a discrete uniform distribution with support $\{3,\dots,15\}$;
- Each component of the smoothing parameters $\boldsymbol{\alpha}$, the initial budget $\mathbf{b}_1$, and the supply $\mathbf{s}$, is sampled from a uniform distribution with support $[0,1]$;
- Every customer is connected to at least one item (customer j is connected to product $i=j$). All other edges in the network are generated with some probability $\delta \in [0,1]$, where for each problem instance δ is sampled from a uniform distribution on $[0,1]$;
- For each customer j, we sample a parameter ρ from a uniform distribution with support $[0,2]$. Then the valuations $\{v_{ij}\}$ and prices and $\{p_{ij}\}$ are uniformly sampled from the circular sector defined by $(1-\gamma)\rho \leq v_{ij}/p_{ij} - 1 \leq \rho$ and $v_{ij}^2 + p_{ij}^2 \leq 25$.

EC.5.2. Numerical Experiments from Section 6.1: Uncertain Customer Arrivals

We examine the numerical performance of the myopic policy with uncertain customer arrivals for randomly generated systems. We find that the myopic policy collects at least 90% of the optimal revenue for 98% of the sampled systems.

More specifically, for each value of L we sample system parameters and customer arrivals as follows:

- n and m are sampled from a discrete uniform with support $\{1,\dots,5\}$ and $\{1,\dots,10\}$, respectively;
- For each customer j, the size of the characteristic set A_j is sampled from a discrete uniform with support $\{1,\dots,n\}$. Then, given the size of A_j, the products in A_j are chosen through random sampling among available products;

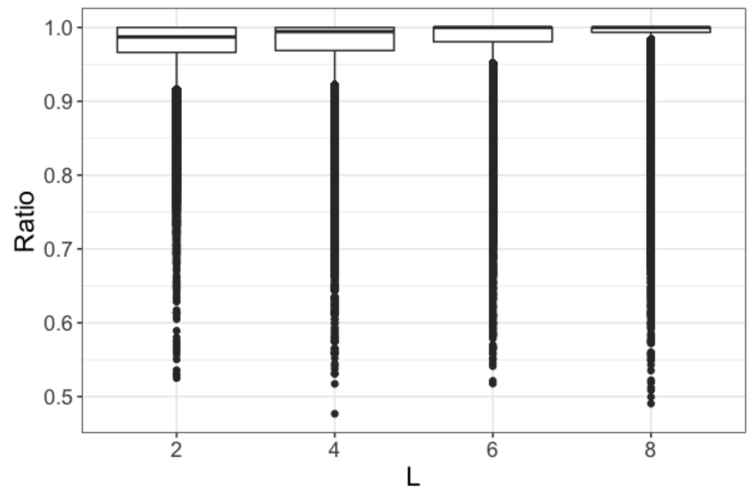

Figure EC.1 Boxplot of the ratio J_T^{MY}/J_T^{relax} when customer arrivals are uncertain and customers arrive at least every L periods. The box plot for each value of L represents the distribution of ratios sampled from 1,000,000 randomly generated problem instances.

- T is sampled from a discrete uniform distribution with support $\{2,\ldots,15\}$;
- We assume customers have an exponentially smoothed budget update as in (4). Each component of the smoothing parameters $\boldsymbol{\alpha}$, the initial budget $\mathbf{b}_1$, and the supply $\mathbf{s}$, is sampled from a uniform distribution with support $[0,1]$;
- For each customer j, we sample a parameter ρ from a uniform distribution with support $[0,2]$. Then the valuations $\{v_{ij}\}$ and prices and $\{p_{ij}\}$ are uniformly sampled from the circular sector defined by $\rho \leq v_{ij}/p_{ij} - 1 \leq \rho$ and $v_{ij}^2 + p_{ij}^2 \leq 25$;
- For each customer we sample interarrival intervals from a discrete uniform distribution with support $\{1,\ldots,L\}$ until the horizon T is reached.

With a slight abuse of notation, we denote by J_T^{relax} the revenue collected by the platform when every time a customer is present in the platform its budget grows at the maximum possible rate (the quality level is one). Figure EC.1 depicts the boxplots of the ratio J_T^{MY}/J_T^{relax} (which is an upper bound for J_T^{MY}/J_T^{*}) for different values of L. For each value of L, we randomly sampled 1 million systems. We note that the myopic policy collect at least 90% of the optimal revenue for 98% of the sampled systems. Furthermore, while Example 2 states that the worst-case performance of the myopic policy scales with L, the worst-case ratios in the simulation do not seem to scale with L. We believe this occurs because randomly sampling a system instance similar to the instance in Example 3 becomes more unlikely as L increases, since it involves having L customers appear sequentially in a problem horizon of length L.

EC.5.3. Numerical Experiments from Section 6.2: Time-Varying Supply

We first observe that the scaled myopic policy, while admitting the theoretical bound from Theorem 6, is potentially quite wasteful. That is, if it scales $\mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$ by $1-\epsilon$, the platform can end up with leftover products and unspent customer budgets which could still be profitably matched; this motivates us to modify the policy so as to perform these additional allocations. We tack on an additional adjustment to the scaled myopic policy as we find it improves performance in numerical experiments, but it does not change the lower bound. The modification proceeds as follows:

The modified scaled myopic allocation policy. In each period t, for customer budgets $\mathbf{b}_t$ and supply $\mathbf{s_t}$, the modified scaled myopic allocation denoted by $\mathbf{x}_t^{MS-MY}(\mathbf{b}_t, \mathbf{s}_t)$, is constructed in three steps:

a) First, calculate the myopic allocation with budgets $\mathbf{b}_t$ and supply $\bar{\mathbf{s}}$ and scale it by a factor $1-\epsilon$. Namely, compute $(1-\epsilon)\cdot \mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$;

b) Let $\mathbf{b}_t^L$ and $\mathbf{s}_t^L$ be, respectively, the budget and supply leftover after allocation $(1-\epsilon)\cdot \mathbf{x}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$. Specifically, for customer j, $b_{j,t}^L = b_{j,t} - (1-\epsilon)\cdot \sum_i p_{ij} x_{ij}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$ and, for good i, $s_{i,t}^L = s_{i,t} - (1-\epsilon)\cdot \sum_j x_{ij,t}^{MY}(\mathbf{b}_t, \bar{\mathbf{s}})$. Compute the myopic allocation with leftover budget and supply, namely, $\mathbf{x}_t^{MY}(\mathbf{b}_t^L, \mathbf{s}_t^L)$;

c) Then, the scaled myopic allocation is defined as the sum of the allocation of the two previous steps, i.e. $\mathbf{x}_t^{MS-MY}(\mathbf{b}_t, \mathbf{s}_t) \triangleq (1-\epsilon)\cdot \mathbf{x}_t^{MY}(\mathbf{b}_t, \bar{\mathbf{s}}) + \mathbf{x}_t^{MY}(\mathbf{b}_t^L, \mathbf{s}_t^L)$.

The modified policy $\mathbf{x}_t^{MS-MY}(\mathbf{b}_t, \mathbf{s}_t)$ differs from the scaled myopic policy $\mathbf{x}_t^{S-MY}(\mathbf{b}_t, \mathbf{s}_t)$ only through the allocation padding that occurs though the term $\mathbf{x}_t^{MY}(\mathbf{b}_t^L, \mathbf{s}_t^L)$. It can be seen that this policy trivially also satisfies Theorem 6; however for simplicity we chose to present our results in Section 6.2 using a more straightforward policy.

We denote by $J_T^{S-relax}$ the revenue when budgets increase at the maximum possible rate (quality level is always one) and supply varies over time. In Figure EC.2, we depict the box plots of the distribution of the ratio $J_T^{S-MY}/J_T^{S-relax}$ for different values of ϵ. Each sample corresponds to a randomly generated allocation problem (in a similar vein to the simulation in Section EC.5.2 but we assume that all customers are present in each period), and for each value of ϵ we sampled one million allocation problems. In general, the scaled myopic policy collects at least 95% of the optimal revenue and, even for large values of ϵ, the scaled myopic policy collects at least half of the optimal revenue. The simulation also indicates that the bound in Theorem 6 is potentially loose. Devising tight bounds for the performance of the scaled myopic policy (and other policies) is a promising direction of future research.

Finally, we examine the performance of the "regular" myopic policy (i.e. we use the policy defined in 12) in a setting where the supply for each good follows a random walk. Namely, the supply of good i in period $t=1$ is $s_{i,1} = \bar{s}_i$ and, for $t>1$, we have $s_{i,t} = \max\{s_{i,t-1} + Z_t, 0\}$, where Z_t is a uniform random variable with support $[-\epsilon\cdot \bar{s}_i, \epsilon\cdot \bar{s}_i]$. We calculate the ratio $J_T^{MY}/J_T^{S-relax}$ (which is an upper bound for the ratio J_T^{MY}/J_T^{*}) for different values of ϵ. For each value of ϵ, one million problem instances were randomly sampled. The boxplot of the ratios is depicted in Figure EC.3.

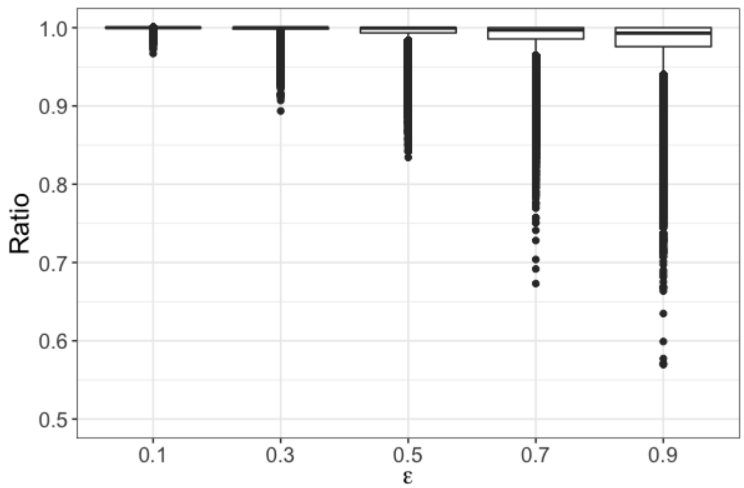

Figure EC.2 Boxplot of the ratio $J_T^{S-MY}/J_T^{S-relax}$ when the supply of each good vary in a bounded range. The box plot for each value of ϵ represents the distribution of ratios sampled from 1,000,000 randomly generated problem instances. We note that the y-axis does not start at zero.

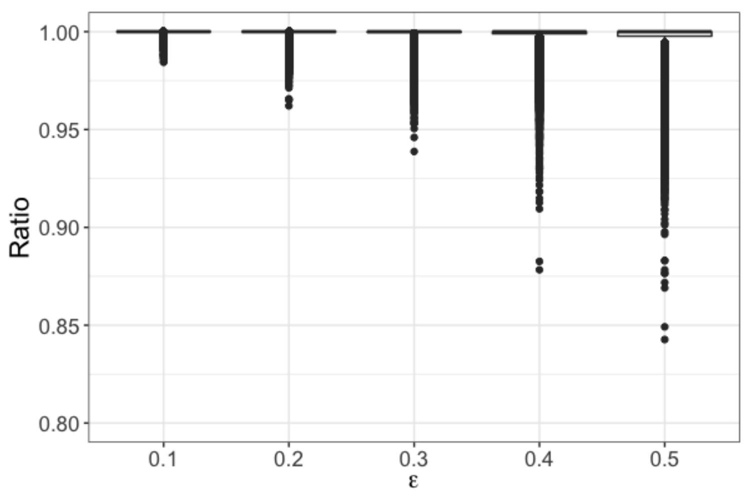

Figure EC.3 Boxplot of the ratio $J_T^{MY}/J_T^{S-relax}$ when supply of each good vary according to a random walk. The box plot for each value of ϵ represents the distribution of ratios sampled from 1,000,000 randomly generated problem instances.

Even in this case, where supply does not vary within a bounded range, the myopic policy has a near-optimal average performance and, in general, captures at least 80% of the optimal revenue.