The Business School for the World®

Working Paper

2021/21/FIN

Valuation of Corporate Tax Assets Under Tax Asymmetry: A Market-Based Approach

Pierre Hillion
INSEAD, pierre.hillion@insead.edu

The paper proposes an analytical framework to value corporate tax assets (carryforwards and carrybacks) subject to refundability risk under tax asymmetry. Building on the existing literature, tax assets are modelled as derivatives contracts. The framework captures earnings' volatility, the dynamic features of the tax code and the complex intertemporal and substitution effects between tax assets. The paper makes five contributions to the existing literature. First, it derives the relationship between the book value and the market value of tax assets. The difference between the two values captures refundability risk, defined as the risk that earnings will not materialize in the future. Second, the paper provides a formal derivation of the cost of tax asymmetry and shows that tax assets decrease the cost of tax asymmetry but leave the convexity of the tax liability function unchanged. Third, the paper highlights the role of earnings' volatility and sheds light on incentives issues that arise from the difference between the market and the book value of tax assets for firms endowed with large stocks of carryforward or carrybacks. Fourth, the paper provides estimates of conditional transition probabilities and tax persistence. They differ from their unconditional counterparts by being firm-specific and by exploiting the information contained in the distance between earnings and the stock of carrybacks or carryforwards scaled by earnings' volatility (akin distance-to-default in credit risk models). The last contribution is to show how to formulate marginal tax rates in terms of conditional transition probabilities. The paper has important implications for empirical work that involves tax assets.

Key words: Market value of tax assets, refundability risk, conditional transition probabilities, marginal tax rates, agency costs of tax asymmetry.

1. Introduction

Tax asymmetry is a key characteristic of corporate tax systems. In contrast to tax symmetry where the government shares in both profits and losses, firms pay corporate taxes when profitable but do not receive tax refunds when unprofitable. The adverse effects of tax asymmetry on firms' investment and financing policy have been subject to significant theoretical and empirical work in the economics, finance, and accounting literature. Tax asymmetry leads to risk-averse behaviour and reduces firms' incentives to invest in risky projects in favor of safer projects that generate more predictable returns.[1] Likewise, tax asymmetry has significant adverse implications on firms' financing policy that leads to sub-optimal capital structures.[2] These negative effects increase firms' incentives to hedge.[3]

The cost of tax asymmetry is even more relevant today than in the past because of an upward trend in losses experienced by U.S. firms. Coincident with the rise of intangible assets, the number of loss-making firms has increased from 2% in the early 1960's to over 30% in recent years, when measured by the number of publicly-listed firms reporting negative operating cash-flows. Scaled by assets, the annual losses of firms in the bottom decile of operating cash-flows have increased from 11% of assets in the 1970s to 58% in the 2000s. The persistence of tax losses has also increased over time. The proportion of negative cash flow firms reporting positive cash flows in the following year has decreased from 80% in the 1970's to 20% in recent years and the average number of consecutive years of negative cash flows has increased from 1 to 4 years.[4]

The relevance of tax asymmetry depends on firms' abilities to mitigate their adverse impact. Various provisions of the tax code alleviate the negative impact of loss offset constraints. Carryforward and carryback provisions allow firms to reduce their tax liabilities by writing off losses against past or future profits. Empirical evidence shows that most U.S. firms carryforward losses. Almost 90% of the large public U.S. firms disclose loss carryforwards in at least one jurisdiction (federal, state, or foreign), a sevenfold increase over the 13% reported

---

[1] Theory and empirical evidence suggest that tax asymmetry discourages risk taking. See Domar and Musgrave (1944), Stiglitz (1969), Majd and Myers (1985, 1987), Green and Talmor (1985), and more recently Ljungqvist *et al.* (2017), Langenmayr and Lester (2018) among others.

[2] See Leland (1994). Interest expenses decrease earnings and increase the likelihood that tax savings will not be fully realized in all states of the world. This lowers the expected tax benefits and decreases the firm's optimal leverage ratio.

[3] Hedging reduces the costs of tax asymmetry and lowers the expected taxes generated by a volatile earnings stream. See Smith and Stulz (1985) and Graham and Smith (1999).

[4] Altshuler *et al.* (2009) report that the fraction of firms with negative net income divided by the fraction of firms with positive net income increased from about 0.67 to more than 0.90 between 1962 and 2007. They argue that the sharp increase is traceable to a sharp general decline in the average rate of return among the firms. See also Denis and McKeon (2020).

in the 1980s and 1990s.[5] Tax provisions endow firms with valuable tax assets, raising issues about their valuation and their effectiveness at reducing the cost of tax asymmetry. Tax assets are typically valued at their "book values" given by the "stock" or "reserve" of tax losses held by the firm. However, the book value approach fails to capture their riskiness. Delays between the generation and utilization of tax losses introduce uncertainty in the tax cash-flows and in their timing. Losses that are carried forward have no value when firms are unable to generate sufficient taxable income to ever utilize their losses. Low recovery rates, estimated to be equal to less than one-half of the value of tax losses on average, underscore their riskiness.[6] This suggests that the book value over-estimates the "market value" of tax assets by failing to adjust for their riskiness.[7]

Refundability risk, defined as the risk that earnings will not materialize in the future, prevents firms from receiving either tax credits or tax refunds, and creates a wedge between the market value and the book value of tax assets.[8] The difference is acknowledged in the economics, accounting, and finance literature but is typically not modelled. In the empirical accounting literature that looks at the impact of tax provisions on the firm's investment policy, the book value is typically used as a proxy for the market value of the tax asset. In the theoretical corporate finance literature, such as dynamic capital structure models, tax provisions are either ignored (as is the case for the carryback provision) or accounted for by introducing an exogeneous asset-dependent boundary below which firms lose tax credits. The paucity of theoretical work on the valuation of tax assets stems from the difficulty of modelling the intertemporal and dynamic effects generated by tax provisions.

The first contribution of the paper is to fill a gap in the literature and to show how to value tax assets at their market values. Consider the (standalone and unconstrained) carryforward provision under a one-period model (without loss of generality). The tax provision allows a firm to decrease its tax liability in taxable states by an amount that depends on prior tax losses. The likelihood that a firm will receive a tax credit is conditional on the occurrence of two consecutive events taking place in the future. Both are driven by future earnings and require the firm to incur a tax loss and to make a subsequent taxable profit. *De facto,* the tax asset granted by the government to the firm is a "put-on-a-call" compound option on earnings, with the put and the call contracts

[5] See Heitzman and Lester (2020).

[6] See Cooper and Knittel (2006) and (2010). Over a ten–year window, they find that approximately 40 to 50% of losses are used as a loss carryforward deduction, approximately 25 to 30% are lost and 10 to 20% remain unused.

[7] An analogy is the difference between the book and the market value of a defaultable bond. The difference depends on the default probability and the expected recovery rate.

[8] The focus of the paper is on refundability risk. Tax assets are subject to other risks, such as the possibility that a future ownership change constrains the firm's ability to use carryforwards. See Erickson and Heitzman (2010). The paper assumes an unlevered firm and no default risk. This issue is left to a companion paper.

capturing the tax loss and the taxable profit, respectively. The two derivative contracts are nested. The firm must exercise both options, starting with the put contract and subsequently the call contract, to receive a tax credit in the future.[9] The argument extends to multiple periods.

More specifically, the put option contract, referred to as the carryforward reserve, or the book value of the tax asset, is written on earnings. The put option is exercised by the firm when it incurs a tax loss. Its strike price is equal to the carryforward reserve at the end of the previous year. The put option captures the dynamics of the carryforward reserve over time. The call contract, referred to as the market value of the tax asset, granted to the firm upon the put's exercise, is a call spread on earnings expiring one year later. It is a risky asset composed of a long call option with a strike price equal to 0 and a short call option with a strike price equal to the book value of the tax asset. Depending on earnings prevailing on the call spread's expiry date one year later, the firm receives a tax credit floored at zero in the worst-case scenario and a tax credit capped at the book value of the tax asset in the best-case scenario (times the expected statutory tax rate). The market value of the tax asset is bounded above by its book value. Refundability risk explains the difference between the value of the put option (the book value) and the value of the call spread (the market value). The difference depends on two offsetting effects, a cash-flow effect, and a likelihood effect. An increase in the book value increases the market value of the tax asset. However, this positive cash-flow effect is offset by an adverse likelihood effect. An increase in the book value decreases the probability that the firm will receive the full tax credit. Refundability risk decreases the marginal value of an additional dollar of tax loss and decreases the present value of tax credits to be received by the firm in the future.

The (standalone) carryback provision obtains by symmetry. The likelihood that a firm will receive a tax refund in the future is conditional on the occurrence of two consecutive events taking place, i.e., a taxable profit followed by a tax loss. *De facto,* the tax asset granted to the firm by the government is a "call-on-a-put" compound option on earnings with the call and the put contracts capturing the taxable profit and the tax loss, respectively. The firm must exercise the call contract and subsequently the put contract to receive a tax refund in the future. The call option, referred to as the carryback reserve or the book value of the tax asset, is exercised by the firm in years when it makes a taxable profit. Its strike price is equal to the carryback reserve at the end of the previous year. The call option captures the dynamics of the carryback reserve over time. The put contract, referred to as the market value of the tax asset, granted to the firm upon the call's exercise, is a put spread on earnings expiring one

[9] This formalizes the argument put forward by Green and Talmor (1985) that loss carryforwards can be viewed as the government holding a call on the firm's cash-flows in each period with a path-dependent exercise price.

year later. The put contract is a risky asset composed of a long put option with a strike price equal to zero and a short put option with a strike price equal to the (negative of the) book value of the tax asset. Depending on earnings prevailing one year later, the firm receives a tax refund floored at zero in the worst-case scenario and a tax refund capped at the book value of the tax asset in the best-case scenario (times the expected statutory tax rate). The market value of the tax asset is bounded above by its book value. The put spread captures refundability risk and the difference between the book and the market value of the tax asset. Like in the carryforward case, refundability risk decreases the marginal value of an additional dollar of taxable profit and decreases the present value of the tax refunds to be received by the firm in the future.

Tax assets can be modelled as a portfolio of compound options on the firm's earnings with path-dependent time-varying strike prices. The benefits of the derivatives' framework are two-fold. First, it provides key insights on the valuation of tax assets and on the intertemporal and substitution effects that arise between the carryforward and the carryback provisions. The framework is useful from a public policy point of view to assess the impact of a change in a tax provision on the market value of tax assets. Any policy change in the constraints imposed on one of the two tax provisions, such as length of the carryback or of the carryforward period, must incorporate the intertemporal and substitution effects that arise between the tax provisions. Second, the market values of the tax assets can be estimated in closed form when the book value, i.e., the carryforward or carryback reserve, is observable (as is the case at time 0). This requires an assumption about the stochastic process followed by earnings, such as an arithmetic Brownian motion, and the estimation of the parameters governing the process.[10] Beyond time 0, the tax assets must be valued by simulation given the path-dependency of the derivative contracts. The usefulness of the derivatives' framework is to impose a parametric model onto an otherwise plain algorithmic approach.

The second contribution of the paper is to use the derivatives' framework to assess the effectiveness of tax provisions at mitigating the negative impact of tax asymmetry. Consider a firm' balance sheet (in market value terms). On the liability side, the firm issues (i.e., is short) a perpetual coupon-paying bond to the government paying an annual floating-rate coupon. Absent tax provisions and tax shields, the annual coupon is a put option on earnings with a strike price equal to zero times the expected statutory tax rate.[11] The reason is that in the tax asymmetry case, and unlike the tax symmetry case, the firm receives no tax refund from the government when

[10] By preventing earnings to be negative, a geometric Brownian motion undervalues both the carryback and carryforward tax assets. Under a GBM, the carryback reserve would be equal to 0 and the market value of the carryback asset would be equal to 0. Further, a GBM would undervalue the carryforward tax asset by under-estimating the carryforward reserve. Dynamic capital structure models typically assume a GBM. One exception is Ammann and Genser (2004).

[11] Tax shields increase the put's strike price by a positive amount equal to depreciation allowances and interest expenses (among other tax deductions). They increase the cost of tax asymmetry.

incurring a tax loss. The shortfall for the firm is a put option on earnings. The annual coupon increases with tax losses and is theoretically unbounded. The market value of the bond depends on the stochastic process governing the firm's earnings and on the future statutory tax rates. Its value increases unambiguously with earnings' volatility. The market value of the bond captures the cost of tax asymmetry to the firm. On the asset side of the balance sheet, the firm holds (is long) a perpetual annual floating-rate coupon-paying bond granted by the government. Depending on which one of the carryforward reserve or carryback reserve is positive, the annual floating-rate coupon is either a call spread or a put spread on earnings times the expected statutory tax rate. Tax provisions mitigate but do not perfectly hedge the cost of tax asymmetry because of the mismatch in the bonds' coupons. Unlike the coupon of the short bond issued to the government given by a put option with a strike price of 0, the annual floating-rate coupon paid by the government to the firm depends on past earnings, is capped at the book value of the tax asset and is subject to refundability risk. The firm bears a net (positive) cost of tax asymmetry equal to the difference in the market value of the two bonds held by the firm on the asset side and liability side of its balance sheet. The difference increases with earnings' volatility and refundability risk, i.e., with a decrease in the probability that the call spread or the put spread will be exercised in the future. In the best-case scenario, the firm recovers its tax loss, albeit with a time lag. The valuation of tax assets at their book values under-estimates the net cost of tax asymmetry to the firm. The net cost of tax asymmetry is unknown ex-ante and leaves the firm exposed to tax uncertainty, i.e., to a change in the tax provisions.

The third contribution of the paper is to shed light on the agency costs of tax asymmetry and on the conflicts of interest arising from the difference between the book and the market value of the tax asset.[12] Under tax asymmetry, firms own on the asset side of their balance sheets a risky bond granted by the government. A unique characteristic of the bond is to pay a coupon that depends on the firm's earnings, i.e., the firm is long a risky bond subject to its own earnings' risk. The firm has incentives to manipulate the earnings generating process to increase (or to preserve) the market value of its tax asset. Refundability risk raises agency problems between the firm and the government akin the agency problems between shareholders and creditors. When the market value of the tax asset is low relative to its book value, i.e., when the call spread or the put spread is out-of-the-money, the firm has incentives to increase earnings' volatility by investing in high-risk projects and by scaling back its financial and

[12] There is empirical evidence that firms take actions to preserve and maximize tax loss benefits. See Maydew (1997), Erickson and Heitzman (2010), Albring, Dhaliwal, Khurana, and Pereira (2011), Sikes, Tian, and Wilson (2014). The literature does not address the issue of how risk-taking can increase the market value of the tax asset.

operational hedges.[13] This is the case when the firm's earnings are negative (positive) and the firm holds a large carryforward (carryback) reserve, respectively.[14] In both cases, the likelihood that the current and future call spreads or put spreads will be in-the-money in the future is low. Conversely, when the call spreads or put spreads are in-the-money, the firm has incentives to decrease earnings' volatility to preserve the value of its tax asset by investing in low-risk projects and by scaling up its financial and operational hedges. Risk taking allows firms to actively manage and to maximize the market value of their tax assets.

The extent to which tax systems incentivize firms to invest in risky projects and affect the amount of risk-taking has been investigated both theoretically and empirically. The literature argues that tax provisions decrease the cost of tax asymmetry and induce firms to increase their overall risk-taking by shifting a portion of investment risk to the government. There are two issues with this argument. First, from a theoretical standpoint, tax provisions do not linearize the tax function and leave its convexity unchanged. They merely translate the convexity "kink" to the left (right) when the carryback (carryforward) reserve is positive, respectively. The firm's expected tax liability remains negatively affected by an increase in earnings' volatility. As a result, tax provisions should not incentivize firms to increase their overall risk-taking. Second, from an econometric standpoint, the literature assumes a causal relationship running from tax loss rules to investment decisions (both level and risk). In contrast, tax asymmetry reverses the causality and suggests a relationship running from investment to tax assets. Firms holding tax assets with a low market value relative to their book values, have incentives to increase risk-taking by investing in risky projects. Conversely, firms holding tax assets with a high market value may be incentivized to decrease risk-taking. The agency cost of tax asymmetry creates an endogeneity problem in the empirical work that looks at the relationship between tax loss rules and investment.[15]

A fourth contribution of the paper is to suggest a novel approach to estimate transition probabilities, defined as the probability that a firm's tax status will change in the future, from a non-taxable to a taxable status or vice-versa. Transition matrices are typically used in the literature to estimate unconditional transition probabilities from historical data.[16] This approach suffers from three weaknesses which have been discussed in the literature.[17]

[13] An additional benefit is to increase the market value of equity at the expense of creditors. This raises interesting issues about interaction effects between the agency costs of debt and the agency costs of tax asymmetry. The bonds held by the firm on both side of its balance sheet are driven by the firm's earnings generating process.

[14] This challenges the traditional argument that hedging decreases the firm's expected tax liability by reducing earnings' volatility.

[15] The endogeneity issue has been discussed in the literature but the role of earnings' volatility has not been highlighted.

[16] See for example Auerbach and Poterba (1987) and Altshuler and Auerbach (1990).

[17] See Shevlin (1990).

First, it assumes that firms face identical, time-invariant, exogenous probabilities of switching from one tax status to another one. Second, it generates average rather than firm-specific transition probabilities. Third, it captures the effect of tax laws during the sample period used to estimate the probabilities which make them sensitive to a change in the tax system. In contrast, the derivatives framework generates firm-specific conditional transition probabilities which can be either estimated in closed-form or by simulation and do not require any other information than the firm's current earnings, current carryforward or carryback reserve and the parameters governing the earnings' process. The conditional transition probabilities exploit information contained in the distance between earnings and the present value of the carryforward reserve or the carryback reserve (all observable at the start of the period) scaled by earnings' volatility. The scaled distance (analogous to the distance-to-default in credit risk models) replaces the simple dummy variable (tax status at the start of the period) used in the literature to estimate transition probabilities. The conditional probability that a non-taxable firm endowed with a carryforward reserve becomes taxable one-period later is given by the probability that the call spread will be exercised. Likewise, the conditional probability that a taxable firm endowed with a carryback reserve becomes non-taxable one-period later is given by the probability that the put spread will be exercised. An assumption about the process followed by earnings, such as an arithmetic Brownian motion allows the scaled distance to be mapped into a probability. Alternatively, historical data and a frequentist approach (*a la* KMV) can be used to transform the scaled distance into an (empirical) probability distribution. Both approaches are complementary, and the latter (frequentist) can be used to gauge the appropriateness of the distributional assumption (i.e., a Normal distribution) made by the former (parametric) approach.

The last contribution of the paper relates to marginal tax rates. They play a key role in financial economics but are difficult to estimate without an algorithmic approach. The challenge is to capture earnings volatility and the dynamic features of the tax code.[18] The benefit of the derivatives' framework is to shed light on the determinants of the marginal tax rates by imposing a parametric model onto the algorithmic approach. Marginal tax rates are expressed as the present value of an incremental dollar of expected tax loss (when the carryforward reserve is positive), or of an incremental dollar of expected tax refund (when the carryback reserve is positive), with the expectation calculated using the conditional transition probabilities. They require the same distributional assumption about the earnings generating process as their algorithmic counterparts.

The paper's findings have important implications for the empirical work that involves tax assets. Consider the econometric work that investigates the ability of tax assets to reduce taxes. The unavailability of market values force researchers to use the book value of tax assets as proxies for market values, resulting in upward-biased

[18] See Shevlin (1990) and Graham (1996a).

estimates of tax benefits. More generally, the paper sheds light on the econometric issues that arise from the failure to account for earnings' risk and refundability risk in the empirical work that involves tax assets. Substituting the market value for the book value of the tax asset alleviates some of the econometric concerns.

The paper is organized as follows. Section 2 introduces the derivatives' framework. Sections 3 and 4 show how to value the carryforward and carryback tax assets, respectively. Section 5 focuses on the intertemporal and substitution effects arising between the two tax assets. Section 6 discusses empirical implications and Section 7 concludes. Constraints imposed by tax systems on the carryforward and the carryback provisions and estimation issues are left to the appendices.

2.0 Preliminary: The Case of No Tax Provision

The paper assumes that the firm has no tax shields, no investment tax credit, no foreign tax credits, and there is no uncertainty in the statutory corporate tax rate T_C in the future.[19] The firm's earnings before interest, tax, depreciation, and amortization at time t are denoted by $E(t)$. The stochastic process followed by earnings must be flexible enough to accommodate negative earnings. An arithmetic Brownian motion (ABM) is assumed throughout the paper. The two parameters governing the earnings' process are the drift term μ and the (dollar) volatility term σ, assumed constant without loss of generality.

2.1 The Case of Tax Symmetry

Consider the tax symmetry case used as benchmark for the more complex tax asymmetry cases examined below. Tax losses face a negative tax levied at the same rate as tax on profits. The firm's (expected) tax liability at time t, denoted by $T_O^*(t)$, is given by:

$$T_O^*(t) = E(t) \times T_C$$

The tax liability function is linear and the firm's marginal tax rate defined as the derivative of the tax function with respect to earnings, denoted by $MTR_O^*(t)$, is equal to T_C.

2.2 The Case of Tax Asymmetry

Tax symmetry does not hold in practice as tax systems treat positive and negative income in an asymmetric manner. Under tax asymmetry, absent tax shields and other tax provisions for firms with negative earnings, the firm's taxable income at time t, denoted by $TI_o(t)$, is given by:

[19] Tax shields are introduced in a companion paper.

$$TI_o(t) = MAX[0, E(t)] = C(0, t)$$

where $C(x, t)$ denotes a call option with a strike price equal to x and maturity t written on earnings. Multiplying $TI_o(t)$ by the statutory tax rate gives the firm's expected tax liability at time t. It is denoted by $T_o(t)$ and is given by:

$$T_o(t) = TI_o(t) \times T_C = C(0, t) \times T_C = [E(t) + P(0, t)] \times T_C = T_O^*(t) + P(0, t) \times T_C$$

with $T_O^*(t) \leq T_o(t)$, and where $P(x, t)$ denotes a put option with a strike price equal to x and maturity t.[20] Only positive earnings are taxed. The tax function has a kink at 0 and its convexity depends on the statutory tax rate, the firm's earnings process and the maturity of the put option. The firm's (expected) marginal tax rate at time t, denoted by $MTR_O(t)$, is given by:

$$MTR_O(t) = \Delta[C(0, t)] \times T_C$$

with $0 \leq MTR_O(t) \leq MTR_O^*(t) = T_C$, and where $\Delta[C(0, t)]$ denotes the derivative of the call $C(0, t)$ with respect to earnings, with $0 \leq \Delta[C(0, t)] \leq 1.0$.[21] The marginal tax rate increases with earnings, and is bounded below by 0 and above by T_C.

The firm's expected tax liability in the symmetric and asymmetric cases differ by the before-tax amount $P(0, t)$. The put option, which captures the annual cost of tax asymmetry, is exercised by the government against the firm in negative earnings states caused either by adverse systematic or firms' specific shocks. This has three implications. First, tax asymmetry discriminates cross-sectionally against risky firms, such as new, small, undiversified firms, and stand-alone projects given the positive relationship between the option premium and volatility.[22] Second, the expected tax liability function is convex given the positive gamma of a long option contract (but decreases with volatility) and its convexity increases with the statutory tax rate. The convexity of the tax function leads to risk-averse behavior and incentivizes firms to underinvest in risky projects. Third, the put premium and the cost of tax asymmetry vary over time. They increase in bad economic conditions at times of high uncertainty and low liquidity, thereby incentivizing firms to hedge.

---

[20] This holds true from the put-call parity relationship for European-style option which states that at time t $E(t) + P(x, t) = C(x, t) + x$.

[21] More generally, $\Delta_{y,z}(x, t)$ denotes the derivative (delta) of an option with strike price x and maturity t with respect to y, where z indicates whether the option is a call or a put option.

[22] These effects are likely to be even more pronounced today than in the past given the empirical evidence suggesting that cash flow risk has increased over time. Average industry cash flow risk more than doubled from 7.0% in the 1980s to 15.9% in the 2000s. See Bates et al. (2009).

The cost of tax asymmetry faced by a firm (modelled as a going concern) is given by the market value of a perpetual floating-rate coupon-paying bond granted by the firm to the government. It is denoted by $M(0)$ with:

$$M(0) = \sum_{t=1}^{\infty} P(0,t) \times T_C$$

The bond $M(0)$ is used as a benchmark to assess the effectiveness of tax provisions at decreasing the cost of tax asymmetry.[23]

3.0 The Unconstrained Standalone Carryforward Provision

The carryforward provision allows firms to carry tax losses forward in time to offset future tax payments. The section shows that the carryforward provision endows firms with a portfolio of path-dependent derivatives contracts that mitigates the cost of tax asymmetry. The mitigation is not perfect because of refundability risk (and time value of money). The section describes the characteristics of the contracts under the assumption that there are no constraints imposed either on the length of the carryforward period or on the taxable income that can be sheltered by carryforwards. This is referred to as the unconstrained standalone carryforward provision. The impact of constraints is discussed in the Appendix A.2.[24]

3.1 The Book Value of The Tax Asset

Consider time $t-1$, and denote by $B_F(t-1)$ the sum of tax losses carried forward by the firm into tax year $t-1$, referred to as the book value of the tax asset (or carryforward reserve).[25] Appendix A.1 shows that the book value $B_F(t-1)$ is a risky asset which can be expressed analytically as a put option on $E(t-1)$:

$$B_F(t-1) = P(B_F(t-2), t-1)$$

where $P(B_F(t-2), t-1)$ denotes a path-dependent put option with a time-varying strike price equal to $B_F(t-2)$ and maturity $t-1$. Tax losses (profits) increase (decrease) the strike price and increase (decrease) the book value of the tax asset, respectively. The put option captures the uncertainty in the future (earnings'-driven)

[23] The time lag between time t and the actual tax payment/refund is ignored. In addition, the model assumes that the firm's first tax liability occurs at time 1.

[24] In the U.S., the Tax Cuts and Jobs Act of 2017 (TCJA) eliminated the carryback provision and allowed tax losses to be carried forward indefinitely with a tax loss carryforward provision limited to 80% of taxable income.

[25] It is referred to as $NOLs$ or $TLCFs$ in the literature.

tax losses and models the dynamics of the carryforward reserve. The latter is initialized at the current $B_F(0)$ assumed to be observable throughout the paper.

Consider the two scenarios prevailing at time $t-1$ on the option's expiry date. When $E(t-1) > B_F(t-2)$, the firm let the put expire worthless. The put's payoff $B_F(t-1)$ is equal to zero and the carryforward reserve is fully depleted. Else, when $E(t-1) \leq B_F(t-2)$, the firm exercises the put option. The carryforward reserve increases or decreases depending on whether $E(t-1)$ is negative or positive, respectively. Unlike a plain vanilla contract, the exercise of the put option at time $t-1$ does not generate the terminal payoff $B_F(t-1)$. Rather, its exercise grants the firm a risky asset with a payoff at time t *capped* at $B_F(t-1)$.[26] The put option is a compound option whose characteristics are discussed below.

3.2 The (Before-Tax) Market Value of The Tax Asset: A One-Period Model

At time $t-1$, the tax asset $B_F(t-1)$ is worthless unless it generates tax credits in the future. The likelihood that the firm will recover past tax losses depends on the firm's earnings generating process and the book value of the tax asset $B_F(t-1)$. Assume momentarily that tax losses can be carried forward for one year only until time t, referred to as the one-period model. Appendix A.1.1 shows that, conditional on exercising the put option at time $t-1$, the firm is granted a one-year derivative contract expiring at time t, denoted by $M_F(t-1,t)$ and referred to as the (before-tax) market value of the tax asset, with the following characteristics:

$$M_F(t-1,t) = C(0,t) - C(B_F(t-1),t)$$

The contract is a path-dependent call spread on earnings maturing at time t, composed of a long call option with a strike price equal to 0 and a short call option with a time-varying strike price equal to $B_F(t-1)$. The derivative's payoff at time t is floored at 0 when both options expire worthless and is capped at $B_F(t-1)$ when they expire in-the-money. In the one-period model, scaling $M_F(t-1,t)$ by $B_F(t-1)$ and multiplying by T_C gives the market value at time $t-1$ of an expected dollars' worth of tax credit to be received at time t.[27]

The contract displays two noticeable characteristics. First, conditional on the put option being exercised at time $t-1$, both the book value $B_F(t-1)$ and the market value $M_F(t-1,t)$ of the tax asset are positive. A special

---

[26] The book value of the tax asset is capped. The cap is given by the sum of all the tax losses generated by the firm until time $t-1$, $B_B(t-1) \leq \sum_{\tau=0}^{\tau=t-1} P(0,\tau)$, i.e., the sum of all the tax refunds the firm would have received under tax symmetry. The cap is binding when the firm experienced tax losses in every single year between time 0 and time $t-1$.

[27] Favilukis, Giammarino and Pizarro (2015) argue that, with a large enough carryforward reserve, a firm will pay no taxes and the riskiness (beta) of its after-tax cash flows is equal to the riskiness of the pre-tax cash flows. The argument ignores the risk of the tax asset. The beta of an out-of-the-money call option increases with the strike price (the book value of the tax asset). The market value of the tax asset has a high beta when the book value is high and the option is out-of-the-money.

case obtains when the firm does not exercise the put option in which case the firm holds no tax asset and $B_F(t-1) = M_F(t-1,t) = 0$. Second, the market value of the tax asset at time $t-1$ is bounded above by its book value:

$$M_F(t-1,t) \leq B_F(t-1)$$

The difference between the market value and the book value captures refundability risk, the risk that earnings will not materialize at time t.

To shed light on the relationship between the market value and the book value of the tax asset, assume for the sake of simplicity that earnings follow a risk-neutral ABM. Under these assumptions, the market value of the tax asset $M_F(t-1,t)$ at time $t-1$ is given by:[28]

$$M_F(t-1,t) = \big[E(t-1) \times [N(d) - N(d_F)] + [PV \times B_F(t-1) \times N(d_F)] + v \times [n(d) - n(d_F)]\big]$$

with $d_F = [E(t-1) - PV \times B_F(t-1)]/v$, $d = E(t-1)/v$, and where $N(\cdot)$ and $n(\cdot)$ denote the cumulative and the probability density function of the standard normal distribution, respectively. At time t, the derivative's payoff is bounded below by 0 and above by $B_F(t-1)$. At time $t-1$, the probability that the firm will exercise the call spread and receive the full tax credit $B_F(t-1) \times T_C$ at time t depends on the distance between $E(t-1)$ and the present value of $B_F(t-1)$ scaled by earnings' volatility.[29] The market value is close to the book value of the tax asset for high positive values of d_F when the call spread is deep-in-the-money. Conversely, its market value is low relative to its book value for negative values of d_F when the call spread is deep-out-of-the-money. Figures 1.1-1.3 illustrate the sensitivity of $M_F(t-1,t)$ to $B_F(t-1)$, v and R_F.

[28] See Bachelier (1900), Smith (1976), Haug (2007), Brooks and Brooks (2016), Thomson (2016), Terakado (2019), among others. Smith's (1976) formula assumes no drift (consistently with the assumption of zero growth in Bachelier's (1900) model). In a risk-neutral world, the value of a one-year call on earnings with a strike price equal to $B_F(t-1)$ is given by: $[E(t-1) - PV \times B_F(t-1))] \times N(d_F) + v \times n(d_F)$, with $d_F = [E(t-1) - PV \times B_F(t-1)]/v$, $PV = e^{-R_F}$ and $v = \sigma \times \sqrt{(1-e^{-2\times R_F})/(2 \times R_F)}$, where R_F is the continuously compounded risk-free rate. This formula is derived under the assumption that, under the risk-neutral measure, earnings follow a Brownian motion given by the following SDE: $dE(t) = R_F E(t)dt + \sigma dW(t)$, where $dW(t)$ is a standard Brownian motion. See Musiela and Rutkowski (2005) for the derivation. A drift can be introduced by assuming the forward process $dF(t) = \sigma dW(t)$ with $F(t,T) = E(t) \times e^{\mu\times(T-t)}$, where μ is the drift term. Under this assumption, the value of a one-year call on earnings is given by $PV \times \big[\big(E(t-1) \times e^{\mu} - B_F(t-1)\big) \times N(d_F) + \sigma \times n(d_F)\big]$ with $d_F = [E(t-1) \times e^{\mu} - B_F(t-1)]/\sigma$. It assumes that earnings follow the process: $dE(t) = \mu E(t)dt + \sigma e^{-\mu\times(T-t)} dW(t)$ (obtained from the forward process using Ito's Lemma).

[29] This is analogous to the calculation of the default probability in credit-risk models. A firm's default probability depends on the distance between its assets and its liabilities (the option's strike price) scaled by its asset volatility. The default probability increases with a decrease in the scaled distance. In the present case, the probability that the firm will lose its tax asset in whole or in part depends on the scaled distance between earnings and the book value of the tax asset. The latter plays the same role as the face value of liabilities in credit-risk models.

At time 0, the current market value of the tax asset, $M_F(0,1)$ depends on the current and observable earnings $E(0)$ and the carryforward reserve $B_F(0)$. It has a simple analytical solution given by the formula above. Else, when $t > 0$, the path-dependent derivative contract $M_F(t-1,t)$ can be valued by simulation. The simulation approach discussed in Appendix A.3 makes the same distributional assumption as the simulation used to calculate the marginal tax rate in the tax literature and requires the estimation of the same two parameters, i.e., earning's drift and earnings' volatility.

3.3 The (After-Tax) Market Value of the Tax Asset and the Cost of Tax Asymmetry

The carryforward provision allows the firm to decrease its tax liability and receive a tax credit in taxable states that depends on prior tax losses realized in non-taxable states. The derivative contract is a "put-on-a-call" compound option on earnings. Conditional on exercising the put option at time $t-1$, the firm is endowed with a one-year call spread maturing at time t with a final payoff capped at the book value $B_F(t-1)$ and worth $M_F(t-1,t)$ at time $t-1$. The one-period model extends to multiple periods. The government grants a call spread once a year to the firm. The contract has a positive value when the put option is exercised at time $t-1$ and is worthless otherwise. Summing the call spreads and multiplying by the statutory tax rate gives the (after-tax) market value of the tax asset at time 0. It is denoted $M_F(0)$ with:

$$M_F(0) = \sum_{t=1}^{\infty} M_F(t-1,t) \times T_C = \sum_{t=1}^{\infty} [C(0,t) - C(B_F(t-1),t)] \times T_C \geq 0$$

The tax asset is a perpetual coupon-paying bond paying an annual floating-rate coupon equal to $M_F(t-1,t) \times T_C$ at time t. The first coupon received by the firm at time 1 depends on the (observable) carryforward reserve $B_F(0)$ and the realized earnings at time 1. The carryforward reserve is updated by forward induction using the dynamic process $B_F(t-1) = P(B_F(t-2), t-1)$.[30] The market value of the bond $M_F(0)$ is positive unless $B_F(t) = 0$ for all t, i.e., in the hypothetical case that the firm is never profitable. Example 1 in Appendix A.1.1 illustrates the process.

The market value of the tax asset is bounded above by $M(0)$, the sum of the tax refunds the firm would receive in the tax symmetry case:

[30] Using the derivatives' jargon, the firm is endowed with a portfolio of forward-start options with a maturity of one year. Forward start options are usually designed to be at-the-money when activated, i.e., to have a strike price equal to the price of the underlying asset on the activation date. In the present case, the forward start option is activated at time $t-1$ when the firm incurs a tax loss and exercises the put option. The activation grants the firm a call spread composed of a short call whose strike price is equal to the book value of the tax asset on the activation date.

$$M_F(0) = \sum_{t=1}^{\infty} M_F(t-1,t) \times T_C = \sum_{t=1}^{\infty} \big[PV \times [B_F(t-1)] + [P(0,t) - P(B_F(t-1),t)]\big] \times T_C < \sum_{t=1}^{\infty} P(0,t) \times T_C = M(0)$$

where, for comparative purposes, the put-call parity is used to transform the call spread into a put spread. The (net) cost of tax asymmetry is given by the difference between the market value of the two bonds:

$$M(0) - M_F(0) = \sum_{t=1}^{\infty} [-E(t) + C(B_F(t-1),t)] \times T_C = \sum_{t=1}^{\infty} \big[P(B_F(t-1),t) - PV \times [B_F(t-1)]\big] \times T_C$$

The cost involves a portfolio of option contracts which increases in value with volatility. This suggests that the carryforward provision does not eliminate the firms' incentives to underinvest in risky projects and still discriminates firms with volatile earnings.[31] In the best-case scenario, when $P(B_F(t-1),t)$ expires worthless at time t, i.e., for high earnings values, the carryforward provision allows the firm to recover its past losses $B_F(t-1)$ at time t, albeit with a time lag. Else, the difference in the two floating-rate coupons, equal to $-E(t) \times T_C$, takes either positive or negative values. The bond $M_F(0)$ pays a higher (lower) coupon than $M(0)$ when earnings are positive (negative), respectively. Refundability risk and time value of money explain the difference between the two market values. The relative difference $M_F(0)/M(0)$ measures the efficiency of the carryforward provision at decreasing the cost of tax asymmetry. The relative cost is immune to a change in the future statutory tax rate but is affected by possible changes in the carryforward tax provision.

3.4 Book Value Versus Market Value: Implications for Tax Exhaustion and Transition Probabilities

The derivatives framework provides key insights on the difference between the book value and the market value of the tax asset and on tax transition probabilities.

Suppose the firm experiences a loss at time $t-1$ and is granted a tax asset with a market value equal to $M_F(t-1,t)$. The latter depends on two offsetting effects, a cash-flow effect, and a probability effect. The market value of the tax asset $M_F(t-1,t)$ increases with the book value $B_F(t-1)$.[32] The positive cash-flow effect is offset by an adverse probability effect. The higher is the book value $B_F(t-1)$ and the lower is the likelihood that the short call will be exercised at time t. As tax losses accumulate, the strike price increases and future earnings

[31] The ability of tax loss rules at incentivizing firms to increase their overall risk-taking has been subject to both theoretical and empirical work. The result above formalizes Green and Talmor's (1985) argument that tax provisions increase the strike price of the option held by the government but do not alter the basic structure of its claim on the firm. The cost of tax asymmetry remains an increasing function of earnings' volatility unless the option is either deep in- or out-of-the money. This suggests that empirical tests that look at the impact of tax provisions on investment must control for the moneyness of the option and for earnings' volatility. Econometric implications are discussed in Section 6.3.

[32] The strike price of the short call option is the book value of the tax asset. An increase in the strike price decreases the call's value and increases the value of the call spread.

must be increasingly higher for the call spread to expire in-the-money and for the carryforward reserve to be fully depleted in the future. "Crowding-out" effects decrease the marginal value of an additional dollar of tax loss and decreases the present value of the future tax credits to be received by the firm. Figure 1.4 illustrates the concave relationship between $M_F(t-1,t)$ and $B(t-1)$ for different levels of earnings' volatility.

Consider the implications for tax exhausted firms which bear the highest cost of tax asymmetry. First, compare a non-taxable to a taxable firm at time $t-1$. The non-taxable firm becomes taxable at time t when the call spread expires in-the-money, i.e., when $E(t) > B_F(t-1)$. In contrast, the taxable firm remains taxable when $E(t) > 0$. At time t, the non-taxable firm has a lower probability of being taxable than the latter because of the higher strike price of the short call. Second, the probability of being taxable at time t conditional on being non-taxable at time $t-1$ decreases with an increase in the carryforward reserve $B_F(t-1)$. The conditional probability varies over time and depends on the state of the economy, i.e., decreases (increases) in recessions (expansions) when firms experience negative (positive) earnings' shocks, respectively.

Surprisingly, earnings' volatility has received little attention in the literature on transition probabilities. The derivatives framework sheds light on the impact of earnings' volatility on the likelihood that a non-taxable firm at time $t-1$ will become taxable at time t. The firm's tax status changes when the one-year call $C(B_F(t-1),t)$ is exercised at time t which requires $E(t)$ to be above $B_F(t-1)$. Under the assumption that earnings follow a risk-neutral ABM, the conditional probability that a non-taxable firm endowed with the carryforward reserve $B_F(t-1)$ at time $t-1$ becomes taxable at time t, denoted by $p_F(t-1,t)$, is given by:

$$p_F(t-1,t) = N(d_F) = N\left(\frac{E(t-1) - PV \times B_F(t-1)}{v}\right)$$

The call spread is out-of-the-money when d_F is negative. An increase in v decreases d_F, increases $p_F(t-1,t)$, and increases the likelihood that the firm will be taxable at time t. Figures 1.5 illustrates the sensitivity of $p_F(t-1,t)$ to earnings and the carryfoward reserve for two different estimates of volatility.[33] Regarding estimation issues, the (spot) conditional probability $p_F(0,1)$ has an analytical solution. The (forward) conditional probability $p_F(t-1,t)$ can be estimated by simulation (See Appendix A.3.). This implies that tax status persistence can be estimated from the spot and forward conditional transition probabilities.[34]

[33] Volatility affects the probability through a second channel. The strike price $B_F(t-1)$ is a put option. As a result, an increase in v which increases $B_F(t-1)$ increases $p_F(t-1,t)$. This effect is irrelevant at time $t-1$ when $B_F(t-1)$ is known with certainty.

[34] For example, a non-taxable firm at time 0 endowed with the carryforward reserve $B_F(0)$ has a conditional probability of being taxable at time 2 equal to $\left(1-p_F(0,1)\right) \times p_F(1,2)$.

3.5 Earnings' Volatility: Implications for Risk Shifting and Incentives

The carryforward provision grants the firm a risky bond. A unique characteristic of the bond is to pay a floating-rate coupon at time t that depends on the book value of the tax asset at time $t-1$ and the firms' realized earnings at time t. The firm is long a risky bond subject to its own earnings' risk. The shareholders versus bondholders' literature argues that shareholders have incentives to invest in risky projects to expropriate bondholders' wealth. Similar incentives arise in the present case. Risk-shifting at time $t-1$ allows the firm to increase the market value of its tax asset at the expense of the government (and possibly other creditors).

Consider the impact of earnings volatility on the bond value. The floating-rate coupon to be received at time t $M_F(t-1,t)$ is a call spread with a maturity of one year. Generally, the impact of volatility on a call spread is indeterminate (See Figure 1.2 as an illustration). It is positive (negative) when the call spread is out-(in-)-the money, respectively. At time $t-1$, the impact of volatility is small when the call spread is either deep out-of-the-money or deep in-the-money, i.e., for low and high earnings values, respectively. When earnings are negative, the short call option is out-of-the-money and the call spread behaves like a long call option with a strike price of 0. The market value of the tax asset is low relative to its book value and increases with earning's volatility. As earnings increase, the long call option moves increasingly in-the-money, and the sensitivity of the call spread to a change in volatility is driven by the short call option. An increase in earnings' volatility decreases the market value of the tax asset. The sensitivity of the call spread to volatility changes is ambiguous, i.e., is positive (negative) when earnings are low (high), respectively.

Consider the implications for investing, financing, and hedging. A firm holding a tax asset with a high book value relative to its current earnings holds an out-of-the-money call spread. The firm has incentives to increase the market value of the tax asset by increasing earnings' volatility. Doing so increases the likelihood that $E(t)$ will be higher than $B_F(t-1)$ at time t. The management of tax assets (tax planning) incentivizes the firm to invest in risky projects, to reduce its cash holdings (if any) and to scale back financial and operational hedges that reduce "right-tail" outcomes. Conversely, a firm endowed with an in-the-money call spread has the opposite incentives, i.e., to decrease earnings' volatility by investing in low-risk projects, by increasing its cash holdings, and by scaling up its hedging activities to preserve the market value of its tax asset.[35]

[35] In addition, when the assumption of an unconstrained carryforward provision is relaxed, possible caps imposed on the length of the carryforward period are likely to trigger "deadline effects," which exacerbate the firm's incentives to actively manage the value of its tax assets.

The discrepancy between the market and the book value of the tax asset generates agency problems and agency costs between the firm and the government. These costs have important implications for the theoretical and the empirical work that looks at the relationships between taxes, investment decisions, financing decisions, and hedging decisions. First, regarding investment decisions, the literature on the effects of tax asymmetry on firms' investment incentives assumes a causal relationship running from taxes to investments. It argues that tax provisions allow firms to shift part of the investment risk to the government and incentivize firms to increase their overall risk-taking. In contrast, the agency costs of tax asymmetry suggest a causal relationship running from investments to taxes. Firms increase investment risk to increase the market value of their tax assets when holding out-of-the-money call spreads. Second, regarding financing decisions, the literature on the agency costs of debt argues that investments in high-risk projects benefit shareholders at the expense of creditors. In contrast, the agency costs of tax asymmetry suggest that firms may have incentives to invest in low-risk projects to preserve the market value of their tax assets (possibly by holding more cash) when holding in-the-money-call spreads.[36] As an implication, the firms' shareholders benefit from wealth transfers from the government but lose from wealth transfers to creditors.[37] The bonds issued by the firm to the creditors and the bond granted by the government to the firm are driven by the same underlying factor, i.e., the firm's earnings. This argues in favor of a balance sheet approach that captures the joint interaction effects between the market value of the firms' liabilities (net of cash) and the market value of firms' tax assets. Third, regarding hedging decisions, the literature argues that the convexity of the tax function incentivizes firms to hedge to decrease their expected tax liabilities. In contrast, the agency costs of tax asymmetry suggest that firms may have incentives to scale back hedging to increase the market value of tax assets when holding out-of-the-money call spreads. This argues in favor of dynamic hedging strategies where firms adjust the amount of hedging as a function of the difference between the market value and the book value of tax assets.

3.6 The Firm's Expected Tax Liability

Assume the firm incurs a tax loss and is granted the tax asset $M_F(t-1,t)$ at time $t-1$. Denote by $T_F(t-1,t)$ and $T_O(t-1,t)$ the firm's expected tax liability at time t under tax asymmetry with and without the carryforward provision, respectively. The tax functions differ by an amount equal to the market value of the tax asset times the statutory tax rate:

[36] One example of tax planning discussed by Shevlin (2001) is for a firm to issue equity and to invest the proceeds in fully taxable bonds.

[37] Green and Talmor (1985) p. 1104 argued that the "*benefits associated with making the tax shields safer flow through debtholders.*" However, incentives are aligned when the market value of the tax asset is low relative to its book value.

$$T_F(t-1,t) = T_O(t-1,t) - M_F(t-1,t) \times T_C = C(B_F(t-1),t) \times T_C \leq C(0,t) \times T_C$$

with $T_F(t-1,t) \leq T_O(t-1,t)$. The firm's expected tax liability is a call option with a time-varying strike price equal to $B_F(t-1)$. The carryforward provision leaves the option-like structure of the firm's expected tax liability intact. The kink in the tax function moves from 0 without the carryforward provision to a positive value given by the time-varying strike price $B_F(t-1)$. Earning's volatility increases the firm's expected tax liability (but decreases its convexity). The tax function becomes linear with a slope equal to 0 only in two hypothetical cases, i.e., either when $B_F(t-1)$, or when σ, goes to infinity. Under the assumption of a risk-neutral ABM, the firm's expected tax liability is given by:

$$T_F(t-1,t) = C(B_F(t-1),t) \times T_C = \big[[E(t-1) - PV \times B_F(t-1)] \times N(d_F) + v \times n(d_F)\big] \times T_C$$

Figures 1.6 illustrates the sensitivity of $T_F(t-1,t)$ to $B_F(t-1)$ and v. Finally, the expected cost of tax asymmetry is given by the difference between $T_F(t-1,t)$ and the linear tax function obtained in the tax symmetry case $T_O^*(t)$:

$$T_F(t-1,t) - T_O^*(t-1) = [C(B_F(t-1),t) - E(t-1)] \times T_C = [P(B_F(t-1),t) - PV \times B_F(t-1)] \times T_C$$

The cost of tax asymmetry in the no carryforward case obtains as a special case by setting $B_F(t-1)$ equal to 0 in the equation above. This shows that the carryforward provision decreases the cost of tax asymmetry by an amount equal to $M_F(t-1,t)$.

3.7 The Firm's Marginal Tax Rate

The literature has suggested many proxies for the true but unobservable marginal tax rate defined as the change in the present value of the expected cash flow paid to (or recovered from) the government generated by one extra dollar of taxable earnings in the current tax period. Simple proxies include the statutory proxy, the NOL proxy, dichotomous variables based on the sign of taxable income or net operating loss carryforward status, and trichotomous variables based on both the sign of taxable income and carryforward status, among others.[38] These

[38] The statutory proxy is equal to T_C when current taxable income is positive and is equal to zero otherwise. This proxy assumes that firms pay tax at T_C if they have current taxable income; otherwise, they pay no tax in the present or future and the tax rate is zero. The NOL proxy uses T_C when the current taxable income is positive and the carryforward reserve is 0, and zero otherwise. The trichotomous proxy of Shevlin (1990) uses T_C when the NOL dummy is the statutory rate and zero when there are losses together with current negative income. However, when either current income is positive and there are losses or vice versa, the dummy is set equal to $T_C/2$. Manzon's (1994) proxy estimates the number of years it would take the firm to exhaust its carryforward reserve and uses that value to discount the tax rate.

simple proxies suffer from two deficiencies. First, they are single-period measures which fail to account for the long-term effects of carryback and carryforward rules in calculating taxable income.[39] Marginal tax rates depend on the firm's taxable income in prior and future years and require the sample path followed by earnings over long time periods. Second, the proxies are deterministic and fail to capture earnings' uncertainty and refundability risk. Their low power at predicting the "perfect foresight" marginal tax rate has been documented in the literature. Not surprisingly, the stochastic simulation-based marginal tax rates that capture earnings uncertainty and the time effects of tax provisions display improved forecasting abilities.[40] They dominate any other proxy but require earnings to be simulated over long time periods which makes them challenging to calculate.

The benefit of the derivatives' framework is to shed light on the determinants of the marginal tax rate by imposing a parametric model onto the simulation. Assume that the observable carryforward reserve $B_F(0)$ is positive at time 0. Consider as a benchmark a simple one-period proxy for the marginal tax rate, such as the NOL dummy, equal to 0 when the carryforward reserve is positive and equal to the statutory tax rate T_C, otherwise. Given the positive carryforward reserve, the firm is currently non-taxable and both the firm's taxable income and the NOL dummy are equal to 0. The NOL proxy under-estimates the firm's marginal tax rate by failing to account for the option value, i.e., the right granted to the firm to use $B_F(0)$ to offset taxable profits in the future. The NOL dummy gives a lower bound on the firm's marginal tax rate. At the other end of the spectrum, an upper bound can be obtained by assuming that the firm will be taxable with certainty at time 1. In this case, the \$1.0 decrease in $B_F(0)$ associated with the \$1.0 increase in $E(0)$ increases the firm's tax liability by \$1.0 at time 1 and gives a marginal tax rate equal to $PV \times T_C$, where PV is the present value of \$1.0 to be received at time 1. However, this proxy over-estimates the marginal tax rate by failing to account for earnings' uncertainty. The firm's true but unobservable marginal tax rate is in-between the lower and the upper bound given by 0 and $PV \times T_C$, respectively.

The usefulness of the derivatives framework is to adjust for earnings' uncertainty and to yield an estimate between these two bounds. Define by t, the time when the firm switches from a non-taxable to a taxable status and assume momentarily that t is known with certainty. As discussed in Section 3.4, this happens when the firm exercises the call option $C(B_F(t-1), t)$ at time t which requires $E(t)$ to be above $B_F(t-1)$. Consider the impact of a \$1.0 increase in $E(0)$ and the resulting \$1.0 decrease in $B_F(0)$ on the firm's future tax liability at time t. The

[39] See discussion in Shevlin (1990).

[40] Graham (1996b) investigates the ability of various proxies to predict the "perfect foresight" marginal tax rate and finds that a carryforward dummy and more complex proxies have a low predictive power (lower than the statutory tax rate). Graham's (1996a) simulation-based proxy which captures both the dynamic effects of the tax code and earnings' uncertainty displays the best predictive power.

\$1.0 decrease in $B_F(0)$ at time 0 implies that the taxable firm will forego an incremental $\$(1 \times T_C)$ worth of tax credit at time t. Appendix A.4 shows that the present value at time 0 of the expected tax loss at time t generated by a \$1.0 increase in $E(0)$ at time 0, denoted by $MTR_F(0,t)$, is given by:

$$MTR_F(0,t) = \left[PV \times \left(\prod_{\tau=1}^{\tau=t} N\big(-d_F(\tau-2)\big)\right) \times N\big(d_F(\tau-1)\big)\right] \times T_C$$

with $N\big(-d_F(-1)\big)$ set equal to 1.0 and where PV is a present value term that discounts back the expected tax loss from time t to time 0. In the formula above, the product term captures the probability that the firm remains non-taxable from time 0 until time $t-1$ and the last term measures the probability that the firm switches from a non-taxable status at time $t-1$ to a taxable status at time t.[41] However, time t is unknown *ex-ante*. This issue can be resolved by simulation which entails generating the earnings' sample path s and recording the time t_s when the firm becomes taxable for the first time on sample path s. It is defined as the time when the call $C(B_F(t-1,s),t)$ is exercised, where $B_F(t-1,s)$ denotes the carryforward reserve at time $t-1$ on sample-path s. Replacing time t by time t_s in the above formula gives a single estimate of the marginal tax rate at time 0. The generation of S simulated samples paths yields S estimates. The final step entails calculating the marginal tax rate at time 0, denoted by $MTR_F(0)$, as the arithmetic average of the S estimates $MTR_F(0,t_s)$:

$$MTR_F(0) = \frac{1}{S} \times \sum_{s=1}^{s=S} MTR_F(0,t_s)$$

This approach requires the same distributional assumption and the same inputs as those estimated in the literature with a plain algorithmic approach.[42]

4. The Unconstrained Standalone Carryback Provision

From a valuation perspective, carrybacks and carryforwards are symmetrical tax provisions. The benefit of the carryback provision is to offset in whole or in part the negative impact of a tax loss by a tax refund. Like the carryforward provision, the carryback provision endows the firm with a portfolio of path-dependent derivative contracts that mitigate the cost of tax asymmetry. In this section, the firm is assumed to be able to claim back

[41] Assume for example that tax losses can be carried forward for one year only. In this case, $MTR_F(0,1)$ has an analytical solution given by $PV \times N\big(d_F(0)\big) \times T_C = PV \times p_F(0,1) \times T_C$. It is bounded below by 0 and above by $PV \times T_C$.

[42] The "parametric" approach is computationally less demanding as it does not require earnings to be "bumped-up" by \$1.0. They also differ with respect to the discount rate. In the literature, the marginal tax rates are obtained by discounting the tax cash-flows at the cost of debt, assumed to be constant for all the firms in the sample. In a risk-neutral world, the discount rate is the risk-free rate.

taxes without constraints but to be unable to carry tax losses forward. This is referred to as the unconstrained standalone carryback provision.[43]

4.1. The Book Value of the Tax Asset

Consider time $t-1$ and denote by $B_B(t-1)$ the sum of the taxable profits generated by the firm prior to time $t-1$, referred to as the book value of the tax asset (or carryback reserve)[44]. Appendix A.1.2 shows that $B_B(t-1)$ is a risky asset which can be expressed analytically as a call option on $E(t-1)$:

$$B_B(t-1) = C(-B_B(t-2), t-1)$$

where $C(-B_B(t-2), t-1)$ denotes a path-dependent call option with a time-varying strike price equal to $-B_B(t-2)$ and maturity $t-1$. Taxable profits (losses) increase (decrease) the book value of the tax asset and decrease (increase) the strike price, respectively. The call option captures the uncertainty in the firm's future taxable profits and models the dynamics of the carryback reserve. The latter is initialized at $B_B(0)$ assumed to be observable throughout the paper.

Consider the call's payoff at time $t-1$ on the option's expiry date. When $E(t-1) \leq -B_B(t-2)$, the firm let the call expire worthless. The payoff is equal to 0 and the carryback reserve is fully depleted. Else, when $E(t-1) > -B_B(t-2)$, the firm exercises the call option. The carryback reserve increases or decreases depending on whether $E(t-1)$ is positive or negative, respectively. Unlike a plain vanilla option contract, the exercise of the call option at time $t-1$ does not generate the terminal payoff $B_B(t-1)$. Rather, the derivative contract grants the firm a risky asset with a payoff at time t capped at $B_B(t-1)$.[45] It is a compound option whose characteristics are discussed below.

4.2 The (Before Tax) Market Value of the Tax Asset: The One-Period Model

[43] Firms are assumed to exercise the carryback option optimally. Empirical evidence suggests that this is not the case because of tax complexity. Zwick (2021) reports evidence showing that only 37% of eligible firms claim their refund. Low take-up holds even when the potential refunds are large relative to a firm's operating cash flows.
[44] This is referred to as NOL carryback in the literature.
[45] The book value of the tax asset is capped. The cap is given by the sum of all the taxable profits generated by the firm until time $t-1$, $B_B(t-1) \leq \sum_{\tau=0}^{\tau=t-1} C(0,\tau)$. The cap is binding when the firm experienced taxable profits in every single year between time 0 and time $t-1$.

Suppose the firm holds a tax asset having a book value equal to $B_B(t-1)$ at time $t-1$. The asset is worthless unless it generates tax refunds in the future. It is a risky asset subject to refundability risk. Assume momentarily that losses can be carried back for one year only. Appendix A.1.2 shows that, conditional on exercising the call option at time $t-1$, the firm is granted a one-year derivative contract maturing at time t, denoted by $M_B(t-1,t)$ and referred to as the (before-tax) market value of the tax asset, with the following characteristics:

$$M_B(t-1,t) = P(0,t) - P(-B_B(t-1),t)$$

The contract is a path-dependent put spread on earnings with maturity t, composed of a long put option with a strike price equal to 0 and a short put option with a time-varying strike price equal to $-B_B(t-1)$. Its payoff at time t is floored at 0 when the put spread expires out-of-the-money and is capped at $B_B(t-1)$ when it expires in-the-money. In the one-period model, scaling $M_B(t-1,t)$ by $B_B(t-1)$ and multiplying by T_C gives the value at time $t-1$ of an (expected) dollars' worth of tax refund to be received at time t.

The contract displays two properties. First, conditional on the call being exercised at time $t-1$, both the book value $B_B(t-1)$ and the market value of the tax asset $M_B(t-1,t)$ are positive. A special case obtains when the firm does not exercise the call option, in which case the firm holds no tax asset and $M_B(t-1,t) = B_B(t-1) = 0$. Second, the market value of the tax asset is bounded above by its book value:

$$M_B(t-1,t) \leq B_B(t-1)$$

The put spread is out-of-the-money for positive earnings and is in-the-money for negative earnings. Under the assumption that earnings follow a risk-neutral ABM, the value of the put spread granted to the firm at time $t-1$ with a one-year maturity is given by:

$$M_B(t-1,t) = \big[E(t-1) \times [N(-d_B) - N(-d)] + [PV \times B_B(t-1) \times N(-d_B)] + v \times [n(-d) - n(-d_B)]\big]$$

with $d_B = [E(t-1) + PV \times B_B(t-1)]/v$.[46] At time t, the put spread is bounded below by 0 and above by $B_B(t-1)$. At time $t-1$, the probability that the firm will exercise the put spread and receive the full tax refund $B_B(t-1) \times T_C$ at time t depends on the distance between $E(t-1)$ and the present value of $-B_B(t-1)$ scaled by earnings volatility. Figures 2.1-2.3 illustrate the sensitivity of $M_B(t-1,t)$ to $B_{FB}(t-1)$, v and R_F, in a risk-neutral economy, respectively.

[46] The value of the put option is obtained from the value of the call option using the put-call-parity.

4.3 The (After-Tax) Market Value of the Asset and The Cost of Tax Asymmetry

The carryback provision allows the firm to receive a tax refund in non-taxable states that depends on past taxable profits. The derivative contract is a "call-on-a-put" compound option on earnings. Conditional on exercising the call option at time $t-1$, the firm is endowed with a put spread maturing at time t with a final payoff capped at the book value $B_B(t-1)$ and worth $M_B(t-1,t)$ at time $t-1$. The derivative contract has a positive value when the call option is exercised and is worthless otherwise. The one-period model extends to multiple periods.[47] Summing the put spreads and multiplying by the statutory tax rate gives the (after-tax) market value of the asset at time 0. It is denoted by $M_B(0)$ and is worth:

$$M_B(0) = \sum_{t=1}^{\infty} M_B(t-1,t) \times T_C = \sum_{t=1}^{\infty} [P(0,t) - P(-B_B(t-1),t)] \times T_C \geq 0$$

The tax asset is a perpetual coupon-paying bond paying an annual floating-rate coupon equal to $M_B(t-1,t) \times T_C$ at time t. The benefits of the bond for the firm are two-fold. First, it supplies liquidity to firms at times when the marginal value of cash is high. Second, it works as a hedge against a possible decrease in the market value of the firms' operating assets in bad times. The bond's first coupon earned by the firm at time 1 depends on the realized earnings at time 1 and the carryback reserve $B_B(0)$. The carryback reserve is updated by forward induction using the dynamic process $B_B(t-1) = C(-B_B(t-2), t-1)$. The market value of the bond $M_B(0)$ is positive unless $B_B(t) = 0$ for all t when the firm is always profitable. The numerical Example 2 in Appendix A.1.2 illustrates the process.

The future expected tax refunds cannot exceed the future expected tax losses. As a result, the market value of the bond $M_B(0)$ is bounded above by $M(0)$:

$$M_B(0) = \sum_{t=1}^{\infty} [P(0,t) - P(-B_B(t-1),t)] \times T_C < \sum_{t=1}^{\infty} P(0,t) \times T_C = M(0)$$

The cost of tax asymmetry is given by the difference in the market value of the two bonds:

[47] Like in the carryforward case, the firm is endowed with a portfolio of forward-start options with a maturity of one year. The forward start option is activated at time $t-1$ when the firm makes a taxable profit and exercises the call option. The activation grants the firm a put spread composed of a short put whose strike price is equal to the negative of the book value of the tax asset on the activation date.

$$M(0) - M_B(0) = \sum_{t=1}^{\infty} P(-B_B(t-1), t) \times T_C$$

The cost is a portfolio of put options which increases in value with earnings' volatility. Tax asymmetry still discriminates firms with volatile earnings and discourages risk taking. The annual cost varies over time as a function of $B_B(t-1)$ and is zero unless $P(-B_B(t-1), t)$ is exercised.[48] The cost arises from the inability of the firm to recover an amount higher than its (past) taxable profits. Refundability risk and time value of money explain the difference between the market values of the two bonds. The ratio $M_B(0)/M(0)$ measures the efficiency of the carryback provision at decreasing the cost of tax asymmetry.

4.4 Book Value Versus the Market Value: Implications for Tax Exhaustion and Transition Probabilities

Suppose the firm is granted at time $t-1$ a tax asset with a market value equal to $M_B(t-1, t)$. Like in the carryforward case, the contract value depends on two offsetting effects, i.e., a cash-flow effect and a likelihood effect that offset one another. The market value of the tax asset $M_B(t-1, t)$ increases with the book value $B_B(t-1)$.[49] This positive cash-flow effect is offset by an adverse likelihood effect. The probability that the put spread will be exercised at time t decreases with an increase in the book value of the tax asset. As taxable profits increase over time, the book value $B_B(t-1)$ increases, the strike price $-B_B(t-1)$ decreases, and earnings must be increasingly more negative for the put spread to expire in-the-money at time t and for the carryback reserve to be fully depleted. Refundability risk decreases the marginal value of an additional dollar of taxable profit and decreases the present value of the future expected tax refunds to be received by the firm. Figure 2.4 illustrates the concavity of the relationship between $M_B(t-1, t)$ and $B_B(t-1)$ for different estimates of earfnings' volatility.

Consider the implications for transition probabilities. Taxable firms at time $t-1$ become non-taxable at time t when $E(t) \leq -B_B(t-1)$. This is unlike non-taxable firms at time $t-1$ which remain non-taxable at time t when $E(t) \leq 0$. As a result, taxable firms endowed with the carryback reserve $B_B(t-1)$ have a lower probability of becoming non-taxable in the future than non-taxable firms. In addition, an increase in $B_B(t-1)$ moves the strike price of the put option to the left and decreases the likelihood that taxable firms will become non-taxable in the

[48] Assume the firm incurs a tax loss of \$100 at time t. Absent the carryback provision, the cost of tax asymmetry at time t is equal to $\$100 \times T_C$. Suppose the tax system allows the firm to carryback losses and the firm's carryback reserve is \$150 at time $t-1$. The firm receives a tax refund equal to $\$100 \times T_C$ which fully offsets the cost of tax asymmetry. Suppose the firm's carryback reserve is \$60 at time $t-1$. In this case, the cost of tax asymmetry is $\$40 \times T_C$.

[49] The strike price of the short put option is the negative of the book value of the tax asset. An increase in the strike price decreases the put's value and increases the value of the put spread.

future.[50] This suggests that the duration of taxable spells varies over time as a function of $B_B(t-1)$, and increases (decreases) in expansions (recessions), respectively.

The probability that a taxable firm at time $t-1$ becomes non-taxable at time t is given by the probability that the put option $P(-B_B(t-1),t)$ will be exercised at time t. Under the assumption of a risk-neutral ABM, the probability, denoted by $p_B(t-1,t)$, is given by:

$$p_B(t-1,t) = N(-d_B) = 1 - N(d_B) = 1 - N\left(\frac{E(t-1) + PV \times B_B(t-1)}{v}\right)$$

The put spread is out-of-the-money when d_B is positive. An increase in σ decreases the scaled distance d_B and increases the probability that the firm will move from a taxable state at time $t-1$ to a non-taxable state at time t.[51] Figures 2.5 illustrates the sensitivity of $p_B(t-1,t)$ to the different parameters.[52] Regarding estimation issues, the (spot) conditional probability $p_B(0,1)$ has an analytical solution. The (forward) conditional probability $p_B(t-1,t)$ can be estimated by simulation like in the carryforward case. Tax status persistence can be estimated from the spot and the forward conditional transition probabilities.

4.5 Earnings' Volatility: Implications for Risk-Shifting and Incentives

The carryback provision grants the firm a risky bond subject to the firm's earnings risk. The floating-rate coupon received by the firm equal to $M_B(t-1,t) \times T_C$ and capped at $B_B(t-1) \times T_C$, is positive when earnings are negative and is equal to 0 otherwise. The firm has incentives to manage the value of its tax asset especially when the difference between the market and the book value is large, i.e., when the put spread is out-of-the-money. There are multiple strategies that a firm can pursue to increase the market value of its tax asset. The loss-shifting hypothesis argues that firms have incentives to accelerate losses to obtain cash refunds. The hypothesis is backed by empirical evidence suggesting that firms which are financially constrained and in a tax loss position benefit the most from accelerating losses.[53] A second strategy is for the firm to increase its financial leverage (or to use complex derivatives such as a step-down swap). Profitable firms with a conservative financial policy and a sub-

---

[50] As documented in the literature, taxable firms are more likely to remain taxable in the future. See Altshuler and Auerbach (1990) and Graham (1996b).

[51] The strike price $B_B(t-1)$ is a put option. As a result, an increase in σ which increases $B_B(t-1)$ decreases $p_B(t-1,t)$. This effect is irrelevant at time $t-1$ when $B_B(t-1)$ is known with certainty.

[52] Volatility affects the probability through a second channel. The strike price $B_F(t-1)$ is a put option. As a result, an increase in v which increases $B_F(t-1)$ increases $p_F(t-1,t)$. This effect is irrelevant at time $t-1$ when $B_F(t-1)$ is known with certainty.

[53] See Scholes *et al.* (1992) and Maydew (1997) for methods of shifting revenues and expenses between years to increase a NOL carryback. They include delaying recognition of year-end sales, accelerating recognition of discretionary expenses, and timing nonrecurring transactions such as the sale of operating assets that have declined in value.

optimal capital structure are likely to hold tax assets with a high book value but a low market value. These firms have incentives to increase their interest expenses by levering-up. The decrease in the firm's taxable income will decrease the carryback reserve, decrease refundability risk and increase the market value of the tax asset.

A third strategy is risk-shifting. Like for call spreads, the impact of volatility on the market value of put spreads is ambiguous (see Figure 2.2 for an illustration). It is positive (negative) when the put spread is out- (in-) of-the-money, respectively. Consider the implications for a firm that makes a taxable profit at time $t-1$. The firm exercises the call option, increases its carryback reserve, and is granted an out-of-the money put spread maturing at time t. The difference between the book and the market value of the tax asset is high. By increasing earning's volatility, the firm decreases refundability risk and increases the market value of its tax asset. This incentivizes the firm to invest in risky projects, to decrease its cash holdings, and to scale back financial and operational hedges. Conversely, the firm has incentives to invest in low-risk project, to increase its cash holdings and to scale up financial and operational hedges when the market value of its tax asset is close to its book value.

4.6 The Firm's Expected Tax Liability

Assume the firm makes a taxable profit and exercises the call option at time $t-1$. The granting of a tax asset worth $M_B(t-1,t)$ decreases the firm's expected tax liability at time t from $T_o(t-1,t)$ without the tax asset to $T_B(t-1,t)$:

$$T_B(t-1,t) = [T_o(t-1,t) - M_B(t-1,t) \times T_C] = [E(t-1) + P(-B_B(t-1),t)] \times T_C$$

with $T_B(t-1,t) \leq T_o(t-1,t)$. The firm's tax liability at time t is equal to $E(t) \times T_C$ when the put expires worthless and is equal to $-B_B(t-1) \times T_C$ otherwise. The carryback provision leaves the option-like structure of the firm's expected tax liability intact and displaces the kink of the tax function from 0 in the no carryback case to $-B_B(t-1)$ in the carryback case. Earning's volatility increases the firm's expected tax liability (but decreases its convexity). The tax function becomes linear with a slope equal to T_C in two hypothetical cases, either when $B_B(t-1)$, or when σ, goes to infinity. Under the assumption of a risk-neutral ABM, the firm's expected tax liability is given by:

$$T_B(t-1,t) = [E(t-1) \times N(d_B) - PV \times B_B(t-1) \times N(-d_B) + v \times n(-d_B)] \times T_C$$

Figures 2.6 illustrates the sensitivity of $T_B(t-1,t)$ to the various parameters. Finally, the expected cost of tax asymmetry is given by the difference between $T_B(t-1,t)$ and the tax function obtained in the tax symmetry case $T_O^*(t)$:

$$T_B(t-1,t) - T_O^*(t-1) = P(-B_B(t-1),t) \times T_C$$

The expected cost of tax asymmetry in the no carryback case obtains as a special case by setting $B_B(t-1)$ equal to 0 in the formula above. This shows that the carryback provision decreases the expected cost of tax asymmetry by an amount equal to $M_B(t-1,t)$. The expected cost increases with a decrease in the book value of the tax asset and with earnings' volatility.

4.7 The Firm's Expected Marginal Tax Rate

Assume the carryback reserve $B_B(0)$ is positive at time 0. The firm is currently taxable and the NOL dummy is equal to the statutory tax rate T_C. As a one-period estimate, the NOL proxy over-estimates the firm's marginal tax rate by ignoring the likelihood that the firm may receive tax refunds in the future. The statutory tax rate T_C provides an upper bound on the marginal tax rate. At the other end of the spectrum, a lower bound is obtained by assuming that the firm will be non-taxable with certainty at time 1. In this case, the \$1.0 increase in $B_B(0)$ associated with the \$1.0 increase in $E(0)$ increases the tax refund by \$1.0 $\times T_C$ at time 1. The tax refund worth \$$PV \times T_C$ at time 0 decreases the firm's marginal tax rate from T_C to $(1-PV) \times T_C$. However, this proxy under-estimates the marginal tax rate by failing to account for earnings' uncertainty. The firm's true but unobservable marginal tax rate is in-between the lower and the upper bound given by $(1-PV) \times T_C$ and T_C, respectively.

The derivatives framework can be used to capture earnings' uncertainty. Define by t, the time when the firm switches from a taxable to a non-taxable status. As discussed in Section 4.4, this happens when the firm exercises the put option $P(-B_B(t-1),t)$, which requires $E(t)$ to be lower than $-B_B(t-1)$ at time t. Consider the impact of a \$1.0 increase in $E(0)$, and the resulting \$1.0 increase in $B_B(0)$, on the firm's tax liability at time t. The \$1.0 increase in $B_B(0)$ at time 0 implies that the non-taxable firm will receive an incremental \$$(1 \times T_C)$ worth of tax refund at time t. Appendix A.4 shows that the marginal tax rate of a firm that switches from a taxable to a non-taxable status at time t, denoted by $MTR_B(0,t)$, is given by:

$$MTR_B(0,t) = \left[1 - PV \times \left(\prod_{\tau=1}^{\tau=t} N\big(d_B(\tau-2)\big)\right) \times N\big(-d_B(\tau-1)\big)\right] \times T_C$$

with $N\big(d_B(-1)\big)$ set equal to 1.0. In the formula above, the product term captures the probability that the firm remains taxable from time 0 until time $t-1$, while the final term captures the probability that the firm switches from a taxable status at time $t-1$ to a non-taxable status at time t.[54] Given the uncertainty about time t, a simulation approach is used to estimate the marginal tax rate as in the carryforward case. It is denoted by $MTR_B(0)$ and is given by:

$$MTR_B(0) = \frac{1}{S} \times \sum_{s=1}^{S} MTR_B(0, t_s)$$

where S is the total number of simulated earnings-paths and where t_s denotes the time when the firm becomes non-taxable for the first time on sample-path s, i.e., when the put option $P(-B_B(t-1, s), t)$ is exercised, where $-B_B(t-1, s)$ denotes the carryback reserve on sample path s.

5.0 The Unconstrained Joint Carryforward and Carryback Provisions

The previous sections analyzed the carryforward and the carryback provisions as standalone provisions. They are not mutually exclusive. Tax systems may allow firms to carryback and to carryforward losses. This section discusses intertemporal and substitution effects between the two tax provisions under the assumption that both provisions are unconstrained.

5.1 Intertemporal and Substitution Effects

The joint provision does not generate incremental tax cash-flows relative to either one of the two standalone provisions. Rather, it generates intertemporal and substitution effects that smooth the tax cash-flows received by the firm. Appendix A.13 shows that the joint tax provision provides two benefits. The first benefit is to decrease refundability risk by allowing the firm to monetize a risky asset, either a call spread or a put spread, for a portfolio composed of cash (in whole or in part) and a less risk asset (if any). The cash component is the immediate tax credit or tax refund received by the firm when it makes a taxable profit or a tax loss, respectively.[55] The asset component is a call spread or a put spread with a lower strike price (than in the standalone case). This increases the likelihood that the firm will exercise the derivative contract in the future and decreases refundability risk. A second benefit is to move forward in time the tax credit and tax refund received by the firm and to increase their

[54] Assume for example that tax losses can be carried back for one year only. In this case, $MTR_B(0,1)$ has an analytical solution given by $[1 - PV] \times T_C \leq MTR_B(0,1) = \left[1 - PV \times N\left(-d_B(0)\right)\right] \times T_C \leq T_C$.

[55] Consider a firm that incurs a tax loss. Under the standalone carryforward provision, the firm exercises the put and is granted a call spread. Under the joint provision, the firm gets a tax refund (conditional on the carryback reserve being positive), exercises the put and is granted a call spread when the tax loss exceeds the carryforward reserve. De facto, the firm swaps a risky asset for a package of cash and a less risky asset. A symmetric argument holds when the firm makes a taxable profit instead of a tax loss.

present values. The two effects (risk and time value) increase the market value of the tax asset relative to its standalone value.[56]

5.2 Book and Market Value of the Tax Assets, Expected Tax liabilities and Marginal Tax Rates

The valuation principles remain unchanged once the carryforward and carryback reserved have been purged from interaction and substitution effects. The derivation assumes that only one of the two reserves is positive at any point in time and ignores the option value granted to the firm to exercise the carryback or the carryforward option when experiencing a tax loss.[57]

5.2.1 Book Value of the Tax Asset

The intertemporal and substitutions effects between the two tax provisions decrease the original book values $B_F(t-1)$ and $B_B(t-1)$ obtained under the standalone provisions. The adjusted book values, denoted by $B_F^*(t-1)$ and $B_B^*(t-1)$, are derived in Appendix A.1.3. Under the joint provision, the book value of the tax assets at time $t-1$ remains path-dependent options (with lower strike prices) given by:

$$\begin{cases} B_F^*(t-1) = P(B_F^*(t-2), t-1) \leq B_F(t-1) \\ B_B^*(t-1) = C(-B_B^*(t-2), t-1) \leq B_B(t-1) \end{cases}$$

The pair $B_F^*(t-1)$, $B_F(t-1)$ differs by the amount of the tax refund received by the firm when incurring a tax loss (see numerical example 3.1 in Appendix A.1.3). Likewise, the pair $B_B^*(t-1), B_B(t-1)$ differs by an amount equal to the tax credit received by the firm when making a taxable profit (see numerical example 3.2 in Appendix A.1.3).

[56] The economic effects of carrybacks on firms' investment and financing policy have been investigated in the literature. See Boynton and Cooper (2003), Graham and Kim (2009) and Dobridge (2016) for example. However, interaction and substitution effects and their impact on the market value of tax assets have not been discussed. An increase in the length of the carryback period increases the current and expected carryback reserve and decreases the marginal benefit of an additional $1.0 of tax refund. Coincidently, it decreases the carryforward reserve and increases the marginal benefit of an additional $1.0 dollar of tax credit.

[57] A positive carryback reserve requires the firm to have exhausted its carryforward reserve and to have paid taxes, i.e., $B_B^*(t-1) > 0$ if $B_F^*(t-1) = 0$. Likewise, a positive carryforward requires the firm to have exhausted its carryback reserve and to have incurred (or to incur) tax losses, i.e., $B_F^*(t-1) > 0$ if $B_B^*(t-1) = 0$. However, both reserves could be positive at the same time because of i. caps imposed on carryforwards or carrybacks, ii. expected changes in the statutory tax rate, iii. the complexity of the tax code. For example, a non-taxable firm that expects an increase in the statutory tax rate may have incentives to increase its carryforward reserve rather than claim a tax refund.

5.2.2 Market Value of the Tax Asset

Conditional on the put option or the call option being exercised at time $t-1$, the firm is granted either a one-year call spread when $B_F^*(t-1) > 0$, or a one-year put spread when $B_B^*(t-1) > 0$. It is denoted by $M_{FB}^*(t-1,t)$ and is given by:

$$M_{FB}^*(t-1,t) = \begin{cases} M_F^*(t-1,t) = [C(0,t) - C(B_F^*(t-1),t)] \ \ IF\ B_F^*(t-1) > 0 \\ M_B^*(t-1,t) = [P(0,t) - P(-B_B^*(t-1),t)]\ IF\ B_B^*(t-1) > 0 \end{cases}$$

which for comparative purposes can be expressed as put spreads only, with:

$$M_{FB}^*(t-1,t) = \begin{cases} M_F^*(t-1,t) = [P(0,t) - P(B_F^*(t-1),t) + PV \times B_F^*(t-1)] \ \ IF\ B_F^*(t-1) > 0 \\ M_B^*(t-1,t) = [P(0,t) - P(-B_B^*(t-1),t)]\ IF\ B_B^*(t-1) > 0 \end{cases}$$

The joint provision endows the firm with a perpetual bond paying an annual floating-rate coupon equal to $M_{FB}^*(t-1,t)$ times the statutory tax rate:

$$M_{FB}^*(0) = \sum_{t=1}^{\infty} M_{FB}^*(t-1,t) \times T_C < \sum_{t=1}^{\infty} P(0,t) \times T_C = M(0)$$

with $M_{FB}^*(0) > M_F(0)$ and $M_{FB}^*(0) > M_B(0)$. Relative to the standalone provision case, the intertemporal smoothing increases the frequency of positive coupon payments and decreases their variability. The coupons are positive except in the two unlikely cases when the firm always makes taxable profits or tax losses.

5.2.3 Transition Probabilities

The transition probabilities are higher than those obtained under either one of the two standalone provisions because of the decrease in the options' strike prices. They are given by:

$$p_{FB}^*(t-1,t) = \begin{cases} p_F^*(t-1,t) = N\left[\left(\dfrac{E(t-1) - PV \times B_F^*(t-1)}{v}\right)\right] \ IF\ B_F^*(t-1) > 0 \\ p_B^*(t-1,t) = N\left[-\left(\dfrac{E(t-1) + PV \times B_B^*(t-1)}{v}\right)\right] IF\ B_B^*(t-1) > 0 \end{cases}$$

The decrease in $B_F^*(t-1)$, relative to $B_F(t-1)$, implies that the non-taxable firm at time $t-1$ is more likely to become taxable at time t. Likewise, the decrease in $B_B^*(t-1)$, relative to $B_B(t-1)$, implies that the taxable firm at time $t-1$ is more likely to become non-taxable at time t.

5.2.4 The Firm's Expected Tax Liability

Except for the lower carryforward or carryback reserve, the firm's expected tax liability at time t, denoted by $T^*_{FB}(t-1,t)$, remain unchanged to:

$$T^*_{FB}(t-1,t) = \begin{cases} T^*_F(t-1,t) = C(B^*_F(t-1),t) \times T_C \ IF\ B^*_F(t-1) > 0 \\ T^*_B(t-1,t) = [E(t-1) + P(-B^*_B(t-1),t)] \times T_C \ \ IF\ B^*_B(t-1) > 0 \end{cases}$$

The kink in the tax function moves to the right to $B^*_F(t-1)$ when the carryforward reserve is positive and to the left to $-B^*_B$ when the carryback is non-zero. Tax functions remain positively convex. The only difference are the two kinks of the tax functions which bracket 0 more closely than in the standalone case. These results question the merits of approximating tax functions by piecewise linear functions, where losses are taxed at a lower rate and profits at a higher rate. The linear approximation fails to capture the convexity of the tax function and fails to identify $B^*_F(t-1)$ and $-B^*_B(t-1)$ as the relevant (time-varying) convexity thresholds.[58]

5.2.5 The Firm's Marginal Tax Rates

The joint tax provision makes it more difficult to express the marginal tax rates as a function of conditional transition probabilities. Assume for example that $B^*_F(0) > 0$, and assume that the firm is taxable at time 1 and non-taxable at time 2 with certainty. A $1.0 increase in earnings at time 0, decreases the carryforward reserve by $1.0 and increases the tax liability by $1.0 at time 1. Without the carryback reserve, the firm's marginal tax rate is equal to $PV(1) \times T_C$ as discussed in Section 3.7. Under the joint tax provision, the $1.0 increase in taxable profits at time 1 increases the carryback reserve by $1.0 and the(before-tax) tax refund by $1.0 at time 2. Under the joint tax provision, the marginal tax rate is equal to $\big(PV(1) - PV(2)\big) \times T_C$. This suggests that the marginal tax rate derived in Section 3.7 under the standalone carryforward provision over-estimates the marginal tax rate that would be obtained under the joint provisions. Conversely, the marginal tax rate derived in Section 4.7 under the standalone carryback provision under-estimates the marginal tax rate that would prevail under the joint provisions.

6.0 Empirical Implications

The derivatives framework has important implications for empirical research that involves tax assets.

6.1 Estimation of Conditional Versus Unconditional Transition Probabilities

[58] See for example Sarkar (2008) and Langenmayr and Lester (2018). Their simple scheme assumes a linear tax function equal to $\theta \times T_C$, where θ is a pre-determined fixed parameter equal to 1.0 when earnings are positive and less than 1.0 when earnings are negative.

Various estimation techniques have been suggested in the literature to estimate the transition probability that a firm will move from a taxable status to a non-taxable status in the future (or vice-versa). Using a first-order and a second-order Markov process, Auerbach and Poterba (1987) document a high level of tax status persistence.[59] As acknowledged by the authors, a limitation of their statistical approach is to assume that a firm's current tax status contains all the relevant information about its transition prospects, i.e., all firms have identical transition probabilities and the probabilities do not vary over the sample period. Under these assumptions, the firm's tax status is assumed to be fully described by its current and previous years' states.[60] These assumptions do not hold in practice. The transition probabilities depend on the distance between earnings and the present value of the carryforward or carryback reserve scaled by earnings' volatility. In addition, the transition matrix approach only captures the effect of tax laws during the sample period used to estimate the probabilities. Transition probabilities are outdated when tax laws change.[61]

The derivatives framework provides a valuable alternative to statistical and econometric models used in the literature. The framework does not impose firms to have identical transition probabilities over time and across firms and can be used to estimate conditional (rather than unconditional) transition probabilities from historical data. To see this, assume that the carryforward reserve $B_{F,i}(t-1)$ and the carryback reserve $B_{B,i}(t-1)$ are observable for a panel dataset of firms composed of N firms over a (past) sample period composed of T years with $i = 1, \dots, N$ and $T = 1, \dots, T$. Denote by $E_i(t-1)$ the earnings of firm i at time $t-1$ assumed to follow a risk-neutral ABM with drift R_F and volatility σ_i, assumed constant over time without loss of generality. Suppose that the goal is to estimate the conditional probability that firms will switch from either a non-taxable status at time $t-1$ to a taxable status at time t when endowed with the carryforward reserve $B_F(t-1)$, or from a taxable status at time $t-1$ to a non-taxable status at time t for firms endowed with carryback reserve $B_B(t-1)$. Denote by $d_{F,i}(t-1)$ and $d_{B,i}(t-1)$ firm's i scaled distance:

[59] They find that it is very unlikely for a firm without a tax loss carryforward to incur one and for a firm with a tax loss carryforward to become taxable again. The probability that a firm with two previous years of loss carryforward would remain in the loss carryforward state is 0.92 while the probability of remaining carryforward-free after two years of being currently taxable is 0.97.

[60] Some of these limitations have been addressed in the literature. Probit models and Poisson regressions have been suggested as an alternative to Markov processes. Using a Poisson regression, Gendron, Anderson and Mintz (2003) regress tax exhaustion on a set of explanatory variables and find that the relationship between a proxy for carryforwards, and debt financing (among other variables) is positive and statistically significant. Further, they find that the restriction that transition probabilities are independent is strongly rejected by the data. A tax-exhausted firm at times $t-1$, t, and $t+1$ does not have independent probabilities of being tax-exhausted in each of those 3 separate years.

[61] See discussion in Shevlin (1990).

$$\begin{cases} d_{F,i}(t-1) = \dfrac{E_i(t-1) - PV \times B_{F,i}(t-1)}{v_i} & \text{if } B_{F,i}(t-1) > 0 \\ d_{B,i}(t-1) = -\dfrac{E_i(t-1) + PV \times B_{B,i}(t-1)}{v_i} & \text{if } B_{B,i}(t-1) > 0 \end{cases}$$

The issue is to convert the scaled distance into a probability. A "frequentist" approach can be used as a substitute to the parametric approach based on the Normal distribution. The first step entails calculating the pair $[d_{F,i}(t-1), d_{B,i}(t-1)]$ for the $N \times T$ firms-years in the sample. The second step entails ranking in ascending order and partitioning all the pairs obtained in step 1 into a prespecified number of groups (ranks), denoted by g_F and g_B, respectively, with $g_F = 1, \dots, G_F$ and $g_B = 1, \dots, G_B$ where G_F and G_B are the total number of groups. The third step entails calculating for all the firms within a group the proportion of firms that (actually) switched from a non-taxable status at time $t-1$ to a taxable status at time t when $B_{F,i}(t-1) > 0$ (or vice versa when $B_{B,i}(t-1) > 0$). Denote by $f(g_F)$ and $f(g_B)$ with $0 \le f(g_F) \le 1.0$ and $0 \le f(g_B) \le 1.0$ the empirical frequencies.[62] The fourth step entails fitting an "empirical" transition probability function by plotting $f(g_F)$ onto g_F and $f(g_B)$ onto g_B, respectively. The "frequentist" approach should generate a conditional transition probability function, denoted by $\widehat{p_F}(t-1,t)$, that increases from 0 to 1.0 as $d_F(t-1)$ increases from negative to positive values for firms endowed with a positive carryforward reserve at time $t-1$ (as in Figure 1.5). Conversely, the conditional transition probability function $\widehat{p_B}(t-1,t)$ should be decreasing from 1.0 to 0 as $d_B(t-1)$ increases from negative to positive values for firms endowed with a carryback reserve at time $t-1$ (as in Figure 2.5).

The benefits of the frequentist approach are four-fold. First, it generates firm-specific estimates of transition probabilities rather than average estimates. Second, it generates conditional transition probabilities rather than unconditional probabilities by mapping firm's i scaled distance $d_{F,i}(t-1)$ or $d_{B,i}(t-1)$ (or their ranks) into $\widehat{p_{F,i}}(t-1,t)$ or $\widehat{p_{B,i}}(t-1,t)$, respectively. Third, the approach allows conditional transition probability functions to be estimated from the pooled sample across firms/years or across firms only. The benefit of the latter approach is to allow the probability function to vary over time and to capture the impact of changes in the tax code on transition probabilities. Fourth, the approach can be used to gauge the validity of the distributional assumption regarding earnings. This can be done by comparing the empirical distribution functions to their analytical counterparts, i.e., by comparing $\widehat{p_F}(t-1,t)$ to $p_F(t-1,t) = N(d_F)$ and $\widehat{p_B}(t-1,t)$ to $p_B(t-1,t) = N(-d_B)$.

[62] This frequentist approach is analogous to KMV $EDFs$ calculations from the distance-to-default DDs.

Regression techniques can also be used to estimate the average conditional transition probability for any cross-section of firms in any given year. Assume for example that firm i holds the positive carryforward reserve $B_{F,i}(t-1)$. Its expected tax liability at time t is given by a call option on earnings with a strike price equal to $B_{F,i}(t-1)$ times the statutory tax rate. As discussed in Section 3.6, under the assumption that earnings follow an ABM and, the firm's expected tax liability is given by:

$$T_{F,i}(t-1,t) = C\big(B_{F,i}(t-1),t\big) \times T_C = \big[E_i(t-1) \times N(d_F) - PV \times B_{F,i}(t-1) \times N\big(d_{F,i}\big) + v_i \times n\big(d_{F,i}\big)\big] \times T_C$$

with $N\big(d_{F,i}\big) = p_{F,i}(t-1,t)$. Under the assumption that taxes at time t are equal to their expected values plus a random noise, the average transition probability (across firms) at time $t-1$ can be estimated by running a cross-sectional regression of taxes paid at time t as the dependent variable onto a set of explanatory variables formed of earnings at time $t-1$, the carryforward reserve at time $t-1$ and an estimate of earnings' volatility:

$$T_{F,i}(t) = a_0 + a_1 \times E_i(t-1) + a_2 \times B_{F,i}(t-1) \ + a_3 \times v_i + \varepsilon_i(t)$$

Under the null hypothesis, the slope coefficient a_1 is an estimate of the average conditional transition probability at time $t-1$ for the sample of firms under consideration.[63] The cross-sectional regression can be performed year by year and for any cross-section of firms sorted as a function of industry or any other firm's characteristics (i.e., size….).

6.2 Estimation of the Tax Benefits of Carryforwards and Carrybacks

Empirical work that investigates the benefits of tax assets typically use the book value as a proxy for the market value, i.e., $B_F(t-1)$ for $M_F(t-1,t)$, and $B_B(t-1)$ for $M_B(t-1,t)$. Failure to adjust for the riskiness of tax assets generates upward biased estimates of tax benefits.

Suppose for example that the goal of an empirical investigation is to estimate the tax benefits of carryforwards and their ability at predicting taxes using a cross-section of firms for which $B_F(t-1) \ > 0$. Assume that after proper scaling of the dependent and independent variables, the null hypothesis is tested by running a cross-sectional regression of taxes paid at time t onto $B_F(t-1)$ observed at time $t-1$ as the explanatory variable:

[63] Under the null hypothesis, the slope coefficient a_2 is an estimate of $PV \times N(d) \times T_C$. Scaling a_2 by a_1 provides an estimate of the PV factor, i.e., the average drift term.

$$T_{F,i}(t) = a_0 + a_1 \times E_i(t) + a_2 \times B_{F,i}(t-1) + \varepsilon_i(t)$$

where under the null hypothesis, the regression parameter a_1 provides an estimate of the statutory tax rate T_C and a_2 provides an estimate of the average tax benefits at time t per \$1.0 of carryforwards available at time $t-1$, with a_2 hypothesized to be negative.

This specification fails to capture refundability risk and the difference between the market value and the book value of the tax asset. To model earnings' uncertainty, assume that earnings follow an ABM. As discussed in Section 3.2, the market value of the tax asset at time $t-1$ is given by:

$$M_F(t-1,t) = \left[E(t-1) \times [N(d) - N(d_F)] + PV \times B_F(t-1) \times N(d_F) + v \times [n(d) - n(d_F)]\right]$$

Replacing the book value $B_F(t-1)$ by the market value of the tax asset $M_F(t-1,t)$ in the regression equation yields:

$$T_{F,i}(t) = b_0 + b_1 \times E_i(t) + b_2 \times B_{F,i}(t-1) + b_3 \times E_i(t-1) + b_4 \times v_i + \varepsilon_i(t)$$

The comparison of the two specifications shows that the first regression model imposes the constraint $b_3 = b_4 = 0$ onto the second regression equation. The constraint holds only when earnings are high relative to $B_{F,i}(t-1)$ but not in the general case.[64] The comparison highlights two serious econometric problems with the first specification. First, the parameter estimate a_2 is upward biased, i.e., over-estimates the tax benefits of carryforwards. Under the null hypothesis, $b_2 = -PV \times N(d_F) \times T_C$. Given that $N(d_F)$ is a probability, the slope coefficient b_2 captures the present value of the expected tax benefits. The bias is time-dependent and is likely to be high in recessions when earnings are low and to be low in expansions when earnings are high (which generates cross-correlated residuals when observations are pooled across years). Second, the regression suffers from an omitted variable problem by leaving out the uncertainty in the value of the tax asset (captured by b_4 in the second specification). Given that earnings' volatility is correlated with the dependent variable and the explanatory variable, the omitted variable problem is likely to bias the regression parameters estimates further. Adding lagged earnings and earnings' volatility as explanatory variables as done in the second regression equation generates unbiased estimates of the tax benefits of carryforwards.

[64] When earnings are high, the market value of the tax asset is close to the book value. The call spread is in-the-money and $N(d) \approx N(d_F) \approx 1.0$, and $n(d) \approx n(d_F)$, giving, $b_2 = PV \times a_2$, and $b_3 = b_4 = 0$.

Finally, the sample can be augmented by all the firms having a positive carryback reserve. Under the assumption that either one of the two reserves is positive at any point in time, the tax benefits of carryforwards and carrybacks can be assessed by running the following regression:

$$T_{FB,i}(t) = c_0 + c_1 \times E_i(t) + c_2 \times M_F(t-1,t) \times \mathrm{I} + c_3 \times M_B(t-1,t) \times (1-I) + \varepsilon_i(t)$$

where I is a dummy variable taking a value of 1.0 when $B_{F,i}(t-1) > 0$ and a value of 0 otherwise.

6.3 Endogeneity Issues and Risk Taking

The theoretical literature argues that tax provisions alleviate the cost of tax asymmetry by allowing firms to shift part of the investment risk to the government which incentivize firms to increase their overall risk-taking.[65] Empirically, the relevant issue is whether firms consider the tax loss treatment rules when making investment decisions. The empirical evidence is mixed. Papers that look at non-tax rate elements of the corporate tax system report only small effects of loss rules on the level of investment (with cash flows having more important effects). Other papers find tax rates have a positive effect on risk-taking for firms that expect to use losses, and a weak negative effect for those that do not have such opportunities.[66] There is also evidence of an asymmetric response whereby firms reduce risk taking in response to an increased tax burden but do not significantly increase risk taking in response to a reduced tax burden.[67] The null hypothesis that tax loss rules affect risk-taking is typically tested by regressing investment-related variables such as the volume of investment or the riskiness of investment, as the dependent variable onto tax-related variables used as explanatory variables (along with control variables). Different tax variables are used in the literature. They include statutory tax rates, the book value of tax assets, dummy variables that indicate whether the carryforward or carryback reserve is equal to 0, or variables that capture key characteristics of the tax provisions, such as the length of the carryforward or carryback period.

This strand of literature raises two issues. First, from a theoretical standpoint, as discussed in the previous sections, tax loss rules do not alter the convexity of the tax function. Tax loss rules merely displace the kink of the tax function away from 0 either to the left by an amount equal to $-B_B(t-1)$ when the carryback reserve is positive, or to the right by an amount equal to $B_F(t-1)$ when the carryforward reserve is positive. From an empirical perspective, the variable of interest is the distance d_B, d and d_F between earnings and the present value of strike price, $-B_B(t-1), 0,\ B_F(t-1)$ scaled by earnings' volatility, respectively. When the (absolute

[65] See for example Langenmayr and Lester (2018) p. 237.

[66] Langenmayr and Lester (2018) find that loss carryback periods and, to a lesser extent, loss carryforward periods are positively and significantly related to the level of firm risk-taking. They conclude that to the extent that governments want to encourage risk-taking, longer tax loss periods, particularly for carrybacks, provide appropriate incentives.

[67] See Devereux, Keen, and Schiantarelli (1994), Ljungqvist, Zhang, and Zuo (2017), and Langenmayr and Lester (2018).

value of the) scaled distance is large, the call or the put option is either deep-in or deep-out-of-the money and the expected tax function is (quasi) linear irrespective of tax loss rules. In this case, tax policies should not affect firms' investment decisions. When the scaled distance is small, the tax function is convex, the option is at- or close to being at-the-money and earnings volatility increases the firms' tax liabilities which, irrespective of tax loss rules, disincentivizes firms to invest in risky projects (and incentivizes them to hedge).[68] This effect is partly mitigated by the fact that, (irrespective of tax loss provisions), the convexity of the tax function decreases with an increase in earning's volatility, i.e., the adverse effects of convexity decrease with volatility. This suggests that the empirical tests should be conditioned on the scaled distance between earnings and the (present value) of the carryforward or carryback reserve and on earnings' volatility.[69]

Second, from an econometric standpoint, the empirical tests suffer from an endogeneity issue. The active management of tax assets creates an endogeneity problem when investment-related variables, such as investment volume or risk, are regressed onto a tax-related variable. Consider a firm at time $t-1$ endowed either with the tax asset $M_F(t-1,t)$ when $B_F(t-1)>0$, or with $M_B(t-1,t)$ when $B_B(t-1)>0$. They are valuable assets that can be used by firms to get either tax credits or tax refunds at time t. As discussed in Sections 3.5 and 4.5, refundability risk incentivizes firms to use their investment policy at time $t-1$ to boost the market values of their tax assets when they are low relative to their book values, i.e., when the call spread or the put spread is out-of-the-money. The active management of tax assets imply that tax-related variables and investment-related variables are determined simultaneously and endogenously.[70]

Finally, the difference between the book value and the market value of tax assets shades light on the relationship between tax incentives and tax asymmetry. Suppose that an econometric model is designed to estimate the relative responsiveness of an investment ratio (the dependent variable) to a change in an investment tax credit (the explanatory variable) for taxable and non-taxable firms. Inferences are likely to be sensitive to the definition of the firm's tax status. The use of a simple dummy variable that indicates whether a firm is taxable or not, versus a variable that exploits information contained in the market value of the tax asset is likely to generate different inferences regarding the relative policy response. The likelihood that a non-taxable firm will become taxable in the future depends on the difference between the market value and the book value of its tax asset. High cash-

[68] The linearity of the tax function increases as earnings move away from the present value of the strike price. However, the range of prices over which linearity holds changes over time as a function of $B_B(t-1)$ and $B_F(t-1)$.

[69] In their Proposition 1, Langenmayr and Lester (2018) argue that "*better loss offset increases firm risk taking.*" This holds true when the tax refund is assumed to be a linear function of the loss as per their assumption. However, as discussed in Section 4, the tax refund $M_B(t-1,t)$ and the tax liability are convex functions of earnings.

[70] Similar endogeneity issues arise when debt-related (rather than investment-related) variables are used as dependent variables.

flows increase the market value of the carryforwards relative to its book value and increase the likelihood that the firm will become taxable in the near term. This may explain the empirical finding that firms are more (less) responsive to investment tax incentives when their cash flows are high (low).[71] The carryforward (carryback) reserve is more valuable when earnings are high (low), respectively. This reduces the cost of tax asymmetry and incentivizes firms to take more risk and to be more responsive to investment tax incentives.

7. Conclusion

The paper suggests an analytical framework to value tax assets as derivative contracts on earnings. Tax assets are modelled as path-dependent compound options, a put-on-a-call option contract for the carryforward and a call-on-a-put option contract for the carryback provision. The framework captures earnings' uncertainty and the dynamic features of the tax code. It shows that refundability risk creates a wedge between the book value and the market value of tax assets. The latter depends on two offsetting effects, a positive cash-flow and an adverse likelihood effect. Refundability risk decreases the marginal value of one additional dollar of tax loss (profit) and decreases the present value of the future tax credits (refunds) to be received by the firm, respectively.

The carryforward and carryback provisions endow firms with valuable tax assets, raising the issue of their effectiveness at decreasing the adverse impact of tax asymmetry. Tax assets decrease the cost of tax asymmetry but leave the convexity of the tax function unchanged. The cost is a long option contract whose value increases with earnings' volatility. Tax liabilities increase with earnings volatility incentivizing firms to shun risky projects all else equal. However, the difference between the market value and the book value of tax assets incentivizes firms to actively manage their values, using their investment, financing, and hedging policies. Firms have incentives to invest in risky projects and to scale back hedging to boost the market value of their tax assets when the difference is large. Conversely, they have incentives to invest in low-risk projects to preserve the market value of tax assets when the difference is small. In the latter case, the wealth transfer from the government to the shareholders is offset by possible wealth transfers from the shareholders to the creditors.

The derivatives framework can be used to estimate conditional transition probabilities and marginal tax rates. Unlike their unconditional counterparts which assume identical, time-invariant and exogeneous probabilities, the conditional probabilities are firm specific and capture all the information contained in the distance between earnings and the present value of the book value of the tax asset scaled by earnings' volatility. Tax exhausted firms endowed with a carryforward reserve are more likely to be tax exhausted when the scaled distance is large. A similar result holds for firms endowed with a large carryback reserve. The conditional probabilities have a closed

[71] See Edgerton (2010).

form solution when earnings are assumed to follow a process such as an ABM. Marginal tax rates can be inferred from the conditional probabilities.

A limitation of the current framework is to exclude the multiple provisions of the tax code that allow firms to decrease their tax liabilities. A companion paper extends the framework to include non-debt and debt tax shields. The results are left unchanged except for the options' higher strike prices. The derivatives framework allows the tax benefits of debt to be valued in a novel way. It shades new light on intertemporal and substitution effects between tax assets (carryfowards and carrybacks) and tax shields. These effects are similar to those that arise between the carryforward and the carryback provision. It shows that substitution effect between the carryback provision and tax shields decrease the tax benefits of debt all else equal.

References

Albring , S., D. Dhaliwal, I. Khurana, and R. Pereira, (2011) “Short-Term Incentive Effects of a Reduction in the NOL Carryback Period,” Journal of the American Taxation Association, 33 (2), 67–88.

Auerbach, A., and J. Poterba, (1987), “Tax loss carryforwards and corporate tax incentives,” in “The Effects of Taxation on Capital Accumulation,” ed. M. Feldstein, 305-42.

Altshuler, R., and A. Auerbach (1990), “The significance of tax law asymmetries: An empirical investigation.” Quarterly Journal of Economics 105 (1), 61–86.

Altshuler, R., A. Auerbach, M. Cooper, and M. Knittel, (2009), “Understanding U.S. corporate tax losses.” Tax Policy and the Economy 23 (1): 73–122.

Ammann, M., and M. Genser, (2004), “A Testable Ebit-Based Credit Risk Model.” Available at SSRN: https://ssrn.com/abstract=498803

Bachelier, L. (1900), "Théorie de la spéculation." Annales Scientifiques de l'École Normale Supérieure, 3 (17), 21-86.

Bates, T., K. Kahle, and R. Stulz, (2009), “Why do U.S. firms hold so much more cash than they used to?” Journal of Finance 44, 1985-2021.

Boynton, C., and M. Cooper, (2003) "The Impact of Changing the Corporate Net Operating Loss Carryback Period." Proceedings of the Ninety-Sixth Annual Conference on Taxation, 2003: 231-239.

Brooks, R., and J. Brooks, (2017), “An option valuation framework based on arithmetic Brownian motion: Justification and implementation issues.” The Journal of Financial Research, 40, 3, 401-427.

Cooper, M., and M. Knittel, (2006), “Partial Loss Refundability: How Are Corporate Tax Losses Used?” National Tax Journal, 2006, 59, 3, 651-63.

Cooper, M. and M. Knittel, (2010), “The implications of tax asymmetry for U.S. corporations.” National Tax Journal, 63 (1): 33–61.

Denis, D. and S. McKeon, (2020), “Persistent Negative Cash Flows, Staged Financing, and the Stockpiling of Cash Balances.” Available at SSRN: https://ssrn.com/abstract=3622453.

Devereux, M. P., M. Keen, and F. Schiantarelli, (1994), “Corporation tax asymmetries and investment. Evidence from U.K. panel data.” Journal of Public Economics 53 (3): 395–418.

Dobridge, C., (2016), “Fiscal Stimulus and Firms: A Tale of Two Recessions.” Finance and Economics Discussion Series 2016-013, Divisions of Research and Statistics and Monetary Affairs, Federal Reserve Board, Washington, DC. Available at: https://www. federalreserve.gov/econresdata/feds/2016/files/2016013pap.pdf.

Domar, E., and R. Musgrave, (1944), “Proportional income taxation and risk-taking.” Quarterly Journal of Economics 58 (3): 388–422.

Edgerton, J., (2010), “Investment incentives and corporate tax asymmetries.” Journal of Public Economics 94, (11/12), 936–952.

Erickson, M., and S. Heitzman, (2010), “NOL poison pills: Selectiva v. Versata.” Tax Notes (June 23).

Favilukis, J., R., Giammarino, and J. Pizarro (2016), “Tax-Loss Carry Forwards and Returns.” Available at SSRN: https://ssrn.com/abstract=2812410

Graham, J., (1996a.), “Debt and the marginal tax rate.” Journal of Financial Economics 41, 41-73.

Graham, J., (1996b.), “Proxies for the corporate marginal tax rate.” Journal of Financial Economics 42, 187-221.

Graham, J., and H. Kim, (2009),"The Effects of the Length of the Tax-Loss Carryback Period on Tax Receipts and Corporate Marginal Tax Rates." National Tax Journal, 62 , 413-427.

Graham, J., and C. Smith (1999), “Tax incentives to hedge.” Journal of Finance 54 (6): 2241–2262.

Green, R. and E. Talmor, (1985), “The structure and incentive effects of corporate tax liabilities.” Journal of Finance, 40, 4, 1095-1114.

Haug, E., (2007), “The Complete Guide to Option Pricing Formulas.” McGraw-Hill, second edition, 2007.

Heitzman, S. and R. Lester, (2020), “Tax Loss Measurement.” Avail. at SSRN: https://ssrn.com/abstract=3553527

Kaymack, B., and I. Schott, (2019), “Loss-offset provisions in the corporate tax code and misallocation of capital.” Journal of Monetary Economics, 105, 1-20.

Langenmayr, D., Lester, R., (2018), “Taxation and corporate risk-taking.” The Accounting Review 93, 237-66.

Leland, H., (1994), “Corporate Debt Value, Bond Covenants, and Optimal Capital Structure.” The Journal of Finance, 49, 4, 1213-1252.

Ljungqvist, A., L. Zhang, and L. Zuo, (2017) “Sharing risk with the government: How taxes affect corporate risk-taking.” Journal of Accounting Research 55 (3), 669–707

Majd, S. and S., Myers, (1985), “Valuing the Government's Tax Claim on Risky Corporate Assets.” No 1553, NBER Working Papers from National Bureau of Economic Research, Inc.

Majd, S. and S., Myers, (1987), “The effects of taxation on capital accumulation.” Ed.:M. Feldstein, University of Chicago Press.

Manzon, G., (1994), “The role of taxes in early debt retirement.” Journal of the American Taxation Association 16, 87–100.

Maydew, E., (1997), “Tax induced earnings management by firms with net operating losses.” Journal of Accounting Research 35 (1), 83– 96.

Musiela, M., and M. Rutkowski, (2005), “Martingale Methods in Financial Modelling,” 2nd Edition, Springer.

Sarkar, S., (2008), “Can Tax Convexity be Ignored in Corporate Financing Decisions.” Journal of Banking & Finance 32(7), 1310-1321.

Scholes, M., G. Wilson, and M. Wolfson, (1992), "Firms' Responses to Anticipated Reductions in Tax Rates: The Tax Reform Act of 1986." Journal of Accounting Research (Supplement 1992): 161-85.

Sikes, S. A., X. Tian, and R. Wilson, (2014), “Investors’ reaction to the use of poison pills as a tax loss preservation tool.” Journal of Accounting and Economics 57 (2/3): 132–148.

Shevlin, T., (1990), “Estimating corporate marginal tax rates with asymmetric tax treatment of gains and losses.” The Journal of the American Taxation Association 11, 51-67.

Shevlin, T., (1999), “A Critique of Plesko's 'an Evaluation of Alternative Measures of Corporate Tax Rates.” Available at SSRN: https://ssrn.com/abstract=190436.

Smith, C., (1976), “Option Pricing: A review.” Journal of Financial Economics, 3, 3–51.

Smith, C., and R. Stulz, (1985), “The Determinants of Firms' Hedging Policies Author(s).” The Journal of Financial and Quantitative Analysis, 20, 4, 391-40

Stiglitz, J., (1969), “The Effects of Income, Wealth, and Capital Gains Taxation on Risk-Taking.” *The Quarterly Journal of Economics*, 83, 2, 263-283.

Terakado, S., (2019), “On the Option Pricing Formula Based on the Bachelier Model.” Available at SSRN: https://ssrn.com/abstract=3428994

Zwick, E. (2021), “The Costs of Corporate Tax Complexity.” American Economic Journal: Economic Policy (Forthcoming).

Appendix A.1: Valuation of Tax Assets as Derivative Contracts

A.1.1 The Unconstrained Standalone Carryforward Provision

Denote by $L(t)$ the tax loss experienced by the firm at time t with:

$$L(t) = MAX[0, -E(t)] = P(0,t) = [C(0,t) - E(t)]$$

where $E(t)$ are earnings before interest, tax, depreciation, and amortization at time t, and $C(0,t)$ and $P(0,t)$ denote a (European) call and put option with a strike price of 0 and maturity t. Denote by $TI_O(t)$ the firm's taxable income at time t without the carryforward provision:

$$TI_O(t) = MAX[0, E(t)] = C(0,t)$$

The Book Value of the Tax Asset (Before Tax)

The dynamics of the carryforward reserve at time t, denoted by $B_F(t)$, is given by:

$$B_F(t) = B_F(t-1) + L(t) - MAX[0, E(t) - TI_F(t)]$$

where $TI_F(t)$ denotes the firm's taxable income at time t with the carryforward provision:

$$TI_F(t) = MAX[0, E(t) - B_F(t-1)] = C(B_F(t-1), t)$$

The carryforward reserve increases with tax losses and decrease with taxable profits. Substituting $L(t)$ and $TI_F(t)$ into $B_F(t)$ and re-arranging yields:

$$B_F(t) = P(B_F(t-1), t)$$

where $P(B_F(t-1), t)$ denotes a path-dependent put option with a time-varying strike price equal to $B_F(t-1)$ and maturity t. The carryforward reserve, or book value of the tax asset, at time t is a put option on earnings with a strike price equal to the carryforward reserve at time $t-1$.

The Market Value of the Tax Asset (Before Tax)

When the firm suffers a tax loss at time $t-1$, the firm exercises the put option and is granted the tax asset $M_F(t-1,t)$ maturing at time t. The asset entitles the firm to receive a (before) tax credit at time t equal to the difference between two non-negative taxable income:

$$M_F(t-1,t) = [TI_O(t) - TI_F(t)] = MAX[0, E(t)] - MAX[0, E(t) - B_F(t-1)]$$

This is the payoff of a path-dependent call spread on earnings with maturity t, composed of a long call option with a strike price of 0 and a short call option with a path-dependent time-varying strike price equal to $B_F(t-1)$. The call spread is denoted by:

$$M_F(t-1,t) = [C(0,t) - C(B_F(t-1), t)]$$

The market value of the tax asset at time $t-1$ is a one-year call spread on earnings with a payoff at time t capped at the book value of the tax asset at time $t-1$.

Example 1: Consider two consecutive years $t-1$ and t, and assume without loss of generality that the firm holds no carryforward reserve at time $t-2$, i.e., $B_F(t-2)=0$.

At time $t-1$, there are two possible scenarios for earnings. When $E(t-1)\geq 0$, the firm let the put option $P(B_F(t-2),t-1)$ expire worthless. Both $B_F(t-1)$ and $M_F(t-1,t)$ are equal to 0. When $E(t-1)<0$, the firm exercises the put option $P(B_F(t-2),t-1)$ and is granted the call spread $M_F(t-1,t)$ maturing at time t composed of a long call option with a strike price of 0 and a short call option with a strike price equal to $B_F(t-1)=|E(t-1)|>0$.

At time t, the firm faces two exercise decisions in each of the three possible scenarios for earnings. When $E(t)\leq 0$, the firm let the call spread $M_F(t-1,t)$ expire worthless, exercises the put option $P(B_F(t-1),t)$ and is granted the new call spread $M_F(t,t+1)$ maturing at time $t+1$ composed of a short call option having a strike price equal to $B_F(t)>B_F(t-1)$. When $0<E(t)\leq B_F(t-1)$, the firm exercises $M_F(t-1,t)$, receives the partial tax credit $E(t)\times T_C$, exercises the put option $P(B_F(t-1),t)$ and is granted the new call spread $M_F(t,t+1)$ maturing at time $t+1$ composed of a short call option having a strike price equal to $B_F(t)<B_F(t-1)$. Finally, when $E(t)\geq B_F(t-1)$, the firm exercises $M_F(t-1,t)$, receives the full tax credit $B_F(t)\times T_C$ and let the put option $P(B_F(t-1),t)$ expire worthless.

Numerical Example 1: Consider a simple three-period model. Suppose the firm generates a stream of earnings equal to $-\$100$, $-\$200$ and $+\$350$ at times 1, 2, and 3, respectively. Assume that the initial carryforward reserve is equal to \$0 at time 0. The strike price of the put option maturing at time 1 is equal to \$0. At time 1, the firm suffers a tax loss of \$100, exercises the put option and is granted a call spread maturing at time 2. The call spread is composed of a long call with a strike price equal to \$0 and a short call with a strike price equal to the put's payoff $MAX[0,0-(-100)]=\$100$. At time 2, the firm suffers a tax loss of \$200. The firm let the call spread maturing at time 2 expire worthless, exercises the put option and is granted a new call spread maturing at time 3. The strike price of the long call is equal to \$0 and the strike price of the short call changes by an amount equal to the put's payoff $MAX[0,100-(-200)]=\$300$. At time 3, the firm makes a taxable profit of \$350, exercises the call spread, receives a tax credit of \$300 (times the statutory tax rate), and pays corporate taxes on the remaining \$50. The (ex-post) book values of the tax asset are equal to \$0, \$100 and \$300 at times 1, 2 and 3, respectively. The (ex-post) market values of the tax asset are equal to \$0, \$0 and \$300 at times 1, 2 and 3, respectively. The ex-ante (time 0) market values of the tax assets are given by the time 0 values of the call spread maturing at time 1, 2 and 3, respectively. Given the uncertainty in earnings, the strike price of the short calls is unknown at time 0. It is revealed one period before the call option's final maturity (say time 2) and is given by the (then) put's payoff (time 1).

A.1.2 The Unconstrained Standalone Carryback Provision

The Book Value of the Tax Asset (Before Tax)

The dynamics of the carryback reserve at time t, denoted by $B_B(t)$, is given by:

$$B_B(t) = B_B(t-1) + TI_O(t) - MIN[L(t), B_B(t-1)]$$

where the $MIN[\cdot]$ operator captures the constraint that the tax refund at time t cannot exceed the carryback reserve at time $t-1$. The carryback reserve increases with taxable profits and decrease with tax losses. Substituting $TI_O(t)$ and $L(t)$ into $B_B(t)$, and re-arranging yields:

$$B_B(t) = C(-B_B(t-1), t)$$

where $C(-B_B(t-1), t)$ denotes a path-dependent call option with a time-varying strike price equal to $-B_B(t-1)$ and maturity t.

The Market Value of the Tax Asset (Before Tax)

When the firm makes a taxable profit at time $t-1$, the firm exercises the call option and is granted the tax asset $M_B(t)$ maturing at time t. The asset entitles the firm to receive a (before) tax refund at time t equal to:

$$M_B(t-1, t) = MIN[L(t), B_B(t-1)] = \Big[MAX[0, -E(t)] - MAX\big[0, \big(-B_B(t-1)\big) - E(t)\,\big]\Big]$$

This is the payoff of a path-dependent put spread on earnings with maturity t, composed of a long put option with a strike price of 0 and a short put option with a path-dependent time-varying strike price equal to $-B_{,B}(t-1)$. The put spread is denoted by:

$$M_B(t-1, t) = [P(0, t) - P(-B_B(t-1), t)]$$

The market value of the tax asset at time $t-1$ is a one-year put spread on earnings with a payoff at time t capped at the book value of the tax asset at time $t-1$.

Example 2: Consider two consecutive years $t-1$ and t, and assume without loss of generality that the carryback reserve $B_B(t-2) = 0$.

At time $t-1$, there two possible scenarios for earnings. When $E(t-1) \leq 0$, the firm let the call option $C(-B_B(t-2), t-1)$ maturing at time $t-1$ with a strike price equal to $-B_B(t-2) = 0$ expire worthless. Both $B_B(t-1)$ and $M_B(t-1, t)$ are equal to 0. When $E(t-1) > 0$, the firm exercises the call option $C(-B_B(t-2), t-1)$ and is granted the put spread $M_B(t-1, t)$ maturing at time t composed of a long put option with a strike price of 0 and a short put option with a strike price equal to the negative of $B_B(t-1) = E(t-1)$.

At time t, the firm faces two exercise decisions in each of the three possible scenarios for earnings. When $0 < E(t)$, the firm let the put spread $M_B(t-1, t)$ expire worthless, exercises the call option $C(-B_B(t-1), t)$, and is granted the new put spread $M_B(t, t+1)$ maturing at time $t+1$ composed of a short put option having a strike price equal to $B_B(t) > B_B(t-1)$. When $-B_B(t-1) < E(t) \leq 0$, the firm exercises the put spread $M_B(t-1, t)$, receives the partial tax refund $E(t) \times T_C$, exercises the call option $C(-B_B(t-1), t)$ and is granted the new put spread $M_B(t, t+1)$ maturing at time $t+1$ composed of a short put option having a strike price equal to $B_B(t) < B_B(t-1)$. Finally, when $E(t) \leq -B_B(t-1)$, the firm exercises the put spread $M_B(t-1, t)$, receives the full tax refund $B_B(t) \times T_C$ and let the call option $C(-B_B(t-1), t)$ expire worthless.

Numerical Example 2: Suppose that the firm generates a stream of earnings equal to +$\$100$, $+\$200$ and $-\$350$ at times 1, 2, and 3, respectively. Assume that the initial carryback reserve is equal to \$0 at time 0. The strike price of the call option maturing at time 1 is equal to \$0. At time 1, the firm pays taxes on its taxable profit of \$100, exercises the call option and is granted a put spread maturing at time 2. The put spread is composed of a long put with a strike price equal to \$0 and a short put with a strike price equal to the negative of the call's payoff $-MAX[0{,}100-0] = -\$100$. At time 2, the firm pays taxes on its taxable profit of \$200, let the put spread maturing at time 2 expire worthless, exercises the call option, and is granted a new put spread maturing at time 3. The strike price of the long put is equal to \$0 and the strike of the short put changes by an amount equal to the (negative of the) call's payoff $-MAX[0{,}200-(-100)] = -\$300$. At time 3, the firm makes a tax loss of \$350, exercises the put spread and receives a tax refund equal to $MAX[0{,}0-(-350)] - MAX[0, -300-(-350)] = \$(350-50) = \$300$ (times the statutory tax rate). The (ex-post) book values of the tax asset are equal to \$0, \$300 and \$300 at times 1, 2 and 3, respectively. The (ex-post) market values of the tax asset are equal to \$0, \$0 and \$300 at times 1, 2 and 3, respectively. The ex-ante (time 0) values of the tax asset are given by the time 0 values of the put spread maturing at time 1, 2 and 3, respectively. Given the uncertainty in earnings, the strike price of the short puts is unknown at time 0. It is revealed one period before the put option's final maturity (say time 2) and is given by the (then) call's payoff (time 1).

A.1.3 The Unconstrained Joint Carryforward and Carryback Provisions

The carryforward reserve $B_F(t)$ and carryback reserve $B_B(t)$ must be adjusted for interaction effects. Their adjusted values, denoted by $B_F^*(t)$ and $B_B^*(t)$, are derived under two scenarios for earnings at time t.

Scenario 1: Tax Loss

Suppose the firm experiences the tax loss $L(t) = MAX[0, -E(t)]$ at time t. Under the standalone carryforward provision, the firm exercises the put option, increases the carryforward reserve from $B_F(t-1)$ to $B_F(t)$ by the amount $L(t)$ and is granted the call spread $M_F(t, t+1)$. In contrast, under the joint provision, the firm receives the (before) tax refund $MIN[L(t), B_B^*(t-1)] = L(t) - MAX[0, L(t) - B_B^*(t-1)]$, exercises the put option, increases the carryforward reserve from $B_F^*(t-1)$ to $B_F^*(t)$ by the amount $L^*(t) = MAX[0, L(t) - B_B^*(t-1)]$, and is granted the call spread $M_F^*(t, t+1)$. The tax asset $M_F^*(t, t+1)$ exhibits less refundability risk than $M_F(t, t+1)$, because of the lower strike price $B_F^*(t) < B_F(t)$. *De facto*, the firm swaps the risky asset $M_F(t, t+1)$ granted under the standalone carryforward provision for a portfolio composed of cash (the before-tax refund) and the less risky asset $M_F^*(t, t+1)$.

Compared to $B_F(t)$, the dynamics of the carryforward reserve $B_F^*(t)$ remains unchanged except for the substitution of $L^*(t)$ for $L(t)$, with $L^*(t) \leq L(t)$:

$$B_F^*(t) = B_F^*(t-1) + MAX[0, L(t) - B_B^*(t-1)] - MAX[0, E(t) - TI_F^*(t)]$$

with $TI_F^*(t) = MAX[0, E(t) - B_F^*(t-1)]$. Substituting $L(t)$ and $TI_F^*(t)$ in the formula yields:

$$B_F^*(t) = \left[P(B_F^*(t-1), t) - [P(0, t) - P(-B_B^*(t-1), t)]\right] = P(B_F^*(t-1), t) - M_B^*(t-1, t) \leq B_F(t)$$

The carryforward reserve is lowered by the amount equal to the (before) tax refund $M_B^*(t-1, t)$. The standalone carryforward provision obtains as a special case by setting $B_B^*(t-1) = 0$, giving $B_F^*(t) = B_F(t) = P(B_F(t-1), t)$. This suggests that the joint tax provision provides three benefits at time t. First, it decreases the risk of the tax asset granted under the standalone carryforward provision by converting in whole or in part a risky asset, i.e., a call spread, into a cash amount equal to the (before) tax refund $M_B^*(t-1, t)$. Second, it increases the likelihood that tax credits will be received in the future by decreasing the carryforward reserve from $B_F(t)$ to $B_F^*(t)$. Third, it provides a time value of money benefit by moving forward in time the (before) tax cash flow $M_B^*(t-1, t)$.

Numerical Example 3.1: Assume that the firm generates earnings equal to $-\$100$ and $+\$150$ at time 1 and at time 2, respectively. Assume that the carryforward reserve and the carryback reserve are both equal to 0 at time 0. At time 1, the firm incurs a tax loss of $\$100$, exercises its put option and is granted a call spread maturing at time 2. The call spread is composed of a long call with a strike price of \$0 and a short call with a strike price equal to \$100. At time 2, the firm makes a taxable profit. It exercises the call spread and receives a (before) tax credit equal to $MAX[0, 150] - MAX[0, 150 - 100] = \$(150 - 50) = +\$100$. In addition, the carryback provision allows the firm to exercise the call option (on the excess profit) and to be granted a put spread maturing at time 3. The put spread is composed of a long put with a strike price of \$0 and a short put with a strike price equal to the negative of the excess profit equal to \$50. All in all, at time 2, the firm receives a portfolio composed of cash \$100 (the tax credit) and a risky asset (the put spread). The maximum tax loss faced by the firm if liquidated at time 3 is equal to \$50. Note that at time 2 under the standalone carryback provision, the firm would exercise the call option and would be granted a put spread maturing at time 3, with a short put having a strike price equal to -

\$150. The firm would receive a portfolio composed of a single risky asset, i.e., the put spread. Tax assets are less risky and are more valuable under the joint provision than under the standalone provision.

Case 2: Taxable Profit

Suppose that the firm makes the taxable profit $TI_O(t) = MAX[0, E(t)]$ at time t. Under the standalone carryback provision, the firm exercises the call option, increases its carryback reserve from $B_B(t-1)$ to $B_B(t)$ by the amount $TI_O(t)$ and is granted the put spread $M_B(t, t+1)$. Under the joint provision, the firm receives a (before) tax credit equal to $MIN[TI_O(t), B_F^*(t-1)] = TI_O(t) - MAX[0, TI_O(t) - B_F^*(t-1)]$, exercises the call option, updates its carryback reserve to $B_B^*(t)$ by the amount $TI_O^*(t) = MAX[0, TI_O(t) - B_F^*(t-1)]$, and is granted the put spread $M_B^*(t, t+1)$. The tax asset $M_B^*(t, t+1)$ exhibits less refundability risk than $M_B(t, t+1)$ because of the lower strike price $B_B^*(t) < B_B(t)$. *De facto*, the firm swaps the risky asset $M_B(t, t+1)$ granted under the standalone carryback provision for a portfolio composed of cash (the before-tax credit) and the less risky asset $M_B^*(t, t+1)$.

The dynamics of the carryback reserve $B_B^*(t)$ remains unchanged compared to $B_B(t)$ except for the substitution of $TI_O^*(t)$ for $TI_O(t)$ with $TI_O^*(t) \leq TI_O(t)$:

$$B_B^*(t) = B_B^*(t-1) + MAX[0, TI_O(t) - B_F^*(t-1)] - MIN[L(t), B_B^*(t-1)]$$

Substituting $L(t)$ and $TI_O(t)$ in the above formula yields:

$$B_B^*(t) = \left[C(-B_B^*(t-1), t) - [C(0,t) - C(B_F^*(t-1), t)]\right] = C(-B_B^*(t-1), t) - M_F^*(t-1, t) \leq B_B(t)$$

The carryback reserve is lowered by the amount of the (before) tax credit $M_F^*(t-1, t)$. The standalone carryback provision obtains as a special case by setting $B_F^*(t-1) = 0$, giving $B_B^*(t) = B_B(t) = C(-B_B(t-1), t)$. This suggests that joint tax provision provides three benefits at time t. First, it decreases the risk of the tax asset granted under the standalone carryback provision by converting in whole or in part a risky asset, i.e., a put spread, into a cash amount equal to the (before) tax credit $M_F^*(t-1, t)$. Second, it increases the likelihood that tax refunds will be received in the future by decreasing the carryback reserve from $B_B(t)$ to $B_B^*(t)$. Third, it provides a time value of money benefit by moving forward the (before) tax cash-flow $M_F^*(t-1, t)$.

Numerical Example 3.2: Assume that the firm generates a stream of earnings equal to $+\$100$ and $-\$150$ at time 1 and at time 2, respectively. Assume that the carryforward reserve and the carryback reserve are both equal to 0 at time 0. At time 1, the firm pays taxes on its taxable profit of \$100, exercises its call option and is granted a put spread maturing at time 2. The put spread is composed of a long put with a strike price of \$0 and a short put with a strike price equal to $-\$100$ the negative of the call's payoff. At time 2, the firm suffers a tax loss. It exercises the put spread and receives a (before) tax refund equal to $MAX[0, 0-(-150)] - MAX[0, -100-(-150)] = \$(150-50) = \$100$. In addition, the carryforward provision allows the firm to exercise the put option (on the excess tax loss) and to be granted a call spread maturing at time 3. The call spread is composed of a long call with a strike price of \$0 and a short call with a strike price equal to the excess loss of \$50. All in all, at time 2, the firm receives a portfolio composed of cash \$100 and a risky asset. Note that at time 2 under the standalone carryforward provision, the firm would exercise the put option and would be granted a call spread maturing at time 3, with a short call having a strike price of \$150. The firm would receive a portfolio composed of a single risky asset, i.e., the call spread. Tax assets are less risky and are more valuable under the joint provision than under the standalone provision.

Appendix A.2: Constrained Tax Provisions

Tax loss carryforwards and carrybacks are typically constrained. There are two types of constraints, i.e., caps on the number of years during which losses can be carried forward or backward and caps on the taxable income. Their impact on the carryforward provision is discussed below. Similar results obtain for the carryback provision.

A.2.1 The Constrained Standalone Carryforward Provision: The Time Constraint

Tax assets are wasting assets subject to refundability risk. In many cases, tax jurisdictions impose that unused tax losses be forfeited after a certain number of years which varies in a typical range between five to twenty years. Define by $L(\tau)$ the tax loss booked in year τ with $L(\tau) = MAX[0, -E(\tau)]$ and $\tau < t$. Assume that unused tax losses are forfeited after τ years. Define by τ the number of years during which losses can be carried forward and by $L(t, \tau)$, the *unused* component of the tax loss booked in year $(t - \tau)$ forfeited at the *start* of year t, with $0 \leq L(t, \tau) \leq L(t - \tau)$ [72].

The Book Value of the Tax Asset (Before Tax)

The constrained carryforward reserve at time t, denoted by $B_{F,C}^{+}(t)$, can be expressed as a put option on earnings with the following characteristics:[73]

$$B_{F,C}^{+}(t) = P\big(B_{F,C}^{+}(t-1) - L(t, t-\tau), t\big) \leq B_F(t)$$

The constrained carryforward reserve $B_{F,C}^{+}(t)$ is lower than its unconstrained counterpart $B_F(t)$ because of the forfeited tax losses. The difference in the carryforward reserves increases with a decrease in τ.

The Market Value of the Tax Asset (Before Tax)

The market value of the constrained tax asset at time t, denoted by $M_{F,C}(t)$, is given by:

$$M_{F,C}(t-1, t) = \big[C(0, t) - C\big(B_{F,C}^{+}(t-1), t\big)\big] \geq 0$$

The derivative contract is a call spread identical to the contract obtained in the unconstrained case except for the lower carryforward reserve. The contract's payoff is floored at 0 when both options expire worthless and is capped at $B_{F,C}^{+}(t-1)$, when both options expire in-the-money. The constraint has a positive and a negative effect on the market value of the tax asset. The constraint decreases the strike price of the short call, increases the likelihood

---

[72] Assume for example that $t = 6$ and $\tau = 5$ years. In this case, $L(1)$ is the tax loss booked in year 1 and $L(6,1)$ denotes the unused component of the tax loss booked in year 1 and forfeited in year 5 (in whole or in part) *after* the calculation of the taxable income in year 5 but before the calculation of the taxable income in year 6, i.e., at the *start* of year 6. $L(6,1)$ satisfies the constraint $0 \leq L(6,1) \leq L(1)$. The unused tax loss $L(6,1)$ is equal to $L(1)$ less the sum of all the tax credits $M_F(\tau)$ that $L(1)$ generated between time $\tau = 2$ and time $\tau = 5$ (inclusive).

[73] Assume $t = 6$ and $\tau = 5$. The carryforward reserve $B_{F,C}^{-}(6)$ at time $t = 6$, (at the start of the year) is equal to $B_{F,C}^{+}(5)$ at time $t = 5$ less $L(6,1)$. More generally, $B_{F,C}^{-}(t) = B_{F,C}^{+}(t-1) - L(t, t-\tau)$. The constrained taxable income at time t is equal to $TI_{F,C}(t) = MAX\big[0, E(t) - B_{F,C}^{-}(t)\big] = C\big(B_{F,C}^{+}(t-1) - L(t, t-\tau), 0\big)$. The dynamics of the carryforward reserve at time t is given by $B_F^{+}(t) = B_F^{-}(t) + L(t) - MAX\big[0, E(t) - TI_{F,C}(t)\big] = B_F^{+}(t-1) - L(t, t-\tau) + L(t) - MAX\big[0, E(t) - TI_{F,C}(t)\big]$. Note that $L(t, t-\tau)$ cannot be higher than $B_{F,C}^{+}(t-1)$.

that the call will be exercised in the future, and moves the receipt of tax credits forward in time. This positive effect is offset by a loss of tax cash-flow which decreases the market value of the tax asset.

The Firm's Expected Tax Liability

The tax asset $M_{F,C}(t-1,t)$ decreases the firm's expected tax liability at time t, originally from $T_O(t-1,t)$ in the no carryforward case, to $T_{F,C}(t-1,t)$ in the constrained carryforward case:

$$T_{F,C}(t-1,t) = C\big(B_{F,C}(t-1),t\big) \times T_C \geq C(B_F(t-1),t) \times T_C$$

with $T_O(t-1,t) \leq T_F(t-1,t) \leq T_{F,C}(t-1,t)$. The constraint decreases the market value of the tax asset, increases the firm's tax liability, and increases the cost of tax asymmetry.

A.2.2 The Constrained Standalone Carryforward Provision: The Income Constraint

Another constraint imposed on the carryforward provision is a cap on the taxable income that can be sheltered by tax loss carryforwards. Denote the cap rate by γ, with $0 < \gamma \leq 1.0$, and assume that the excess $(1-\gamma)$ is not forfeited but is carried forward without additional constraints.

The Book Value of the Tax Asset

The constrained taxable income, denoted by $TI_{F,C}(t)$, is given by:

$$TI_{F,C}(t) = MAX\left[0, E(t) - MIN\big[\gamma \times E(t), B_{F,C}(t-1)\big]\right]$$

where $B_{F,C}\ (t-1)$ is the constrained carryforward reserve, equivalently:

$$TI_{F,C}(t) = (1-\gamma) \times MAX[0, E(t)] + \ \gamma \times MAX\left[0, E(t) - \frac{B_{F,C}(t-1)}{\gamma}\right] = (1-\gamma) \times C(0,t) + \gamma \times C\left(\frac{B_{F,C}(t-1)}{\gamma}, t\right)$$

The constrained carryforward reserve is a value-weighted average of two taxable incomes weighted by the parameter γ. The unconstrained carryforward case obtains as a special case by setting the parameter γ equal to 1.0. The dynamics of the carryforward reserve at time t is given by:

$$B_{F,C}(t) = B_{F,C}(t-1) + L(t) - MAX\big[0, E(t) - TI_{F,C}(t)\big]$$

with $B_{F,C}\ (t) \geq B_F(t)$. When the firm's taxable earnings are positive, $B_{F,C}(t)$ decreases by γ% in the former case rather than by 100% in the latter case. As a result, the constrained carryforward reserve is larger than its unconstrained counterpart.

Substituting $L(t)$ and $TI_{F,C}(t)$ into $B_{F,C}(t)$ and re-arranging yields:

$$B_{F,C}(t) = (1-\gamma) \times P(0,t) + \gamma \times P\left(\frac{B_{F,C}(t-1)}{\gamma}, t\right)$$

where $P\left(\frac{B_{F,C}(t-1)}{\gamma}, t\right)$ denotes a path-dependent put option with a time-varying strike price equal to $B_{F,C}(t-1)/\gamma$ and maturity t. The carryforward reserve is a value-weighted average of two put options. It is fully depleted when the put options expire worthless one year later at time t i.e., when $E(t) \geq B_{F,C}(t-1)$. Conversely, it increases by the full loss amount $E(t)$ when they expire in-the-money, i.e., when earnings are negative. When earnings are positive but less than $B_{F,C}(t-1)/\gamma$, the carryforward reserve decreases by $\gamma \times E(t)$.

The Market Value of the Tax Asset (Before Tax)

When the firm incurs a tax loss at time $t-1$, the firm exercises the put option and is granted the tax asset $M_{F,C}(t-1,t)$ maturing at time t. The asset entitles the firm to receive a (before) tax credit at time t equal to the difference between two non-negative taxable income:

$$M_{F,C}(t-1,t) = \left[TI_O(t) - TI_{F,C}(t)\right] = \gamma \times \left[MAX[0, E(t)] + MAX\left[0, E(t) - \frac{B_{F,C}(t-1)}{\gamma}\right]\right]$$

When expressed as a derivative contract, the market value of the tax asset is given by:

$$M_{F,C}(t-1,t) = \gamma \times \left[C(0,t) - C\left(\frac{B_{F,C}(t-1)}{\gamma}, t\right)\right]$$

The derivative contract is a (weighted) call spread identical to the contract obtained in the unconstrained case save for the higher carryforward reserve. The unconstrained tax provision obtains as a special case for $\gamma = 1$. The contract's payoff is floored at 0 when both options expire worthless and is capped at $B_{F,C}(t-1)$, when both options expire in-the-money. In the best-case scenario, the firm recovers its entire carryforward reserve. However, the increase in the strike price decreases the likelihood that the call spread will be exercised which increases refundability risk and delays the receipts of tax credits. The constraint generates two adverse effects which are not compensated by higher incremental tax cash-flows relative to the unconstrained case. The constraint decreases the market value of the tax asset.

The Firm's Expected Tax Liability

The tax asset $M_{F,C}(t-1,t)$ decreases the firm's expected tax liability at time t, from $T_O(t-1,t)$ without carryforwards to $T_{F,C}(t-1,t)$:

$$T_{F,C}(t-1,t) = \left[(1-\gamma) \times C(0,t) + \gamma \times C\left(\frac{B_{F,C}(t-1)}{\gamma}, t\right)\right] \times T_C \geq C(B_F(t-1), t)$$

with $T_o(t-1,t) \leq T_F(t-1,t) \leq T_{F,C}(t-1,t)$. The constraint decreases the market value of the tax asset, increases the firm's tax liability, and increases the cost of tax asymmetry.

Appendix A.3: Estimation Issues

Tax assets are complex path-dependent derivatives which do not have analytical solutions and must be valued by simulation. This requires an assumption about the stochastic process followed by earnings and the estimation of the parameters governing the process. The process must allow earnings to become negative (which precludes a geometric Brownian motion) and possibly to mean-revert.[74] Assume a process has been specified and the parameters, i.e., the drift and the volatility of the firm's earnings have been estimated. Assume also that the firm's earnings, carryforward reserve and carryback reserve are observable at time 0. They are denoted by $E(0)$, $B_F(0)$ and $B_B(0)$, respectively.

The simulation requires the generation of earnings' sample paths (on an annual frequency) using the earnings generating process starting from time 0 until the pre-specified time T in the future (in principle infinity). The market value of tax assets, tax liabilities, the costs of tax asymmetry and the transition probabilities are estimated from the $(T \times S)$ simulated earnings, denoted by $E(t,s)$, with $s = 1, \cdots S$, and $t = 1, \cdots, T$, where S is the total number of sample paths.

A.3.1 (Before Tax) Market Value of Tax Assets

The first step entails updating the carryback and the carryforward reserves along each sample path s starting from $B_F(0)$ and $B_B(0)$, and generating their simulated values $B_F^*(t,s)$ and $B_B^*(t,s)$ using:

$$\begin{cases} B_F^*(t,s) = P(B_F^*(t-1,s),t) = MAX[0, B_F^*(t-1,s) - E(t,s)] \\ B_B^*(t,s) = C(-B_B^*(t-1,s),t) = MAX[0, E(t,s) + B_B^*(t-1,s)] \end{cases}$$

The second step entails calculating the simulated market value of the tax asset in year t on sample path s, denoted by $M_{FB}(t-1,t,s)$ using:

$$M_{FB}(t-1,t,s) = \begin{cases} M_F^*(t-1,t,s) = [C(0,t) - C(B_F^*(t-1,s),t)] \ IF\ B_F^*(t-1,s) > 0 \\ M_B^*(t-1,t,s) = [P(0,t) - P(-B_B^*(t-1,s),t)] \ IF\ B_B^*(t-1,s) > 0 \end{cases}$$

On sample path s, the option values at time t are given by their intrinsic values equal to:

$$M_{FB}(t-1,t,s) = \begin{cases} M_F^*(t-1,t,s) = \big[MAX[0, E(t,s)] - MAX[0, E(t,s) - B_F^*(t-1,s)]\big] \ IF\ B_F^*(t-1,s) > 0 \\ M_B^*(t-1,t,s) = \big[MAX[0, -E(t,s)]\big] - MAX[0, -B_B^*(t-1,s) - E(t,s)] \ IF\ B_B^*(t-1,s) > 0 \end{cases}$$

At time t, the firm receives a tax refund (when the put spread is exercised) or a tax credit (when the call spread is exercised), or zero. The last case happens when the firm's earnings are always positive or always negative between

[74] A popular process used in the finance and accounting literature is the random walk with drift. Graham and Kim (2009) suggests an alternative $AR(1)$ income to capture mean-reversion in earnings.

time 0 and time t on sample path s. The market value of the tax asset at time t, denoted by $\overline{M_{FB}(t-1,t)}$, is estimated as the cross-sectional average of the S-simulated market values $M_{FB}(t-1,t,s)$ obtained at time t:

$$\overline{M_{FB}(t-1,t)} = E[M_{FB}(t-1,t)] = \frac{1}{S} \times \sum_{s=1}^{S} M_{FB}(t-1,t,s)$$

The final step entails calculating the (before-tax) market value of the tax asset at time 0 by discounting back $\overline{M_{FB}(t-1,t)}$ from time t to time 0 at the risk-free rate :

$$M_{FB}(0,t) = PV \times \left[\overline{M_{FB}(t-1,t)}\right]$$

The four steps can be repeated for all times t, with $t = 1, \cdots, T$, to calculate the after-tax market value of the bond granted to the firm under the carryforward and carryback tax provisions. It is given by:

$$M_{FB}(0) = \sum_{t=1}^{T} M_{FB}(0,t) \times T_C$$

A.3.2 Tax Liability

The (simulated) tax liability at time t on sample path s, denoted by $T_{FB}(t-1,t,s)$, is calculated as follows:

$$T_{FB}(t-1,t,s) = \begin{cases} T_F^*(t-1,t,s) = C(B_F^*(t-1,s),t) \times T_C = MAX[0, E(t,s) - B_F^*(t-1,s)] \times T_C \quad IF\ B_F^*(t-1,s) > 0 \\ T_B^* t-1,(t,s) = [E(t,s) + P(-B_B(t-1,s),t)] \times T_C \quad IF\ B_B^*(t-1,s) > 0 \end{cases}$$

The firm's expected tax liability at time t, denoted by $\overline{T_{FB}(t-1,t)}$, is estimated as the cross-sectional mean of the S simulated tax liabilities at time t:

$$\overline{T_{FB}(t-1,t)} = E[T_{FB}(t-1,t)] = \frac{1}{S} \times \sum_{s=1}^{S} T_{FB}(t-1,t,s)$$

The market value of the tax liability at time 0 is obtained by discounting back $\overline{T_{FB}(t-1,t)}$ from time t to time 0:

$$T_{FB}(0,t) = PV \times \left[\overline{T_{FB}(t-1,t)}\right]$$

A.3.3 Cost of Tax Asymmetry

Without tax provisions, the (gross) cost of tax asymmetry at time t is given by $P(0,t) \times T_C$. Along sample path s, the (before-tax) cost is given by:

$$P(0,t,s) = MAX[0, -E(t,s)]$$

The (before-tax) cost of tax asymmetry at time t, denoted by $\overline{P(0,t)}$, is estimated as the cross-sectional average of the S-simulated market values $P(0,t,s)$ obtained at time t:

$$\overline{P(0,t)} = E[P(0,t)] = \frac{1}{S} \times \sum_{s=1}^{S} P(0,t,s)$$

Discounting back $\overline{P(0,t)}$ from time to time 0 and multiplying by the statutory tax rate gives the present value of the expected cost of tax asymmetry at time t $PV \times \left[\overline{P(0,t)}\right] \times T_C$. Repeating the steps for all times t, with $t = 1, \cdots, T$, gives the after-tax cost of tax asymmetry as:

$$M(0) = \sum_{t=1}^{T} PV \times \left[\overline{P(0,t)}\right] \times T_C$$

The net cost of tax asymmetry is given by $\left(M(0) - M_{FB}(0)\right)$.

A.3.4 Conditional Transition Probabilities

As discussed in Sections 3.4 and 4.4, the (spot) conditional probability to switch from a non-taxable status at time 0 to a taxable status at time 1 for firms endowed with a carryforward reserve (or vice versa for firms endowed with a carryback reserve) is given by:

$$p_{FB}^*(0,1) = \begin{cases} p_F^*(0,1) = N\left[\left(\dfrac{E(0) - PV \times B_F^*(0)}{v}\right)\right] & IF\ B_F^*(0) > 0 \\ p_B^*(0,1) = N\left[-\left(\dfrac{E(0) + PV \times B_B^*(0)}{v}\right)\right] & IF\ B_B^*(0) > 0 \end{cases}$$

The (spot) conditional probability $p_{FB}^*(0,1)$ has an analytical solution at time 0. The (forward) conditional transition probabilities at time $t-1$, with $t > 1$ can be obtained by simulation. This entails, calculating at time $t-1$, on sample path s, the simulated conditional probability $p_{FB}^*(t-1,t,s)$ using:

$$p_{FB}^*(t-1,t,s) = \begin{cases} p_F^*(t-1,t,s) = N\left[\left(\dfrac{E(t-1,s) - PV \times B_F^*(t-1,s)}{v}\right)\right] & IF\ B_F^*(t-1,s) > 0 \\ p_B^*(t-1,t,s) = N\left[-\left(\dfrac{E(t-1,s) + PV \times B_B^*(t-1,s)}{v}\right)\right] & IF\ B_B^*(t-1,s) > 0 \end{cases}$$

and calculating $p_{FB}^*(t-1,t)$ as the arithmetic average of the simulated conditional probabilities $p_{FB}^*(t-1,t,s)$:

$$p_{FB}^*(t-1,t) = \begin{cases} \dfrac{1}{S_{NT}(t-1)} \times \displaystyle\sum_{s=1}^{S_{NT}(t-1)} p_F^*(t-1,t,s)\ \ IF\ B_F^*(t-1,s) > 0 \\ \dfrac{1}{S_T(t-1)} \times \displaystyle\sum_{s=1}^{S_T(t-1)} p_B^*(t-1,t,s)\ \ IF\ B_B^*(t-1,s) > 0 \end{cases}$$

where $S_{NT}(t-1)$ is the number of sample paths where $B_F^*(t-1,s) > 0$ at time $t-1$ and $S_T(t-1)$ is the number of sample paths for where $B_B^*(t-1,s) > 0$ at time $t-1$, with $\left(S_{NT}(t-1) + S_T(t-1)\right) = S$.

Appendix A.4: Marginal Tax Rates

A.4.1 The Standalone Carryforward Provision

Consider a non-taxable firm at time 0. A \$1.0 increase in earnings at time 0 results in a \$1.0 decrease in the carryforward reserve $B_F(0)$ and a \$1.0 tax credit loss at time t when the firm becomes taxable for the first time.

Assume momentarily that the tax provision allows tax losses to be carried forward for one year only. Under this assumption, the present value at time 0 of the firm's expected tax loss at time 1 is equal to $\$[PV \times p_F(0,1) \times T_C]$, where $p_F(0,1)$ denotes the probability that the firm moves from a non-taxable status at time 0 to a taxable status at time 1 and PV is the one-year discount rate. This happens when the call option $C(B_F(0), 1)$ expires in-the-money at time 1. Under the assumption that earnings follow an ABM, the present value of the expected tax loss generated by a \$1.0 increase in $E(0)$ at time 0, denoted by $MTR_F(0,1)$, is equal to:

$$MTR_F(0,1) = \left[PV \times N\left(d_F(0)\right)\right] \times T_C$$

with $N\left(d_F(0)\right) = N\left[E(0) - PV[B_F(0)]\right]/v$. The derivatives-based proxy $MTR_F(0,1)$ exploits the information in the scaled distance between the time 0 earnings and the present value of the carryforward reserve. The marginal tax rate is bounded below by 0 and above by the present value of T_C and increases with an increase in earnings.[75]

Suppose now that the tax provision allows the firm to carry tax losses forward for 2 years only. The firm's tax status can switch from a non-taxable to a taxable status either at time 1 or at time 2. The firm incurs the tax loss $\$(1 \times T_C)$ at time 2 only to the extent that it remained non-taxable at time 1, which happens with probability $\left(1 - p_F(0,1)\right)$. The present value at time 0 of the expected tax loss at time 2, denoted by $MTR_F(0,2)$, is given by:

$$MTR_F(0,2) = \left[PV \times \left(1 - p_F(0,1)\right) \times p_F(1,2)\right] \times T_C = \left[PV \times N\left(-d_F(0)\right) \times N\left(d_F(1)\right)\right] \times T_C$$

with $N\left(d_F(1)\right) = N\left(\left[E(1) - PV[B_F(1)]\right]/v\right)$ and where the PV is the two-year discount rate. The two conditional probabilities $p_F(0,1)$ and $p_F(1,2)$ are not independent from one another given the relationship $B_F(1) = \left(B_F(0) - E(1)\right)$, with $p_F(0,1) < p_F(1,2)$ when $E(1) > 0$ and $p_F(0,1) \geq p_F(1,2)$ when $E(1) \leq 0$. Depending on $E(1)$, $MTR_F(0,2)$ could be either higher or lower than $MTR_F(0,1)$. The argument can be generalized to any time t in the future.

[75] Ignoring time value of money, it is equal to half of the statutory tax rate when earnings are equal to the present value of the carryforward reserve.

A.4.2 The Standalone Carryback Provision

Consider a taxable firm at time 0. A \$1.0 increase in earnings at time 0 results in a \$1.0 increase in the carryback reserve $B_B(0)$ and a \$1.0 tax refund at time t when the firm becomes non-taxable for the first time.

Assume momentarily that the tax provision allows tax losses to be carried back for one-year only. Under this assumption, the present value of the firm's expected tax refund at time 0 is equal to $\$[PV \times p_B(0,1)] \times T_C$, where $p_B(0,1)$ denotes the probability that the firm moves from a taxable status at time 0 to a non-taxable status at time 1. This happens when the put option $P(-B_B(0),1) \times T_C$ expires in-the-money at time 1. Under the assumption that earnings follow an arithmetic Brownian motion, the present value of the expected tax refund generated by a \$1.0 increase in $E(0)$ at time 0 is given by $\left[PV \times N\left(-d_B(0)\right)\right] \times T_C$, with $N\left(-d_B(0)\right) = N[-(E(0) + PV[B_B(0)])/v]$. The firm's marginal tax rate at time 0, denoted by $MTR_B(0,1)$, is equal to the statutory tax T_C faced by the firm in the absence of a carryback provision less the present value of the expected tax refund generated by the carryback provision:

$$MTR_B(0,1) = T_C - PV \times N\left(-d_B(0)\right) \times T_C = \left[1 - PV \times N\left(-d_B(0)\right)\right] \times T_C$$

The marginal tax rate is bounded below by $(1 - PV) \times T_C$ and above by T_C and decreases with a decrease in earnings.[76]

Suppose now that the tax provision allows the firm to carry back losses for 2 years only. The firm's tax status can switch at either time 1 or time 2. The firm receives the incremental tax refund $\$(1 \times T_C)$ at time 2 when it remained taxable at time 1 which happens with probability $\left(1 - p_B(0,1)\right)$. The marginal tax rate, denoted by $MTR_B(0,2)$, is obtained by deducting the present value at time 0 of the expected tax refund to be received at time 2 from the statutory tax rate T_C, giving:

$$MTR_B(0,2) = \Big(1 - PV \times \left(1 - p_B(0,1)\right) \times p_B(1,2)\Big) \times T_C = \left[1 - PV \times N\left(d_B(0)\right) \times N\left(-d_B(1)\right)\right] \times T_C$$

with $N\left(-d_B(1)\right) = N(-(E(1) + PV[B_B(1)])/v)$ and where the PV factor is adjusted for the two-year period. The two conditional probabilities $p_B(0,1)$ and $p_B(1,2)$ are not independent from one another given the relationship $B_B(1) = \left(B_B(0) + E(1)\right)$, with $p_B(0,1) < p_B(1,2)$ when $E(1) < 0$ and $p_B(0,1) \geq p_B(1,2)$ when $E(1) \geq 0$. This implies that $MTR_B(0,2)$ could be higher or lower than $MTR_B(0,1)$. The argument can be extended to any time t in the future.

[76] Ignoring time value of money, it is equal to half of the statutory tax rate when earnings are equal to the negative of the present value of the carryback reserve.

Figures

The figures below illustrate the valuation principles. They assume a risk-neutral Arithmetic Brownian Motion (ABM) for illustrative purposes only.

Standalone Unconstrained Carryforward Provision

Figures 1.1-1.4 display the market value of the tax asset $M_F(t-1,t)$ at time $t-1$, valued as a one-year call spread on earnings expiring at time t. The strike price of the long call is 0 and the strike price of the short call is the carryforward reserve at $B_F(t-1)$ at time $t-1$. It is obtained from:

$$M_F(t-1,t) = \big[E(t-1)\times[N(d)-N(d_F)] + [PV\times B_F(t-1)\times N(d_F)] + v\times[n(d)-n(d_F)]\big]$$

where $N(\cdot)$ and $n(\cdot)$, denote the cumulative and the probability density function of the standard normal distribution, respectively, with $d_F = [E(t-1) - PV\times B_F(t-1)]/v$, $d = E(t-1)/v$, $PV = e^{-R_F}$, where R_F is the continuously compounded risk-free rate and $v = \sigma\times\sqrt{(1-e^{-2\times R_F})/(2\times R_F)}$.

Figure 1.5 displays the conditional transition probability $p_F(t-1,t)$ that a firm moves from a non-taxable state at time $t-1$ to a taxable state at time t for two different estimates of earnings' volatility, obtained from:

$$p_F(t-1,t) = N(d_F) = N\left(\frac{E(t-1) - PV\times B_F(t-1)}{v}\right)$$

Figure 1.6 displays the firm's (one-year ahead) expected tax liability $T_F(t-1,t)$ at time $t-1$ for two different estimates of earnings' volatility. It is expressed as a one-year call option on earnings with a strike price equal to the carryforward reserve at time $t-1$ (times the statutory tax rate):

$$T_F(t-1,t) = C(B_F(t-1),t)\times T_C.$$

Standalone Unconstrained Carryback Provision

Figures 2.1-2.4 displays the market value of the tax asset $M_B(t-1,t)$, valued at time $t-1$ as a one-year put spread on earnings expiring at time t. The strike price of the long put is 0 and the strike price of the short put is the negative of the carryback reserve at $B_B(t-1)$ at time $t-1$. It is obtained from:

$$M_B(t-1,t) = \big[E(t-1)\times[N(-d_B)-N(-d)] + [PV\times B_B(t-1)\times N(-d_B)] + v\times[n(-d)-n(-d_B)]\big],$$

with $d_B = [E(t-1) + PV\times B_F(t-1)]/v$.

Figure 2.5 displays the conditional transition probability $p_B(t-1,t)$ that a firm moves from a taxable state at time $t-1$ to a non-taxable state at time t for two different estimates of earnings' volatility. It is given by:

$$p_B(t-1,t) = N(-d_B) = N\left(-\frac{E(t-1) + PV\times B_B(t-1)}{v}\right)$$

Figure 2.6 displays the (one-year ahead) expected tax liability $T_B(t-1,t)$ at time $t-1$, calculated as a portfolio composed of a long one-year call option with a strike price equal to the negative of the book value of the tax asset at time $t-1$ less a cash amount equal to $B_B(t-1)$, times the statutory tax rate:

$$T_B(t-1,t) = [E(t-1) + P(-B_B(t-1),t)]\times T_C = [C(-B_B(t-1) - B_B(t-1),t)]\times T_C.$$

Figure 1.1: Carryforward Provision

Market Value of the Tax Asset M(t-1,t) Modelled as a Call Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Carryforward Reserve B(t-1)

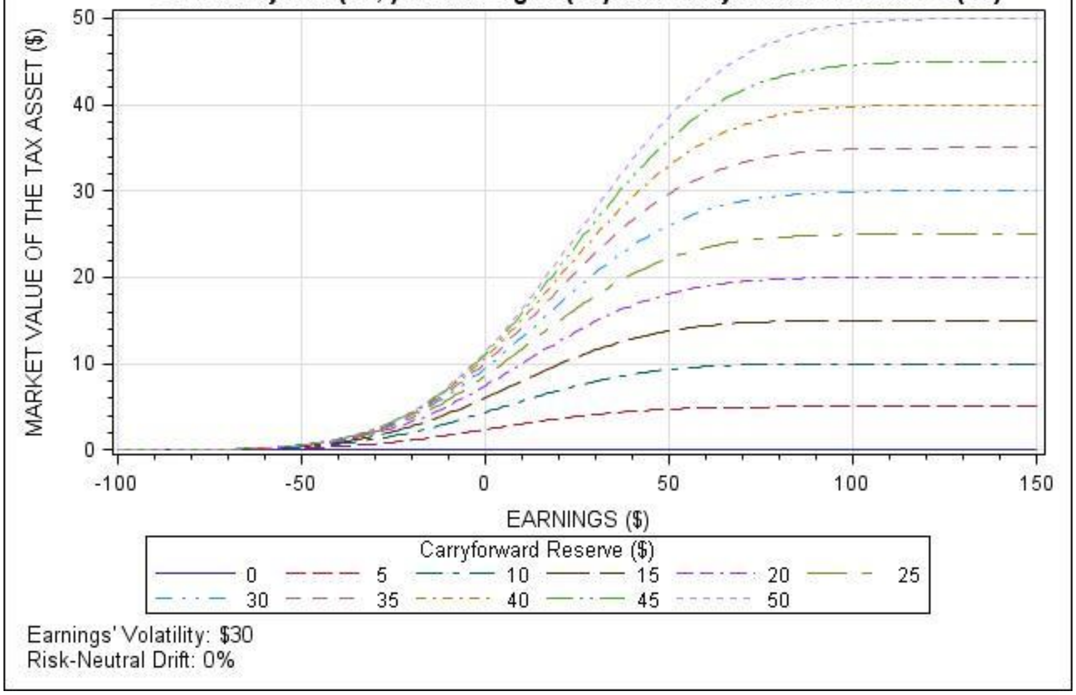
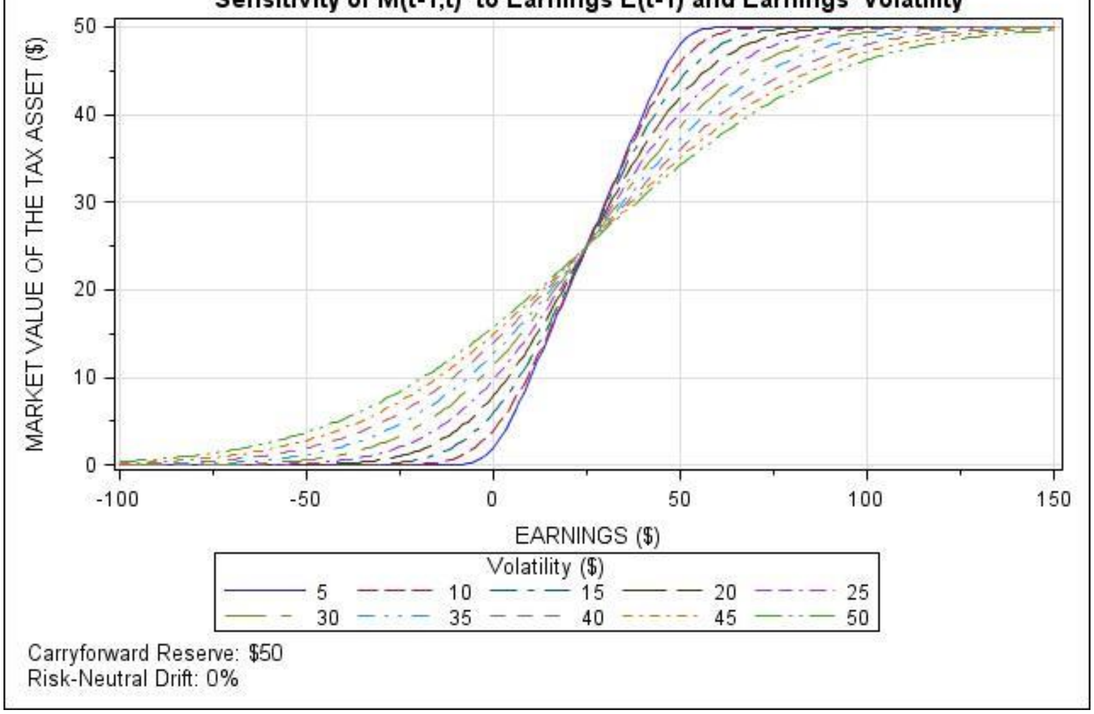


Figure 1.2: Carryforward Provision

Market Value of the Tax Asset M(t-1,t) Modelled as a Call Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Earnings' Volatility



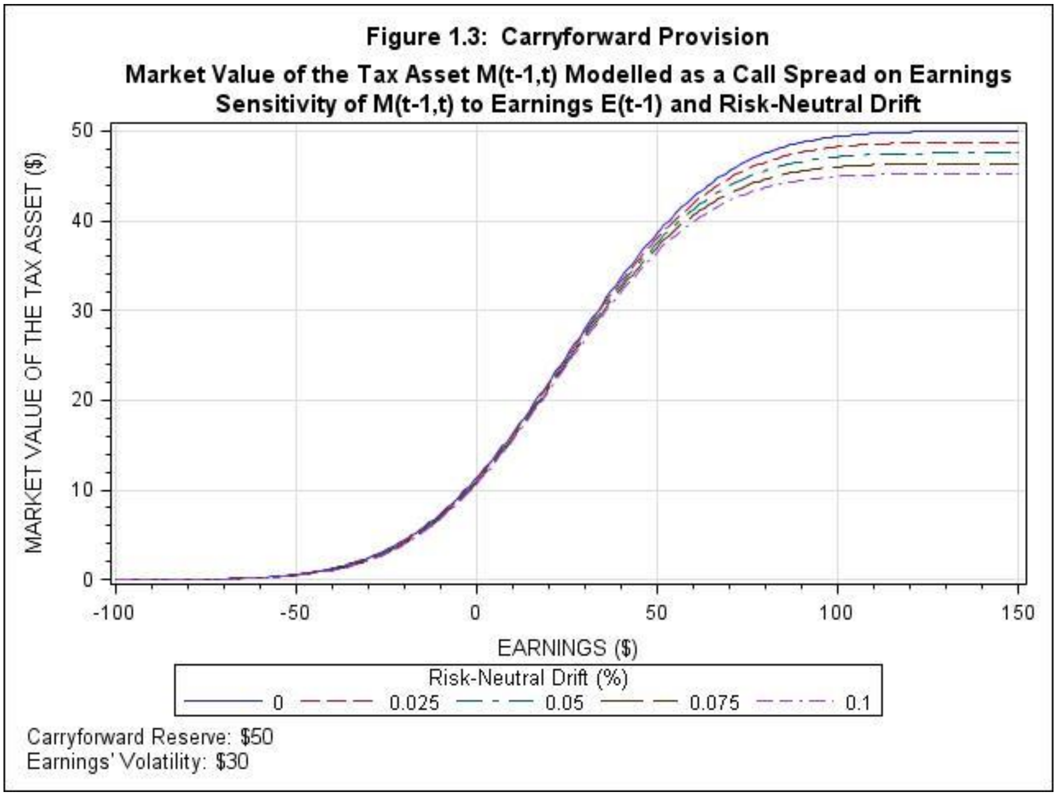


Figure 1.3: Carryforward Provision
Market Value of the Tax Asset M(t-1,t) Modelled as a Call Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Risk-Neutral Drift
MARKET VALUE OF THE TAX ASSET ($)
EARNINGS ($)
Risk-Neutral Drift (%)
0
0.025
0.05
0.075
0.1
Carryforward Reserve: $50
Earnings' Volatility: $30

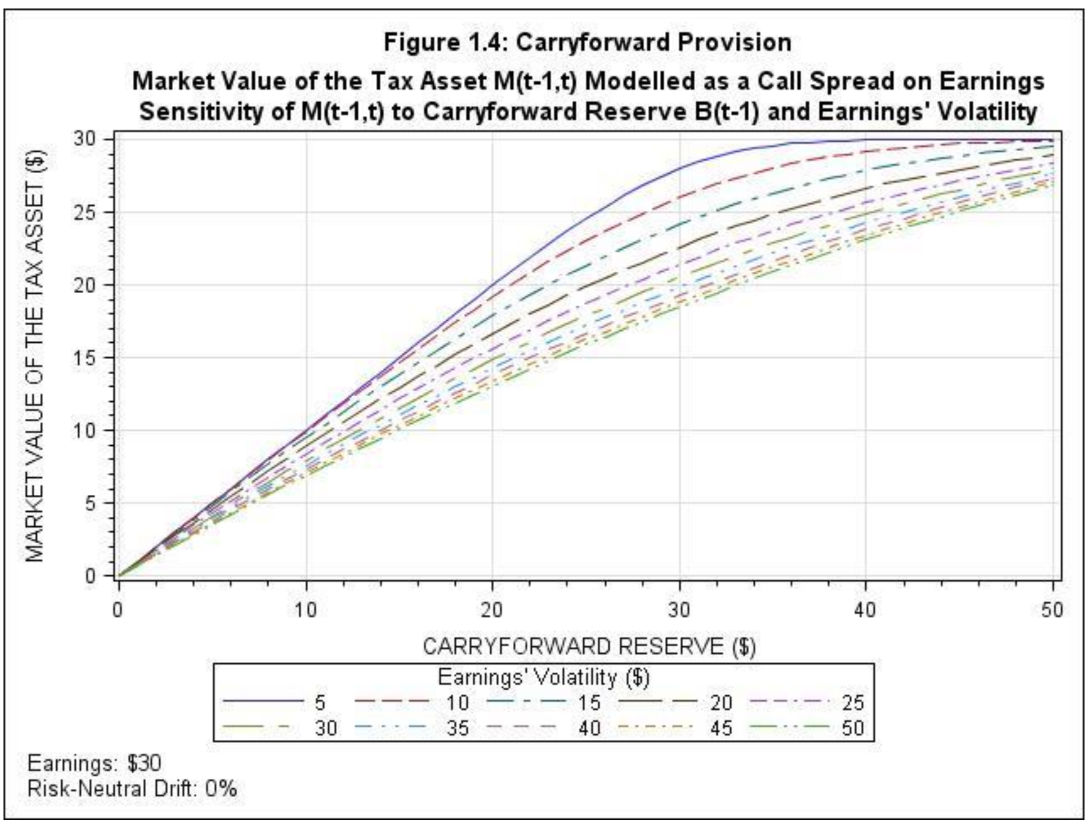
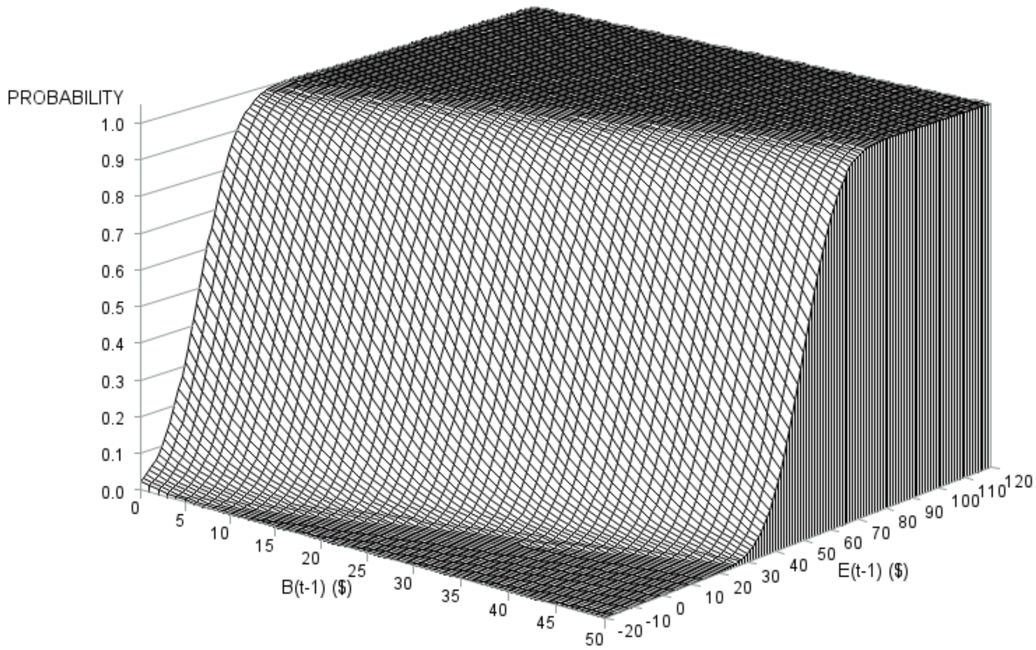


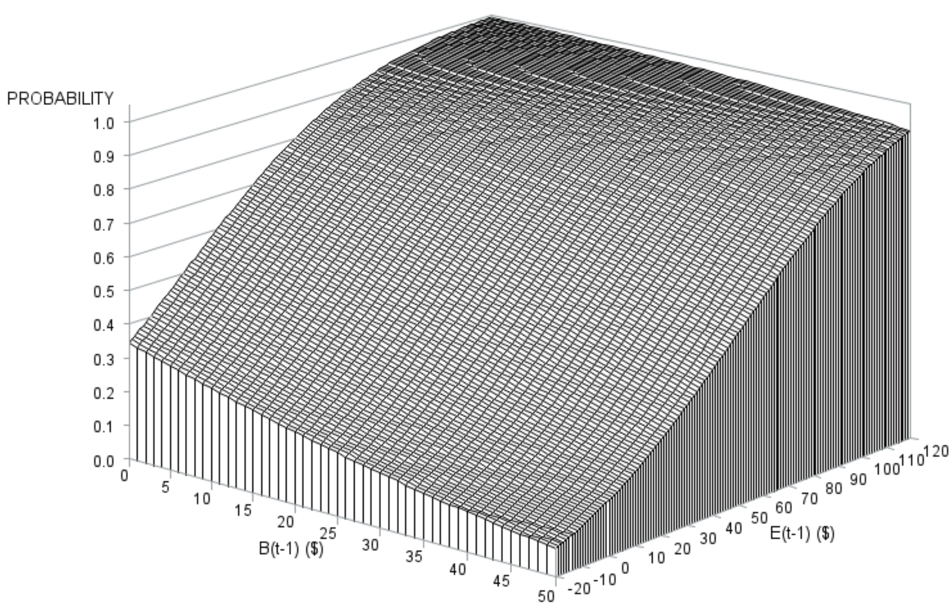
Figure 1.4: Carryforward Provision
Market Value of the Tax Asset M(t-1,t) Modelled as a Call Spread on Earnings
Sensitivity of M(t-1,t) to Carryforward Reserve B(t-1) and Earnings' Volatility
MARKET VALUE OF THE TAX ASSET ($)
CARRYFORWARD RESERVE ($)
Earnings' Volatility ($)
5
10
15
20
25
30
35
40
45
50
Earnings: $30
Risk-Neutral Drift: 0%

Figure 1.5: Carryforward Reserve
Conditional Transition Probability from Non-Taxable Status at Time t-1 to Taxable Status at Time t
Sensitivity to Earnings E(t-1) and Carryforward Reserve B(t-1)
VOLATILITY=10



Risk-neutral Drift:0%

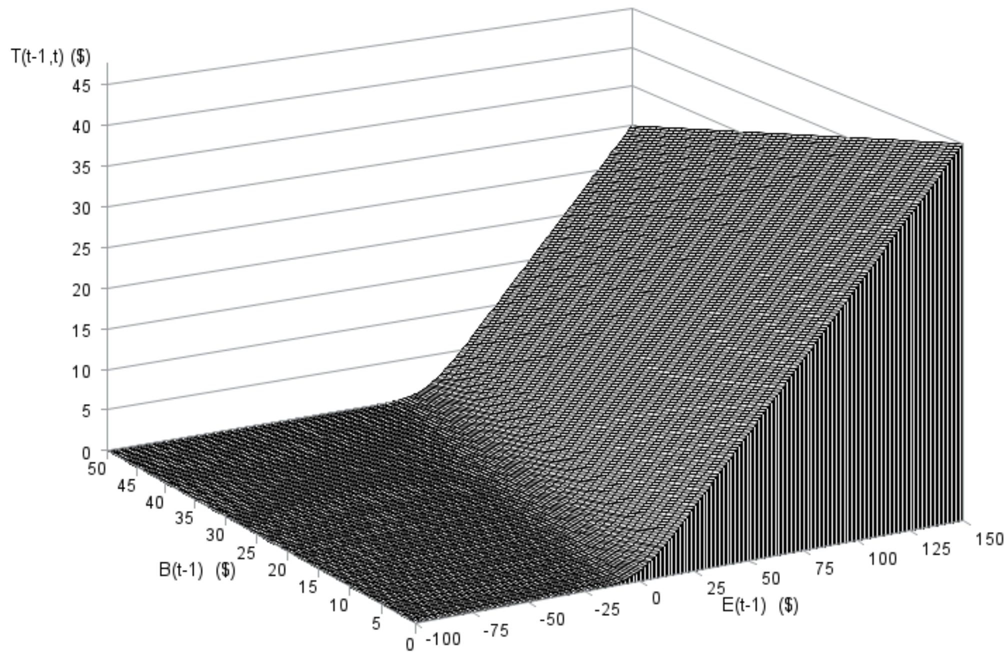
Figure 1.5: Carryforward Reserve
Conditional Transition Probability from Non-Taxable Status at Time t-1 to Taxable Status at Time t
Sensitivity to Earnings E(t-1) and Carryforward Reserve B(t-1)
VOLATILITY=50



Risk-neutral Drift:0%

Figure 1.6: Carryforward Provision

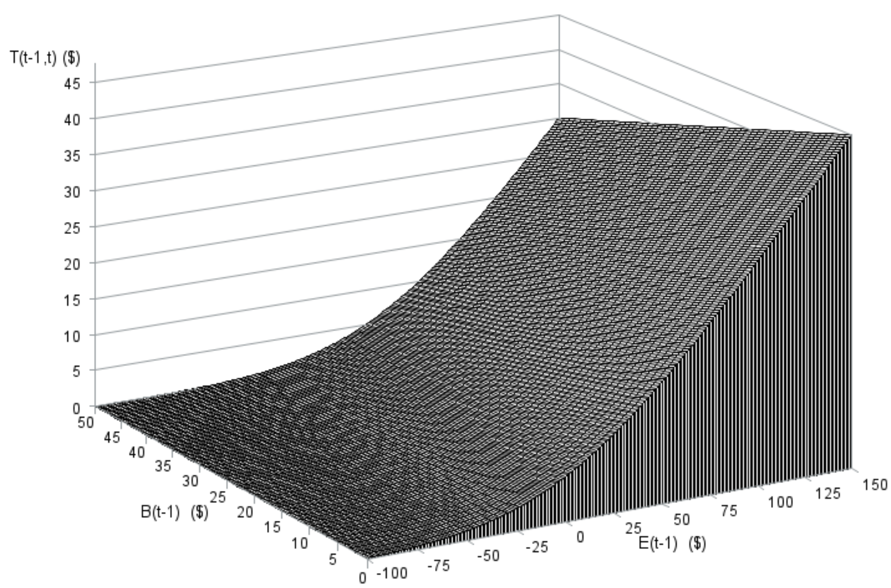
Expected Tax Liability T(t-1,t) Modelled as a Call Spread on Earnings with Tc=30.0%
Sensitivity to Carryforward Reserve B(t-1)and Earnings E(t-1)
Volatility ($)=10



Risk-Neutral Drift: 0%

Figure 1.6: Carryforward Provision

Expected Tax Liability T(t-1,t) Modelled as a Call Spread on Earnings with Tc=30.0%
Sensitivity to Carryforward Reserve B(t-1)and Earnings E(t-1)
Volatility ($)=50



Risk-Neutral Drift: 0%

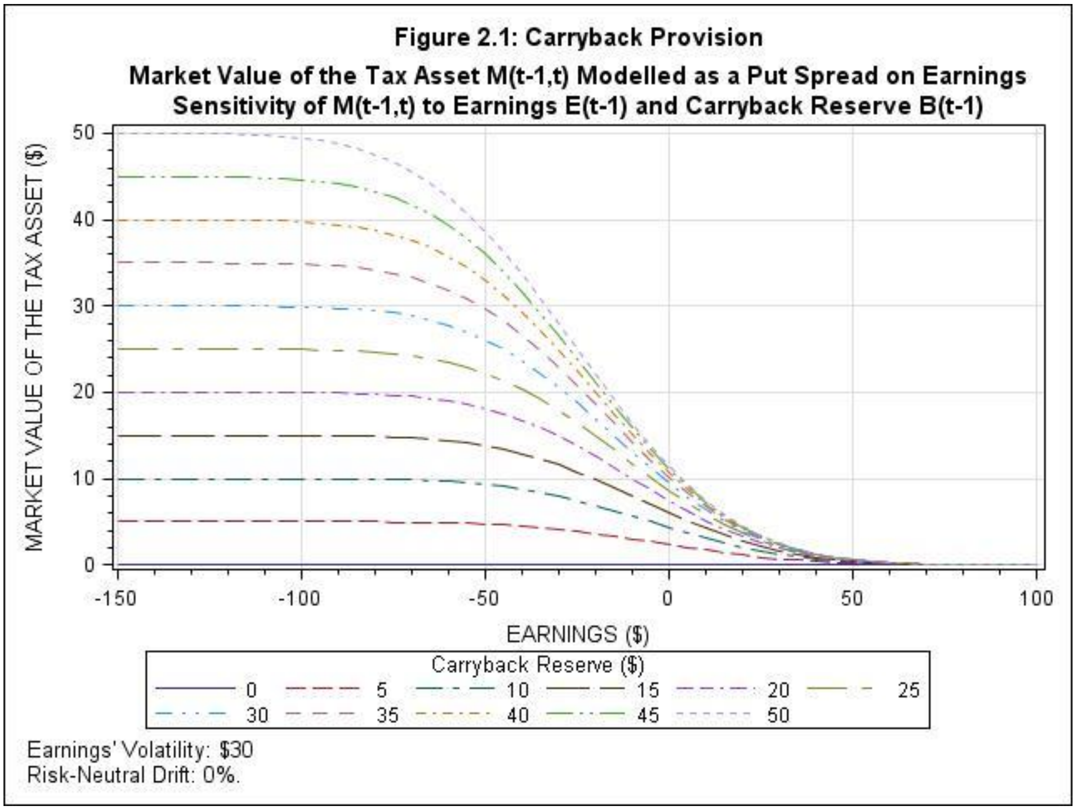


Figure 2.1: Carryback Provision
Market Value of the Tax Asset M(t-1,t) Modelled as a Put Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Carryback Reserve B(t-1)

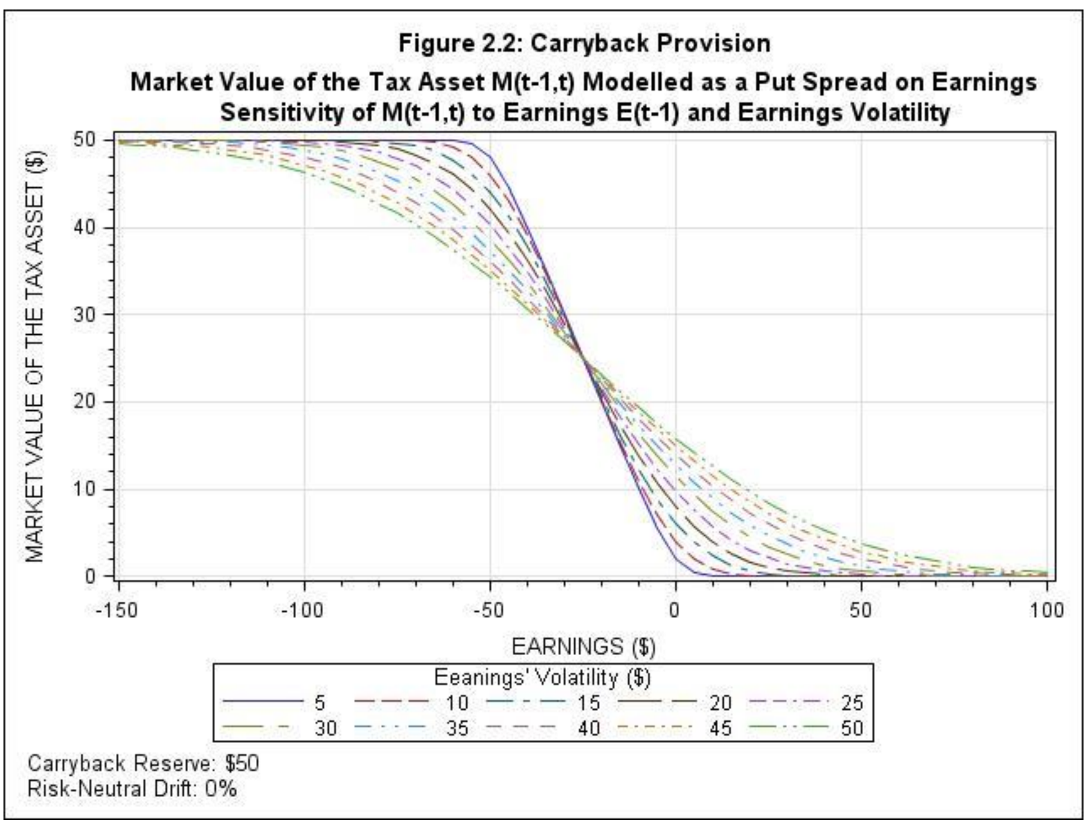


Figure 2.2: Carryback Provision
Market Value of the Tax Asset M(t-1,t) Modelled as a Put Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Earnings Volatility

Figure 2.3: Carryback Provision

Market Value of Tax Asset M(t-1,t) Modelled as a Put Spread on Earnings
Sensitivity of M(t-1,t) to Earnings E(t-1) and Risk-Neutral Drift

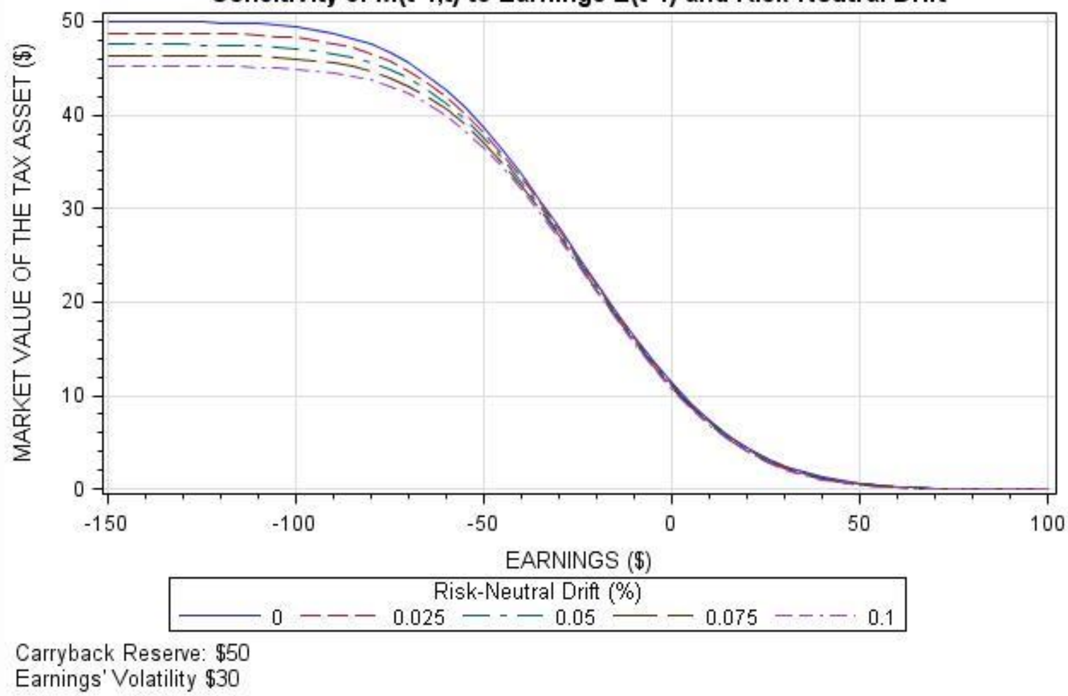


Figure 2.4: Carryback Provision

Market Value of the Tax Asset M(t-1,t) Modelled as Put Spread on Earnings
Sensitivity of M(t-1,t) to Carryback Reserve B(t-1) and Earnings' Volatility

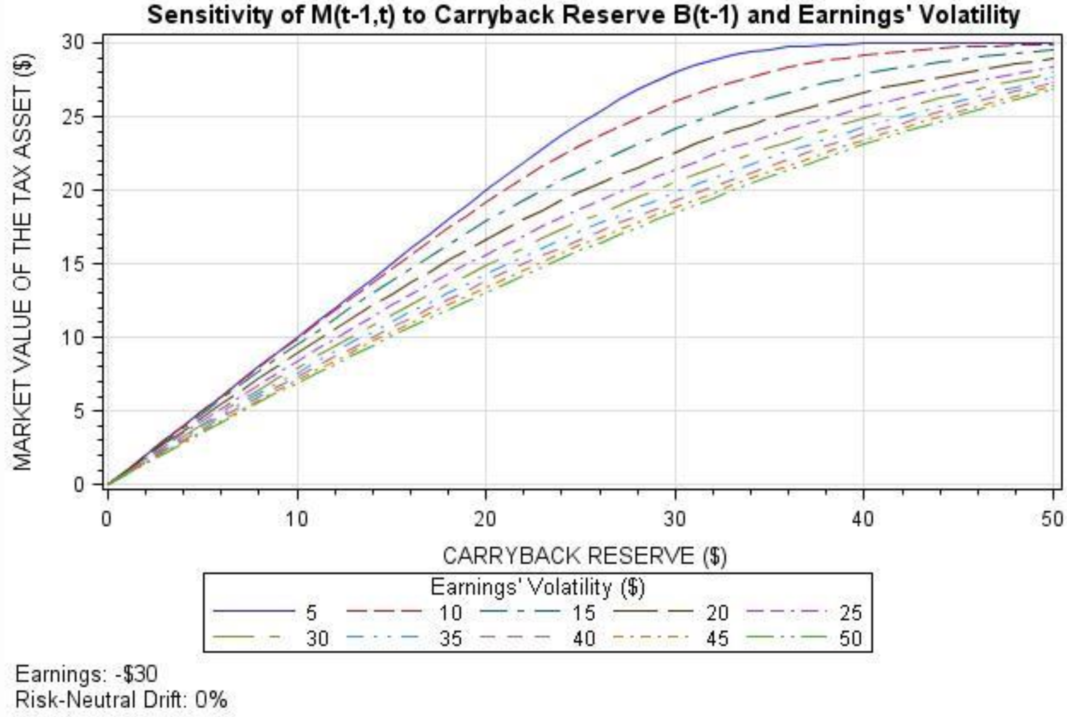
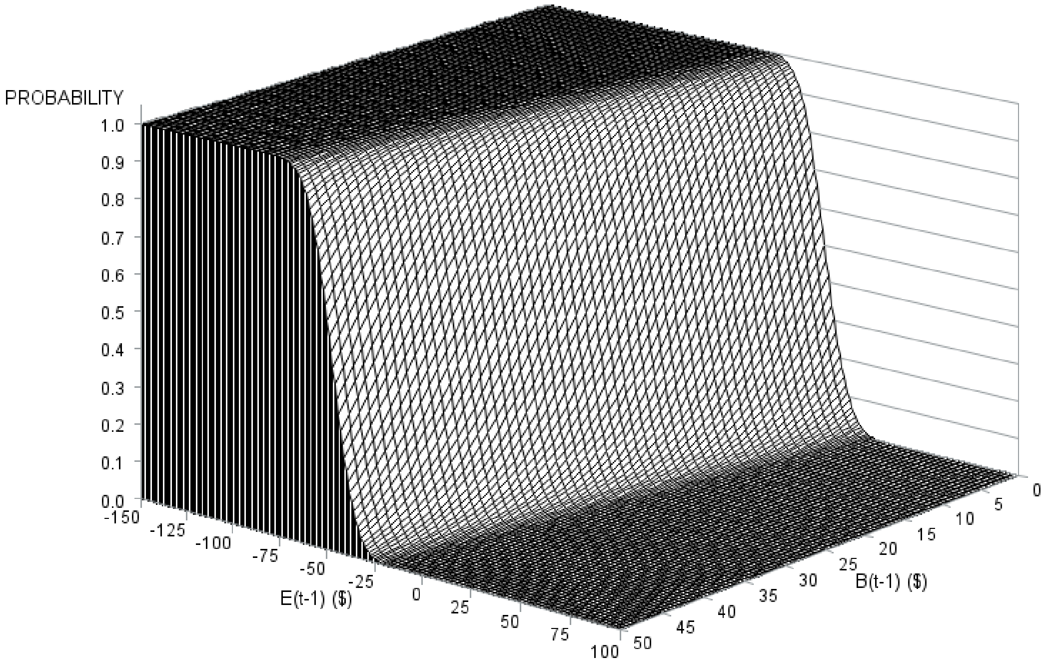
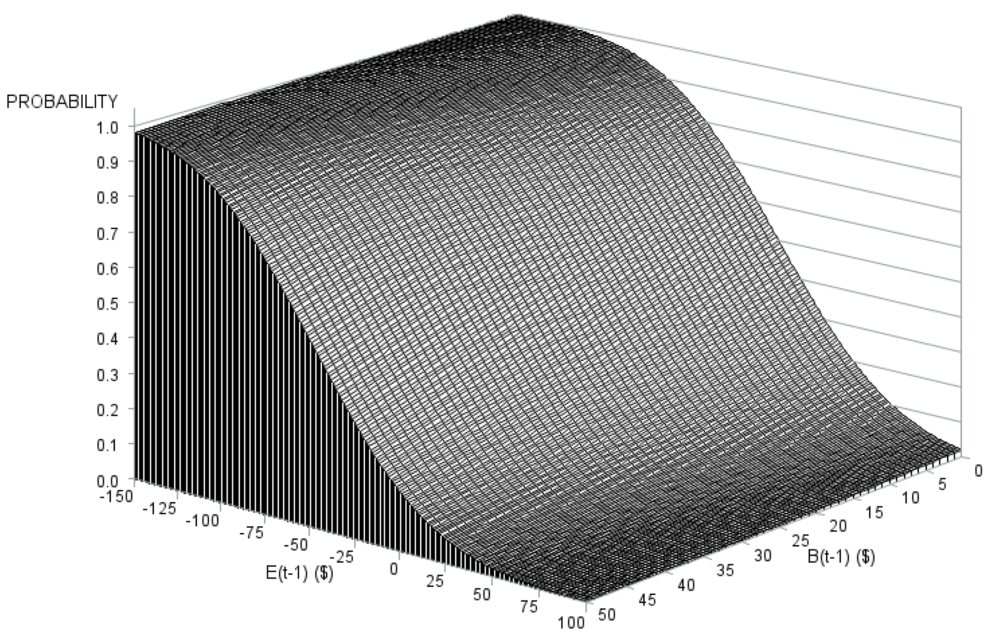


Figure 2.5: Carryback Provision
Conditional Transition Probability p(t-1,t) from Taxable Status at Time t-1 to Non-Taxable Status at Time t
Sensitivity of p(t-1,t)to Earnings E(t-1) and Carryforward Reserve B(t-1)
Volatility ($)=10



Risk-neutral Drift:0%

Figure 2.5: Carryback Provision
Conditional Transition Probability p(t-1,t) from Taxable Status at Time t-1 to Non-Taxable Status at Time t
Sensitivity of p(t-1,t)to Earnings E(t-1) and Carryforward Reserve B(t-1)
Volatility ($)=50



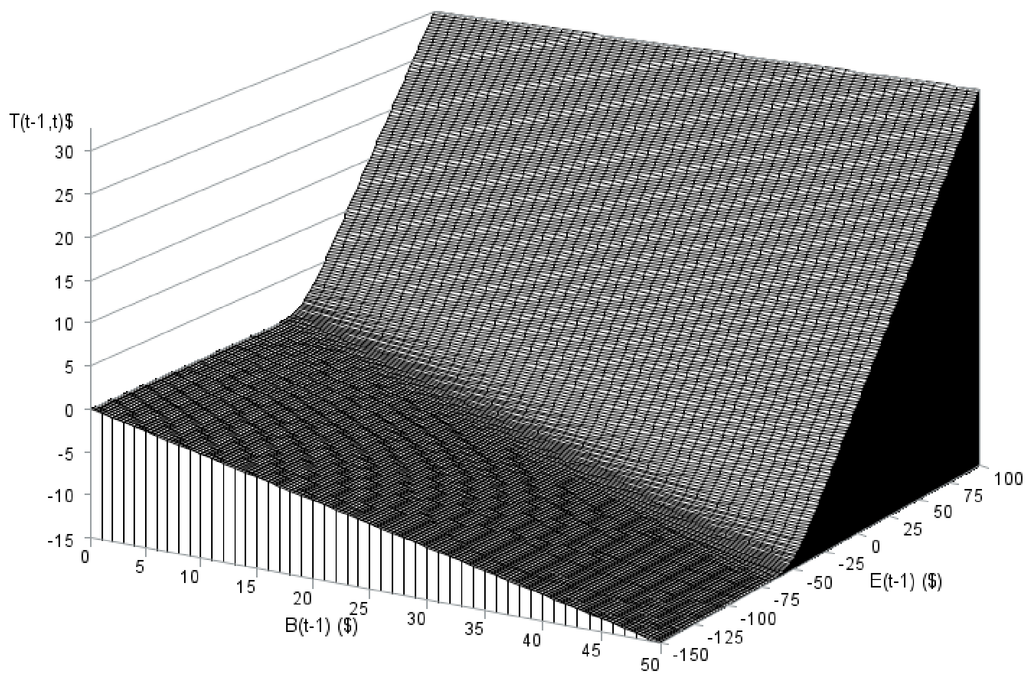
Risk-neutral Drift:0%

Figure 2.6: Carryback Provision

Expected Tax Liability T(t-1,t) Modelled as a Call Spread on Earnings (minus Tax Refund) with Tc=30.0%

Sensitivity of T(t-1,t) to Carryback Reserve B(t-1)and Earnings

Volatility ($)=10



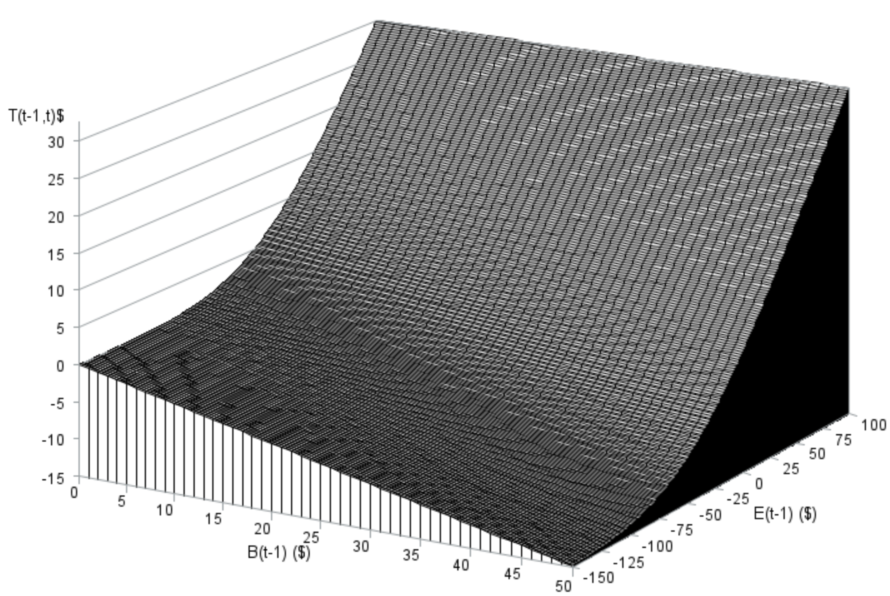
Risk-Neutral Drift: 0%

Figure 2.6: Carryback Provision

Expected Tax Liability T(t-1,t) Modelled as a Call Spread on Earnings (minus Tax Refund) with Tc=30.0%

Sensitivity of T(t-1,t) to Carryback Reserve B(t-1)and Earnings

Volatility ($)=50



Risk-Neutral Drift: 0%