



Decisions with Several Objectives under Uncertainty: Sufficient Conditions for Multivariate Almost Stochastic Dominance

Alfred Müller

Universität Siegen, mueller@mathematik.uni-siegen.de

Marco Scarsini

Luiss University, mscarsini@luiss.it

Ilia Tsetlin

INSEAD, ilia.tsetlin@insead.edu

Robert L. Winkler

Duke University, rwinkler@duke.edu

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Important business, public policy, and personal decisions typically involve multiple objectives, which in turn can be represented by multiple attributes, and uncertainty. Assessing both multiattribute utility and multivariate distributions for the attributes can be challenging. Moreover, big decisions are often made by boards or committees with members holding divergent views and preferences and facing pressures from different stakeholders. Thus, a full-blown traditional decision analysis that leads to the computation of expected utility is very difficult at best and often not possible. We develop sufficient conditions for multivariate almost stochastic dominance (MASD) based on marginal distributions of the attributes or just on their means and variances. To apply MASD, one only needs to assess bounds on marginal utilities. Alternatively, preferences can be explained and elicited via transfers. Realistic examples and a case study using real data illustrate our results, which provide tools for “fast and frugal” screening and evaluation of the available options, while properly accounting for tradeoffs and riskiness. Such tools, consistent with normative decision analysis, are useful when making important decisions in today’s fast-moving and often complex world.

Subject classifications: Decision analysis: stochastic dominance, utility, risk. Probability: distribution comparisons.

Keywords: Multivariate Almost Stochastic Dominance; Transfers; Sufficient Conditions for Dominance; Choice Between Lotteries; Mean and Variance.

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Alfred Müller¹, Marco Scarsini², Ilia Tsetlin³, and Robert L. Winkler⁴

¹Department Mathematik, Universität Siegen, 57072 Siegen, Germany

²Dipartimento di Economia e Finanza, Luiss University, 00197 Roma, Italy

³INSEAD, Singapore 138676

⁴Fuqua School of Business, Duke University, Durham, North Carolina 27708

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Abstract

Important business, public policy, and personal decisions typically involve multiple objectives, which in turn can be represented by multiple attributes, and uncertainty. Assessing both multi-attribute utility and multivariate distributions for the attributes can be challenging. Moreover, big decisions are often made by boards or committees with members holding divergent views and preferences and facing pressures from different stakeholders. Thus, a full-blown traditional decision analysis that leads to the computation of expected utility is very difficult at best and often not possible. We develop sufficient conditions for multivariate almost stochastic dominance (MASD) based on marginal distributions of the attributes or just on their means and variances. To apply MASD, one only needs to assess bounds on marginal utilities. Alternatively, preferences can be explained and elicited via transfers. Realistic examples and a case study using real data illustrate our results, which provide tools for “fast and frugal” screening and evaluation of the available options, while properly accounting for tradeoffs and riskiness. Such tools, consistent with normative decision analysis, are useful when making important decisions in today’s fast-moving and often complex world.

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1 Introduction

When faced with an important choice, decision makers are typically interested in more than just one attribute. For example, a company choosing between two risky projects, A and B, might be interested

in the net present value (NPV) of profits for the first five years and the market share (MS) at the end of the fifth year. Traditional decision analysis would suggest: a) assessing the bivariate distribution of NPV and MS for each project and b) eliciting the two-attribute utility function of the company. Our approach partially bypasses both of these steps.

The following scenarios illustrate the type of situations in which our approach can be helpful. We will return to these examples as we present our results.

- (a) The board of a company makes decisions together. Some board members may have their own assessed utility functions for the company, but these assessments are not identical. Other board members are not sure about their risk preferences, but they can agree about some constraints on tradeoffs among the attributes or bounds on the marginal utilities. Moreover, we assume that they can agree about the marginal distributions or at least the means and variances. We provide conditions under which the board can unanimously rank risky projects A and B.
- (b) A data analytics startup develops inventory and allocation solutions, with the objective of maximizing profit (P) and net promotion score (NPS), as well as some other attributes if requested by a client. Their current approach is to maximize the expected weighted sum (e.g., $P + w\text{NPS}$, where w represents the NPS/P tradeoff). However, often a client is not satisfied with this solution, arguing that this approach does not consider the risks associated with different options or the client's attitude toward these risks. At the same time, the members of the startup team feel that they cannot apply a full-blown decision analysis approach, which would require assessing a multiattribute utility function and the joint distribution of the attributes under each option. As part of their analysis, they have estimates of means and variances of different options. Using our results, they can narrow down the choice to a few non-dominated options and see how the optimal strategies vary with different parameters.
- (c) A company has the possibility of an investment in a photovoltaic solar power system and wants to evaluate possible different locations. The value of such a PV system is a difficult function of the solar irradiance at the location, but there are natural bounds for marginal utilities caused by the prices for buying and selling electricity. Partial knowledge of the multivariate distribution of solar irradiance over the course of a day is available from historical weather observations. We will consider this problem in a case study at the end of this paper.

The assumption of having only partial knowledge of the utility function of a decision maker or of more general preferences is the classical topic of stochastic dominance (SD). This is a very well established topic in the univariate case, but extensions of SD to the multivariate case have also received some attention. Such extensions are tricky, as there are many multivariate stochastic orders (see, e.g., Müller and Stoyan, 2002, Shaked and Shanthikumar, 2007). Studies of multivariate stochastic dominance (MSD) include Levy and Paroush (1974), Levhari et al. (1975), Mosler (1984), Scarsini (1988a), and Baccelli and Makowski (1989). Denuit et al. (2013) develop MSD, using a stochastic

order that is a natural extension of the standard order typically used for univariate SD. The theory of SD has a counterpart in the literature about inequality measurement. Recent multivariate analyses of it can be found in [Faure and Gravel \(2021\)](#) and [Mosler \(2021\)](#).

The SD order provides a partial ranking of distributions that can be helpful when only partial information is known about a decision maker’s utility function. There is a big jump from first-degree stochastic dominance (FSD) (with increasing utility) to second-degree stochastic dominance (SSD) (with increasing and concave utility). Many decision makers are mostly risk averse but cannot assert that they would dislike any risk, an indication of convex segments in their utility functions. The almost stochastic dominance (ASD) relation can provide a continuum of SD rules covering preferences from FSD to SSD ([Leshno and Levy, 2002](#), [Müller et al., 2017](#), [Huang et al., 2020](#), [Mao and Wang, 2020](#)). The importance of MASD is due to the fact that it allows us to rank multivariate distributions when utility does not satisfy multivariate SSD but is “close” to doing so. [Tsetlin and Winkler \(2018\)](#) develop MASD, considering both concave and convex versions. When multiple decision makers are involved, it helps us rank distributions “by most decision makers” in the multivariate context.

Often only partial information is known about the distributions that we want to rank for decision-making purposes. For example, we might know the means and variances of the distributions but not their shapes. The seminal paper by [Markowitz \(1952\)](#) inspired a strong focus on the mean and variance for decision making in finance. [Müller et al. \(2021\)](#) provide a ranking in the single-attribute case when only means and variances are known by bounding how much marginal utility can change.

Even if the distributions we want to rank are known, we usually do not have easy conditions to check for MSD and MASD. In such cases sufficient conditions, which are easy to check, are helpful. The sufficient conditions that we develop in this paper are especially practical, as they require knowing only the marginal distributions of the attributes, or just their means and variances.

In [Section 2](#) we present a review of multivariate first-order stochastic dominance and the difficulties to assess it; this section also motivates our sufficient conditions based on marginal distributions. In [Section 3](#) we provide definitions for limiting how much marginal utilities can change, for dominance based on these limitations, and we develop the corresponding transfers. In [Section 4](#) we develop sufficient conditions for the case where the full marginal distributions are known and for the more common case where we only know their means and variances. We also provide bounds on multiattribute utilities that are additive across the multiple attributes, which is important because it allows us to develop sufficient conditions for MASD involving only marginal distributions of the multivariate random variables associated with the alternatives. In [Section 5](#) we develop a path to a complete order and the corresponding transfers. Throughout we provide examples to illustrate our results, and in [Section 6](#) we present a case study that involves a decision on investing in photovoltaic power systems and analyzes real data using the concepts developed in the paper. Concluding comments are given in [Section 7](#). [Appendix A](#) shows a generalization of some characterizations to distributions with nonfinite support. The proofs of our results can be found in [Appendix B](#).

2 A review of multivariate first order stochastic dominance

The concept of multivariate stochastic dominance was first studied in the statistics literature by [Lehmann \(1955\)](#) and then introduced in the economic literature by [Levhari et al. \(1975\)](#). In general, given a class \mathcal{U} of utility functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ and two random vectors \mathbf{X} and \mathbf{Y} , we say that $\mathbf{X} \leq_{\mathcal{U}} \mathbf{Y}$ if

$$\mathbb{E}[u(\mathbf{X})] \leq \mathbb{E}[u(\mathbf{Y})] \quad \text{for all } u \in \mathcal{U}. \quad (2.1)$$

If \mathcal{U} consists of all componentwise increasing functions, then we speak of *first order stochastic dominance* (FSD). Using the notation of [Müller and Stoyan \(2002\)](#) and [Shaked and Shanthikumar \(2007\)](#), this relation is denoted by $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$.

The following notation for cumulative distribution functions (CDFs) will be used throughout the paper:

$$F(t_1, \dots, t_N) := \mathbb{P}(X_1 \leq t_1, \dots, X_N \leq t_N), \quad G(t_1, \dots, t_N) := \mathbb{P}(Y_1 \leq t_1, \dots, Y_N \leq t_N), \quad (2.2)$$

with the corresponding notation for the marginal distributions:

$$F_i(t_i) := \mathbb{P}(X_i \leq t_i), \quad G_i(t_i) := \mathbb{P}(Y_i \leq t_i). \quad (2.3)$$

In the univariate case the condition in [Eq. \(2.1\)](#) is equivalent to pointwise ordering of the corresponding CDFs, which is typically easy to check:

$$X \leq_{\text{st}} Y \iff F(t) \geq G(t) \quad \text{for all } t \in \mathbb{R}. \quad (2.4)$$

Unfortunately, in the multivariate case the situation is much more complex. In this case, the condition $F_{\mathbf{X}}(\mathbf{t}) \geq F_{\mathbf{Y}}(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^N$ is only necessary for $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$. Moreover, even this condition is typically not easy to verify. The characterization of FSD corresponding to [\(2.4\)](#) becomes

$$\mathbf{X} \leq_{\text{st}} \mathbf{Y} \iff \mathbb{P}(\mathbf{X} \in A) \leq \mathbb{P}(\mathbf{Y} \in A) \quad \text{for every upper set } A \subset \mathbb{R}^N, \quad (2.5)$$

where a set $A \subset \mathbb{R}^N$ is upper if $\mathbf{x} \in A$ and $\mathbf{x} \leq \mathbf{y}$ imply $\mathbf{y} \in A$. This condition is hard to verify. For discrete distributions it is possible to use a characterization via transfers, from which a transportation problem can be derived and solved (see [Range and Østerdal, 2019](#), for details). But even this case is not easy to deal with.

Notice that $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$ implies $X_i \leq_{\text{st}} Y_i$ for all $i = 1, \dots, N$, but the converse implication is false. This is due to the fact that the dependence within the vectors \mathbf{X} and \mathbf{Y} has a relevant role for their dominance conditions. [Sklar \(1959\)](#) proved that, for every N -variate distribution function F with marginals F_1, \dots, F_N , we have

$$F(t_1, \dots, t_N) = C(F_1(t_1), \dots, F_N(t_N)), \quad \text{for all } (t_1, \dots, t_N) \in \mathbb{R}^N, \quad (2.6)$$

where C is a *copula*, i.e., a multivariate CDF with uniform marginals on $[0, 1]$. The function C describes the dependence structure of the random vector. The following theorem shows that, when two random vectors have the same copula, then they are stochastically ordered if and only if their marginals are (Rüschendorf, 1981, Scarsini, 1988b).

Theorem 2.1. *Let \mathbf{X} and \mathbf{Y} have a common copula. Then $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$ if and only if $X_i \leq_{\text{st}} Y_i$ for all $i = 1, 2, \dots, N$.*

Equality of the two copulas is a quite restrictive condition. It is interesting to notice that, if \mathbf{X} and \mathbf{Y} both have multivariate normal distributions, then having the same copula is necessary for being stochastically ordered. Let $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The following result can be found, e.g., in Müller and Stoyan (2002, theorem 3.3.13).

Theorem 2.2. *Let $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$ and $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}_Y, \boldsymbol{\Sigma}_Y)$. Then $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$ if and only if $\mu_i \leq \mu'_i$ for all $i = 1, \dots, N$, and $\boldsymbol{\Sigma}_X = \boldsymbol{\Sigma}_Y$.*

In practical applications the parameters in the covariance matrix $\boldsymbol{\Sigma}$ are typically estimated and then it is almost impossible to get exactly the same values. Therefore dominance conditions for the important case of multivariate normal distributions are almost never satisfied.

In general we are not aware of sufficient dominance conditions except the following very restrictive condition of separated supports for the marginals, which trivially yields FSD.

Theorem 2.3. *Consider the random vectors \mathbf{X} and \mathbf{Y} , and assume that there exists a vector $\boldsymbol{\delta} \in \mathbb{R}^N$ such that $X_i \leq_{\text{st}} \delta_i$ and $\delta_i \leq_{\text{st}} Y_i$ for all $i = 1, \dots, N$. Then $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$.*

If we can verify this very restrictive condition on the marginals, then we can completely ignore the dependence structures of the random vectors. Notice that, in contrast to this, in Theorem 2.1 we have to assume equal covariance matrices even if the means are very different. This shows that it is hopeless to find general weaker sufficient conditions on the marginals that imply FSD for all dependence structures, when the supports overlap, in particular when they are unbounded. However, for uniform marginals with overlapping supports this is possible, as the following example shows.

Example 2.4. Consider the random vectors $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, and assume that the marginal distributions of \mathbf{X} are uniform on $[0, 4]$ and the marginals of \mathbf{Y} are uniform on $[3, 7]$. Then we have $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$, no matter what the copulas of the two random vectors are. To see this, define the following four sets:

$$A = [0, 3]^2, \quad B = [0, 4]^2 \setminus A, \quad D = [4, 7]^2, \quad C = [3, 7]^2 \setminus D,$$

Thus, the support of \mathbf{X} is $A \cup B$, the support of \mathbf{Y} is $C \cup D$, and we have

$$\mathbb{P}(\mathbf{X} \in B) \leq \mathbb{P}(X_1 > 3) + \mathbb{P}(X_2 > 3) = \frac{1}{2}$$

and

$$\mathbb{P}(\mathbf{Y} \in C) \leq \mathbb{P}(Y_1 \leq 4) + \mathbb{P}(Y_2 \leq 4) = \frac{1}{2}.$$

Therefore $\mathbb{P}(\mathbf{Y} \in D) = 1 - \mathbb{P}(\mathbf{Y} \in C) \geq 1/2$. Now assume that u is an increasing function and define

$$a := u(3, 3), \quad b := u(4, 4).$$

On the support of \mathbf{X} we get $u(\mathbf{x}) \leq a \cdot \mathbf{1}_A(\mathbf{x}) + b \cdot \mathbf{1}_B(\mathbf{x})$; therefore

$$\mathbb{E}[u(\mathbf{X})] \leq a \cdot (1 - \mathbb{P}(\mathbf{X} \in B)) + b \cdot \mathbb{P}(\mathbf{X} \in B) \leq a + \frac{1}{2}(b - a) = \frac{1}{2}(a + b)$$

as $a \leq b$. Similarly, on the support of \mathbf{Y} we have $u(\mathbf{x}) \geq a \cdot \mathbf{1}_C(\mathbf{x}) + b \cdot \mathbf{1}_D(\mathbf{x})$; therefore

$$\mathbb{E}[u(\mathbf{Y})] \geq a \cdot (1 - \mathbb{P}(\mathbf{Y} \in D)) + b \cdot \mathbb{P}(\mathbf{Y} \in D) \geq \frac{1}{2}(a + b).$$

This implies that $\mathbb{E}[u(\mathbf{X})] \leq \mathbb{E}[u(\mathbf{Y})]$ for all increasing u .

[Range and Østerdal \(2019, section 5\)](#) contains an interesting example, where first two bivariate normal distributions with different covariance matrices and significantly different means are discretized and then the algorithm for the discrete case is used. As we know from [Theorem 2.2](#), FSD cannot hold in this case. However, if the discretization is done on a grid of at most 16×16 points, then FSD holds for the discretization! Only a very fine discretization shows that FSD is false for the original distributions. This indicates that some version of *almost* FSD should hold for any multivariate normal distribution if the difference of the means is relatively large compared to the variances. Our results in the next sections will show that this is indeed the case. Notice also that all examples in this section concerned distributions from location-scale families (normal, uniform). An explicit treatment of these families will be provided in [Section 4.3](#).

3 Defining γ -multivariate almost stochastic dominance

We consider a decision maker whose utility function u depends on N attributes $\mathbf{x} = (x_1, \dots, x_N)$. The function $u: \mathbb{R}^N \rightarrow \mathbb{R}$ is assumed to be differentiable, and u'_i denotes its partial derivative with respect to its i -th argument:

$$u'_i(\mathbf{x}) := \frac{\partial u(\mathbf{x})}{\partial x_i}.$$

We now define γ -multivariate almost stochastic dominance (γ -MASD) for N -variate random vectors. Recall that, given a class \mathcal{U} of utility functions, we say that $\mathbf{X} \leq_{\mathcal{U}} \mathbf{Y}$ if

$$\mathbb{E}[u(\mathbf{X})] \leq \mathbb{E}[u(\mathbf{Y})] \quad \text{for all } u \in \mathcal{U}.$$

Definition 3.1. For a given vector $\gamma := (\gamma_1, \dots, \gamma_N) \in [0, 1]^N$, the symbol \mathcal{U}_γ denotes the set of utility functions such that, for all $i \in \{1, \dots, N\}$, we have

$$0 \leq \gamma_i u'_i(\mathbf{y}) \leq u'_i(\mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^N. \quad (3.1)$$

Notice that the condition in inequality (3.1) is equivalent to

$$\frac{\inf u'_i(\mathbf{x})}{\sup u'_i(\mathbf{x})} \geq \gamma_i. \quad (3.2)$$

For $\gamma \in [0, 1]^N$, the random vector \mathbf{X} is dominated by the random vector \mathbf{Y} in the sense of γ -MASD if $\mathbf{X} \leq_{\mathcal{U}_\gamma} \mathbf{Y}$. For the sake of simplicity, we will write $\mathbf{X} \leq_\gamma \mathbf{Y}$ instead of $\mathbf{X} \leq_{\mathcal{U}_\gamma} \mathbf{Y}$.

Definition 3.1 corresponds to MASD, as defined by Tsetlin and Winkler (2018), with $\gamma_i = \varepsilon_i / (1 - \varepsilon_i)$ for all $i \in \{1, \dots, N\}$. In the univariate case ($N = 1$), it corresponds to almost first-degree stochastic dominance (AFSD), as defined by Leshno and Levy (2002).

Notice that, if $\gamma \leq \lambda$ componentwise, then $\mathcal{U}_\lambda \subset \mathcal{U}_\gamma$. Therefore $\mathbf{X} \leq_\gamma \mathbf{Y}$ implies $\mathbf{X} \leq_\lambda \mathbf{Y}$ and obviously for $\gamma = \mathbf{0}$ we get $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$.

Example 3.2. The board of a company evaluates projects by focusing on two attributes x_1 and x_2 , where x_1 is the NPV of profits for the next five years and x_2 is the MS in percentage at the end of the period. The members of the board have different risk preferences, but they all agree about the following bounds on marginal utilities:

$$\frac{\inf u'_1(\mathbf{x})}{\sup u'_1(\mathbf{x})} \geq 0.2, \quad \frac{\inf u'_2(\mathbf{x})}{\sup u'_2(\mathbf{x})} \geq 0.4.$$

Therefore, if two projects can be ranked as $\mathbf{X} \leq_{(0.2, 0.4)} \mathbf{Y}$, then all board members prefer \mathbf{Y} to \mathbf{X} .

We have defined an SD rule by a set of utility functions with bounded marginal utilities, and illustrated how one can check that a particular utility belongs to this set. The corresponding preferences can also be characterized via transfers, which might be easier to explain and use for elicitation of decision makers' preferences. The idea of using transfers to characterize SD can be traced back to the seminal paper by Rothschild and Stiglitz (1970), who have shown that increasing risk can be decomposed into a sequence of mean-preserving spreads. The name *transfer* for such operations like mean-preserving spreads was originally more common in the related literature on inequality measurement, where these transfers have the meaning of real transfers of income or wealth; see Atkinson (1970), a famous companion paper to Rothschild and Stiglitz (1970). It can be shown for many types of SD that, in the case of distributions assuming only a finite number of values, the dominance rule holds if and only if one distribution can be obtained from the other by a sequence of simple transfers. For multivariate FSD Østerdal (2010) shows that this holds for increasing transfers, i.e., transfers that shift some probability mass from some point \mathbf{x} to some point $\mathbf{y} > \mathbf{x}$, meaning that we have a *good transfer* to a better situation. For first-degree or second-degree ASD one typically also allows for decreasing transfers shifting

some probability mass from some point \mathbf{x} to some point $\mathbf{y} < \mathbf{x}$ as long as this is compensated or overcompensated by corresponding increasing transfers. See, e.g., Müller et al. (2017) for the univariate case or Müller and Scarsini (2012) for the multivariate case of inframodular transfers. Other related concepts of transfers have been considered in Kamihigashi and Stachurski (2020) and Elton and Hill (1992). A general theory of transfers has been developed in Müller (2013). We will show now that such a characterization also holds for the multivariate versions of SD considered in this paper.

Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ we use the notation $\mathbf{x} < \mathbf{y}$ to indicate

$$x_i \leq y_i, \quad \text{for } i = 1, \dots, N, \quad \text{and} \quad \mathbf{x} \neq \mathbf{y}.$$

The symbol \mathbf{e}_i denotes the i -th vector of the canonical basis.

Definition 3.3. Consider two discrete cumulative distribution functions F and G with respective mass functions f and g .

- (a) We say that G is obtained from F via an *increasing transfer* if there exist $\mathbf{x}_1 < \mathbf{x}_2$ and $\eta > 0$ such that

$$\begin{aligned} g(\mathbf{x}_1) &= f(\mathbf{x}_1) - \eta, \\ g(\mathbf{x}_2) &= f(\mathbf{x}_2) + \eta, \\ g(\mathbf{z}) &= f(\mathbf{z}) \quad \text{for all other values } \mathbf{z}. \end{aligned}$$

- (b) We say that G is obtained from F via a γ_i -transfer along dimension i if there exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^N$, $h, \eta_1, \eta_2 > 0$ such that

$$\mathbf{x}_2 = \mathbf{x}_1 + h\mathbf{e}_i, \quad \eta_2(\mathbf{x}_4 - \mathbf{x}_3) = \gamma_i\eta_1(\mathbf{x}_2 - \mathbf{x}_1), \quad (3.3)$$

and

$$\begin{aligned} g(\mathbf{x}_1) &= f(\mathbf{x}_1) - \eta_1, \\ g(\mathbf{x}_2) &= f(\mathbf{x}_2) + \eta_1, \\ g(\mathbf{x}_3) &= f(\mathbf{x}_3) + \eta_2, \\ g(\mathbf{x}_4) &= f(\mathbf{x}_4) - \eta_2, \\ g(\mathbf{z}) &= f(\mathbf{z}) \quad \text{for all other values } \mathbf{z}. \end{aligned}$$

We say that G is obtained from F via a γ -transfer if G is obtained from F via a γ_i -transfer along some dimension $i \in \{1, \dots, N\}$.

Fig. 1 gives an example of γ_1 -transfer with $N = 2$, $\gamma_1 = 2/3$, $\eta_1 = \eta_2$. This multivariate transfer is the natural generalization of the univariate (convex or concave) γ -transfer (or equivalently the

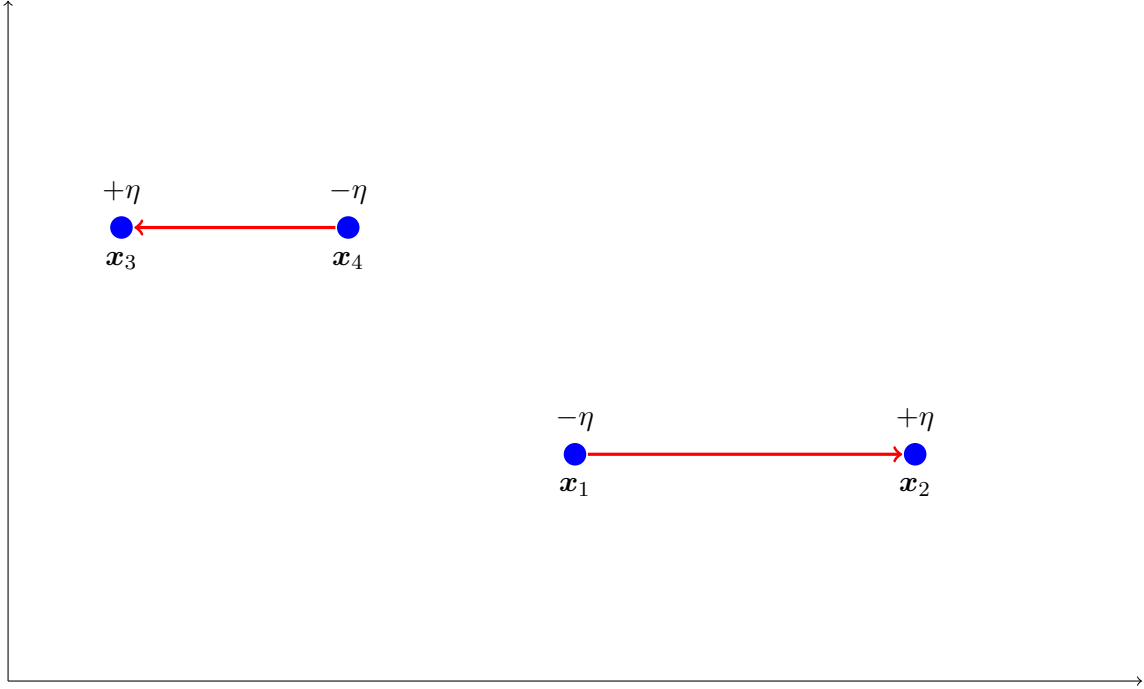


Figure 1: Example of a γ -transfer with $\gamma_1 = 2/3$, $\eta_1 = \eta_2 = \eta$.

univariate AFSD transfer (Müller et al., 2017)). It simply consists of a decreasing transfer from \mathbf{x}_4 to \mathbf{x}_3 which is compensated by an increasing transfer from \mathbf{x}_1 to \mathbf{x}_2 concerning the same component i . It leads to a univariate γ -transfer of the i -th marginal as described in Müller et al. (2017)) and does not affect any of the other marginals.

We now characterize the order \leq_γ in terms of probability transfers.

Theorem 3.4. *Let the random vectors \mathbf{X} and \mathbf{Y} assume only a finite number of values. Then $\mathbf{X} \leq_\gamma \mathbf{Y}$ if and only if the distribution of \mathbf{Y} can be obtained from the distribution of \mathbf{X} by a finite number of increasing transfers and γ -transfers.*

Theorem 3.4 illustrates that preferences consistent with γ -MASD can be thought of as preferences for multivariate γ -transfers. Later we state a similar result in Theorem 5.5 and discuss a generalization to distributions with nonfinite support in Theorem A.1.

Eliciting a multiattribute utility function is notoriously difficult. However, a decision maker might feel comfortable answering this question: “For any fair lottery (say, a coin flip), would increasing attribute i by one unit if the outcome is heads or reducing this attribute by $t < 1$ units if it is tails improve this lottery for you or make it worse for you?” This question can be asked for different values of t . A typical strategy for doing that in decision analysis is to ask the question for a very low value of t (expecting the decision maker will prefer the lottery) and for a high value of t (expecting that the lottery will not be preferred). Then values of t higher than the low value and values lower than the high value can be used to narrow in on an indifference point. This should provide a reasonable estimate of the indifference point, which is the bound γ_i for MASD. Her preference for γ_i -transfers is,

by [Theorem 3.4](#), consistent with γ -MASD. Note that such preferences extend beyond the framework of expected utility over the distributions of the alternatives to settings with dependent background risk ([Section 4.5](#)) and with payoffs that are expressed as suprema of expected utilities ([Section 6](#)).

The following theorem is the building block in the proofs of the subsequent results. The basic idea is that increments of functions $u \in \mathcal{U}_\gamma$ can be bounded above and below by separable piecewise linear utility functions that depend on γ . This fact will allow us to find sufficient conditions for γ -dominance that do not depend on the joint distributions of the random vectors \mathbf{X} and \mathbf{Y} , but only on the marginal distributions of their components. They will be much less restrictive than the ones mentioned in [Theorem 2.3](#).

Theorem 3.5. *Let*

$$v_U(x; \gamma) := \begin{cases} \gamma x & \text{if } x \leq 0, \\ x & \text{if } x > 0, \end{cases}$$

$$v_L(x; \gamma) := \begin{cases} x & \text{if } x \leq 0, \\ \gamma x & \text{if } x > 0. \end{cases}$$

For any $u \in \mathcal{U}_\gamma$, let $b_i := \sup_{\mathbf{x} \in \mathbb{R}^N} u'_i(\mathbf{x})$ and fix some $\mathbf{z} \in \mathbb{R}^N$. Then, for any $\mathbf{x} \in \mathbb{R}^N$, we have

$$\sum_{i=1}^N b_i v_L(x_i - z_i; \gamma_i) \leq u(\mathbf{x}) - u(\mathbf{z}) \leq \sum_{i=1}^N b_i v_U(x_i - z_i; \gamma_i). \quad (3.4)$$

4 Sufficient dominance conditions

In this section we consider sufficient conditions for γ -dominance. Most of the existing literature on stochastic dominance deals with necessary conditions.

In the whole paper the random vectors \mathbf{X}, \mathbf{Y} are assumed to have components with finite means and variances:

$$\mu_{X_i} := \mathbb{E}[X_i], \quad \mu_{Y_i} := \mathbb{E}[Y_i], \quad \sigma_{X_i}^2 := \mathbb{V}[X_i], \quad \sigma_{Y_i}^2 := \mathbb{V}[Y_i]. \quad (4.1)$$

4.1 Conditions for γ -dominance when \mathbf{Y} is degenerate

We start considering the case where one of the alternatives has a degenerate distribution (i.e., is a sure payoff vector). In this special case we can obtain necessary and sufficient dominance conditions, as the following propositions show.

It clarifies that, if the payoff \mathbf{Y} is equal to a sure vector \mathbf{c} , then the dominance conditions do not depend on the joint distribution of the random vector \mathbf{X} , but only on its marginals. Therefore the dependence structure of \mathbf{X} has no role.

Proposition 4.1. *Assume that the marginal distributions of the components of \mathbf{X} are known and that \mathbf{c} is a sure payoff vector.*

- (a) Let $c_i \leq \mu_{X_i}$ for all $i = 1, \dots, N$. Then $\mathbf{c} \leq_\gamma \mathbf{X}$ if and only if $c_i \leq_{\gamma_i} X_i$ for all $i = 1, \dots, N$.
- (b) Let $\mu_{X_i} \leq c_i$ for all $i = 1, \dots, N$. Then $\mathbf{X} \leq_\gamma \mathbf{c}$ if and only if $X_i \leq_{\gamma_i} c_i$ for all $i = 1, \dots, N$.

The next proposition makes the necessary and sufficient conditions for γ -dominance explicit. Notice that the integral condition for $X_i \leq_{\gamma_i} c_i$ reduces to $\mathbb{E}[(X_i - c_i)_+] \leq \gamma_i \mathbb{E}[(c_i - X_i)_+]$ in this degenerate case. Therefore we immediately get the following result.

Proposition 4.2. *Assume that the marginal distributions of the components of \mathbf{X} are known and that \mathbf{c} is a sure payoff vector.*

- (a) Let $c_i \leq \mu_{X_i}$ for all $i = 1, \dots, N$. Then $\mathbf{c} \leq_\gamma \mathbf{X}$ if and only if

$$\gamma_i \geq \frac{\mathbb{E}[(c_i - X_i)_+]}{\mathbb{E}[(X_i - c_i)_+]}, \quad i = 1, \dots, N. \quad (4.2)$$

- (b) Let $\mu_{X_i} \leq c_i$ for all $i = 1, \dots, N$. Then $\mathbf{X} \leq_\gamma \mathbf{c}$ if and only if

$$\gamma_i \geq \frac{\mathbb{E}[(X_i - c_i)_+]}{\mathbb{E}[(c_i - X_i)_+]}, \quad i = 1, \dots, N, \quad (4.3)$$

In this subsection we established sufficient conditions for γ -MASD for the case that one distribution is degenerate. These conditions are based on marginal distributions only, which make them especially easy to implement. In the next subsections we discuss how powerful this is and the corresponding intuition, given that usually the comparison of marginal distributions provides only necessary conditions for MSD.

4.2 Conditions for γ -dominance in the general case

We now provide a sufficient condition for $\mathbf{X} \leq_\gamma \mathbf{Y}$ for general \mathbf{X} and \mathbf{Y} that only uses the marginal distributions and holds for any dependence structures. The basic idea is to find a constant vector $\boldsymbol{\delta}$ that γ -dominates \mathbf{X} and is γ -dominated by \mathbf{Y} . According to Proposition 4.2 we can use any $\boldsymbol{\delta}$ between the means of \mathbf{X} and \mathbf{Y} if these are ordered. Among all these constant vectors $\boldsymbol{\delta}$, we choose the one that produces the smallest γ . Using this idea, we can derive the following result.

Theorem 4.3. *Assume that the marginal distributions of the components of \mathbf{X} and \mathbf{Y} are known and that $\mu_{X_i} \leq \mu_{Y_i}$ for all $i = 1, \dots, N$. Let $\delta_i := \inf\{x: F_i(x) + G_i(x) \geq 1\}$ and let*

$$\gamma_i := \frac{\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]}{\mathbb{E}[(Y_i - \delta_i)_+] + \mathbb{E}[(\delta_i - X_i)_+]}, \quad (4.4)$$

for $i = 1, \dots, N$. Then $\mathbf{X} \leq_\gamma \mathbf{Y}$.

Notice that δ_i is just the median of the mixture distribution $\frac{1}{2}F_i + \frac{1}{2}G_i$. Also note that if supports of Y_i and X_i don't overlap, then we are in the conditions of Theorem 2.3, and the corresponding γ_i equals zero.

Given that the proof of [Theorem 4.3](#) is based on [Theorem 3.5](#), which provides separable bounds for the utility functions in \mathcal{U}_γ , one may suspect that it is always true that checking whether $\mathbf{X} \leq_\gamma \mathbf{Y}$ holds is equivalent to separately checking whether $X_i \leq_{\gamma_i} Y_i$ for each $i \in \{1, \dots, N\}$. The following counterexample shows that this is not the case.

Example 4.4. Let $N = 2$ and $\gamma = (1/2, 1/2)$. Consider the binary random vectors $\mathbf{X}, \mathbf{X}', \mathbf{Y}, \mathbf{Y}'$ having the following distributions:

$$\begin{aligned} \mathbb{P}(\mathbf{X} = (0, 0)) = \mathbb{P}(\mathbf{X} = (5, 2)) &= \frac{1}{2}, & \mathbb{P}(\mathbf{X}' = (0, 2)) = \mathbb{P}(\mathbf{X}' = (5, 0)) &= \frac{1}{2}, \\ \mathbb{P}(\mathbf{Y} = (2, 0)) = \mathbb{P}(\mathbf{Y} = (4, 2)) &= \frac{1}{2}, & \mathbb{P}(\mathbf{Y}' = (2, 2)) = \mathbb{P}(\mathbf{Y}' = (4, 0)) &= \frac{1}{2}. \end{aligned}$$

Then \mathbf{X} and \mathbf{X}' have the same marginal distributions as well as \mathbf{Y} and \mathbf{Y}' . With the characterizations via transfers one can easily see that

$$\mathbf{X} \leq_\gamma \mathbf{Y} \quad \text{and} \quad \mathbf{X}' \leq_\gamma \mathbf{Y}',$$

but

$$\mathbf{X} \not\leq_\gamma \mathbf{Y}'.$$

For a proof of the last statement consider the following utility function u :

$$u(x_1, x_2) = x_1 + x_2 + \max\{x_1 + x_2 - 4, 0\}.$$

All partial derivatives of this function u are bounded between 1 and 2, so we have $u \in \mathcal{U}_\gamma$, but

$$\mathbb{E}[u(\mathbf{X})] = 5 > 4 = \mathbb{E}[u(\mathbf{Y}')].$$

This shows that the ordering \leq_γ in general depends not only on the marginal distributions, but on the whole joint distributions of the random vectors.

4.3 Marginal location-scale families

In this section we will show that, if the marginal distributions of the vectors that we want to compare, have nice properties, then the bounds in [Theorem 4.3](#) become easier to compute. In particular, if the marginal distributions are symmetric and belong to a location-scale family, such as normal or uniform, then we can derive easy explicit formulas for the sufficient bounds in [Theorem 4.3](#), as shown in [Proposition 4.5](#). A univariate distribution function F is said to belong to the symmetric location-scale H -family if

$$F(x) = H\left(\frac{x - \mu}{\sigma}\right), \quad \text{with } H(x) = 1 - H(-x).$$

In other words, H is the distribution function of a random variable Z as well as of $-Z$, and F is the distribution function of $\mu + \sigma Z$.

Proposition 4.5. *Let F_i and G_i belong to the same symmetric location-scale H -family and let*

$$\eta(t) := \frac{\mathbb{E}[(Z - t)_+]}{\mathbb{E}[(t - Z)_+]},$$

where Z has distribution function H . If

$$\tau_i = \frac{\mu_{Y_i} - \mu_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}},$$

then, in (4.4), we have $\gamma_i = \eta(\tau_i)$.

Remark 4.6. If $\mathbf{Y} = \mathbf{c}$, then $\tau_i = (\mu_{Y_i} - \mu_{X_i})/\sigma_{X_i}$ and the dominance conditions in Proposition 4.5 are necessary and sufficient.

In the next proposition we deal with sufficient conditions for marginal dominance of Y_i over X_i .

Proposition 4.7. *Let F_i and G_i belong to the same symmetric location-scale H -family and let*

$$\gamma_i^M := \eta\left(\frac{\mu_{Y_i} - \mu_{X_i}}{|\sigma_{Y_i} - \sigma_{X_i}|}\right). \quad (4.5)$$

Then $X_i \leq_{\gamma_i^M} Y_i$.

Remark 4.8. Propositions 4.5 and 4.7 can be extended to the case where the distribution H is not the same for all the marginals. If F_i and G_i belong to the same symmetric location-scale H_i -family, then

$$\eta_i(t) := \frac{\mathbb{E}[(Z_i - t)_+]}{\mathbb{E}[(t - Z_i)_+]},$$

where Z_i has distribution function H_i and, in Proposition 4.5, $\gamma_i = \eta_i(\tau_i)$. Proposition 4.7 ends similarly.

Remark 4.9. A particular case of distributions with marginals in a location-scale family is given by elliptical distributions, such as the multivariate normal, the multivariate t -distribution, etc. (see, e.g., Cambanis et al., 1981).

It is important to notice that γ_i^M in Eq. (4.5) is smaller than γ_i in Proposition 4.5. This larger γ_i is the price to pay to have sufficient conditions when the covariance matrices are possibly different. This is relevant, as standard dominance allows the comparison of multinormal random vectors only when they have the same covariance matrix. If one distribution is degenerate, then γ_i^M and γ_i are equal.

4.4 Bounds when only means and variances are known

We now consider the case where the marginal distributions of the random vectors \mathbf{X} and \mathbf{Y} are not completely specified, but only the means and variances are known. For univariate almost stochastic

dominance this problem has been considered in Müller et al. (2021). Using the separable bounds from Theorem 3.5 we can extend some of the results there to the multivariate case considered here.

Define

$$\zeta(t) := \frac{1}{1 + 2t(t + \sqrt{t^2 + 1})}. \quad (4.6)$$

Theorem 4.10. *Let the two random vectors \mathbf{X} and \mathbf{Y} have finite means and variances. Moreover, for all $i = 1, \dots, N$, let $\mu_{X_i} \leq \mu_{Y_i}$ and let*

$$\tau_i = \frac{\mu_{Y_i} - \mu_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}.$$

If $\gamma_i = \zeta(\tau_i)$, $i = 1, \dots, N$, then $\mathbf{X} \leq_{\gamma} \mathbf{Y}$.

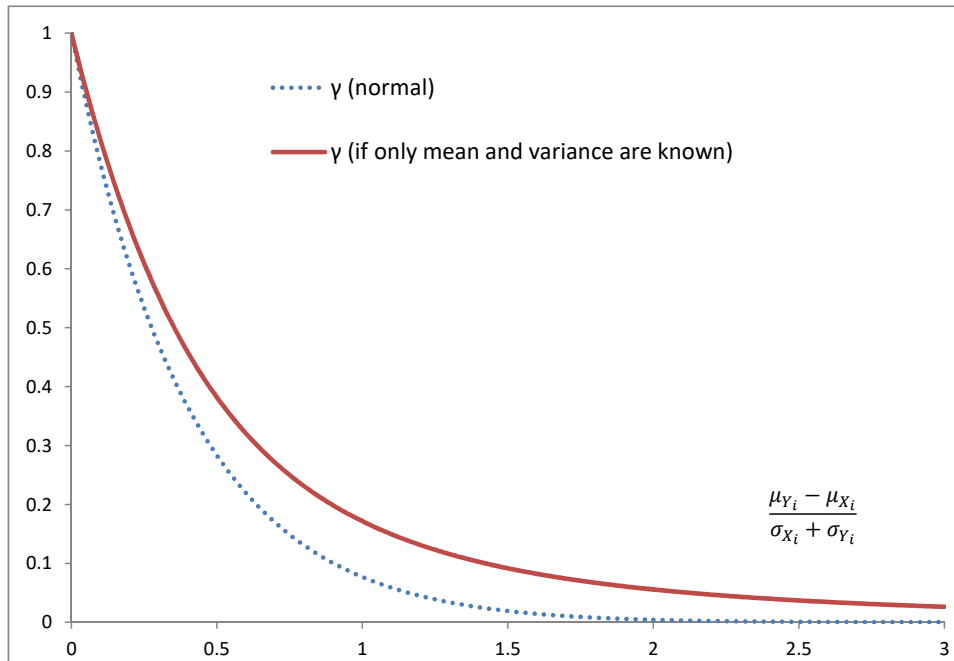


Figure 2: γ_i as a function of $(\mu_{Y_i} - \mu_{X_i})/(\sigma_{X_i} + \sigma_{Y_i})$.

As discussed in Müller et al. (2021), these bounds are not sharp. Fig. 2 shows the values of γ_i as functions of $(\mu_{Y_i} - \mu_{X_i})/(\sigma_{X_i} + \sigma_{Y_i})$ when the distributions of X_i and Y_i are normal (according to Proposition 4.5) and when only their means and variances are known (Theorem 4.10). Fig. 2 extends Fig. 3 in Müller et al. (2021), which deals with the univariate case when the dominated distribution is degenerate.

Proposition 4.11. Assume that the random vector \mathbf{X} has finite means and variances and that \mathbf{c} is a sure payoff. Define

$$t_i = \frac{\mu_{X_i} - c_i}{\sigma_{X_i}}. \quad (4.7)$$

- (a) Let $c_i \leq \mu_{X_i}$ for all $i = 1, \dots, N$. Then $\mathbf{c} \leq_\gamma \mathbf{X}$ if and only if $\gamma_i \geq \zeta(t_i)$, as defined in Eq. (4.6).
- (b) Let $\mu_{X_i} \leq c_i$ for all $i = 1, \dots, N$. Then $\mathbf{X} \leq_\gamma \mathbf{c}$ if $\gamma_i \geq \zeta(-t_i)$, as defined in Eq. (4.6).

Proof. The result is an immediate corollary of Theorem 4.10 and Proposition 4.2. \square

Remark 4.12. Notice that in Eq. (4.3) the right hand side is equal to the Omega ratio $\Omega_{X_i}(c_i)$, as defined in Shadwick and Keating (2002), whereas in Eq. (4.2) the right hand side is $1/\Omega_{X_i}(c_i)$. The right hand side of Eq. (4.7) can be interpreted as the Sharpe ratio. The connection between univariate ASD, the Omega ratio, and the Sharpe ratio is discussed in Müller et al. (2021).

Remark 4.13. There exist various necessary conditions for SD based on moments both in the univariate and the multivariate case (see, e.g., Fishburn, 1980, O'Brien, 1984, O'Brien and Scarsini, 1991). The perspective we take here is completely different, since we provide sufficient conditions.

Example 4.14. To continue Example 3.2, let \mathbf{X} denote the return from project A and \mathbf{Y} the return from project B, and recall that the utility functions of all board members belong to $\mathcal{U}_{(0.2,0.4)}$. According to Theorem 4.10, $\mathbf{X} \leq_{(0.2,0.4)} \mathbf{Y}$ if

$$\frac{\mu_{Y_1} - \mu_{X_1}}{\sigma_{X_1} + \sigma_{Y_1}} \geq 0.9 \quad \text{and} \quad \frac{\mu_{Y_2} - \mu_{X_2}}{\sigma_{X_2} + \sigma_{Y_2}} \geq 0.48.$$

In this case the choice of project B over A is unanimous knowing only the projects' means and variances.

If we know in addition that the marginal distributions are normal distributions, then we get better bounds from Proposition 4.5. In this case $\mathbf{X} \leq_{(0.2,0.4)} \mathbf{Y}$ if

$$\frac{\mu_{Y_1} - \mu_{X_1}}{\sigma_{X_1} + \sigma_{Y_1}} \geq 0.64 \quad \text{and} \quad \frac{\mu_{Y_2} - \mu_{X_2}}{\sigma_{X_2} + \sigma_{Y_2}} \geq 0.37.$$

4.5 Dependent background risks

In many situations, a decision about a risky project must be made in the presence of other important uncertainties. Pratt (1988, p. 395) makes this point very nicely: “Most real decision makers, unlike those portrayed in our popular texts and theories, confront several uncertainties simultaneously. They must make decisions about some risks when others have been committed to but not resolved. Even when a decision is to be made about only one risk, the presence of others in the background complicates matters.”

Many decisions involve some form of background risk that cannot be eliminated. Imagine a department in a company that is contemplating a choice between risks \mathbf{X} or \mathbf{Y} ; the background risk \mathbf{Z} would depend on projects undertaken by other departments. For other examples, see Tsetlin and Winkler

(2005) and papers cited there. In individual decision making, \mathbf{Z} might correspond to health and/or family situation. It is difficult to analyze situations where the random variables that are being compared and the background risk are not stochastically independent. For instance, assume that the marginal distribution of each of the random variables X, Y, Z is uniform on $[0, 1]$. Moreover let $Z = X = 1 - Y$ and let $u(x, z) = xz$. Then $v(X) := \mathbb{E}[XZ | X] = X^2$ and $\mathbb{E}[YZ | Y] = Y(1 - Y) \neq v(Y)$.

In many cases, the joint distribution of \mathbf{X}, \mathbf{Z} is impossible to estimate, and even the distribution of \mathbf{Z} might be hard to assess. Our bounds make it possible to handle such situations.

Theorem 4.15. *Consider \mathbf{X} and \mathbf{Y} as in Theorem 4.3, and let γ_i be given by Eq. (4.4). Let \mathbf{Z} be a K -dimensional multivariate background risk. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N, 0, \dots, 0) \in \mathbb{R}_+^{N+K}$. For any $u(\cdot, \cdot) \in \mathcal{U}_\gamma$ we have that $\mathbb{E}[u(\mathbf{X}, \mathbf{Z})] \leq \mathbb{E}[u(\mathbf{Y}, \mathbf{Z})]$.*

As an illustration, consider univariate X, Y and Z having joint normal distributions with $\mu_Y > \mu_X$. Then

$$\gamma_1^M = \eta\left(\frac{\mu_Y - \mu_X}{|\sigma_X - \sigma_Y|}\right)$$

is given in Proposition 4.7, and dominance with this γ_1^M holds for $X + Z$ and $Y + Z$ if X, Y and Z are independent. In particular, if $\sigma_X = \sigma_Y$, then $Y + Z$ first-order dominates $X + Z$, but this result fails under dependence.

In general, dominance for $X + Z$ and $Y + Z$ will hold with

$$\gamma = \eta\left(\frac{\mu_{Y_i} - \mu_{X_i}}{|\sigma_{X+Z} - \sigma_{Y+Z}|}\right),$$

which in turn depends on correlations between X, Z and between Y, Z . Note, however, that

$$|\sigma_{X+Z} - \sigma_{Y+Z}| \leq \sigma_X + \sigma_Y,$$

and therefore $Y + Z$ dominates $X + Z$ for any correlations with γ_1 given by Proposition 4.5. Thus, our sufficient bounds for dominance of Y over X are also sufficient for dominance of $Y + Z$ over $X + Z$ regardless of the dependence.

As mentioned earlier, our sufficient conditions are based on marginal distributions only, which makes them especially easy to implement. They are also useful in settings with background risk (Theorem 4.15), as we can establish dominance of (\mathbf{Y}, \mathbf{Z}) over (\mathbf{X}, \mathbf{Z}) by comparing only marginal distributions of \mathbf{X} and \mathbf{Y} .

5 (γ, β) -dominance

In the univariate case, when $\gamma = 1$, we have

$$X \leq_\gamma Y \iff \mathbb{E}[X] \leq \mathbb{E}[Y].$$

This means that, in this case, there exists a complete order on the set of random variables with finite expectation. In the multivariate case the situation is more complicated, due to the fact that \mathbb{R}^N is not completely ordered, so there is no natural way to order random vectors by their expectations. One possible way would be to consider weighted expectations $\mathbb{E}\left[\sum_{i=1}^N \beta_i X_i\right]$, as in portfolio analysis. In this section we will consider a version of multivariate almost stochastic dominance with a parameter $\gamma \in [0, 1]$ and a parameter vector β such that we get complete ordering based on weighted expectations $\mathbb{E}\left[\sum_{i=1}^N \beta_i X_i\right]$ for the case $\gamma = 1$ and classical first order stochastic dominance for the case $\gamma = 0$.

5.1 Defining (γ, β) -dominance

To achieve a complete order of random vectors, we consider a new class of utility functions defined in terms of two parameters: a scalar γ and a vector β . Then we define the corresponding SD relation, (γ, β) -multivariate almost stochastic dominance $((\gamma, \beta)$ -MASD).

Definition 5.1. For $\gamma \in [0, 1]$ and $\beta \in \mathbb{R}_+^N$, let $\mathcal{U}_{\gamma, \beta}$ be the class of utility functions $u: \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$0 < \gamma \beta_i \leq u'_i(\mathbf{x}) \leq \beta_i \quad \text{for all } i \in \{1, \dots, N\}. \quad (5.1)$$

The random vector \mathbf{X} is dominated by the random vector \mathbf{Y} in the sense of (γ, β) -MASD ($\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$) if

$$\mathbb{E}[u(\mathbf{X})] \leq \mathbb{E}[u(\mathbf{Y})], \quad \text{for all } u \in \mathcal{U}_{\gamma, \beta}.$$

Notice that, for any $\alpha > 0$, we have $\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$ iff $\mathbf{X} \leq_{\gamma, \alpha\beta} \mathbf{Y}$. This is coherent with the fact that two utility functions represent the same preferences if one is proportional to the other one.

If $\gamma = 1$, then we get a complete ordering by comparing $\mathbb{E}[\sum \beta_i X_i]$ and $\mathbb{E}[\sum \beta_i Y_i]$ and for $\gamma = 0$ we get $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$.

5.2 Characterization via (γ, β) -transfers

We now consider the class $\mathcal{U}_{\gamma, \beta}$ of utility functions of Definition 5.1 and define the corresponding transfers. We first discuss why it is basically sufficient to consider the case of $\beta_i = 1$ for all $i \in \{1, \dots, N\}$, which yields a more intuitive concept than the general case.

Notice that β_i is a scale factor that depends on the units that are used. Indeed, if $\tilde{u}: \mathbb{R}^N \rightarrow \mathbb{R}$ is a function such that

$$0 < \gamma \leq \tilde{u}'_i(\mathbf{x}) \leq 1 \quad \text{for all } i \in \{1, \dots, N\}, \quad (5.2)$$

then the function

$$u(x_1, \dots, x_N) := \tilde{u}(\beta_1 x_1, \dots, \beta_N x_N)$$

fulfills (5.1). Thus, by changing units we can assume without loss of generality that u is a function with the property (5.2), i.e., with the property that all marginal utilities are bounded between γ and 1.

A function u that satisfies (5.2) also satisfies

$$\gamma u'_i(\mathbf{x}) \leq u'_j(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \text{ and for all } i, j. \quad (5.3)$$

Vice versa, if a function satisfies (5.3), we can define

$$\beta := \sup_{i, \mathbf{x}} u'_i(\mathbf{x})$$

and then

$$\gamma\beta \leq u'_j(\mathbf{y}) \leq \beta \quad \text{for all } \mathbf{y} \text{ and for all } j;$$

thus u/β satisfies (5.2). Hence the functions satisfying (5.3) build the convex cone generated by the functions satisfying (5.2) and therefore define the same SD rule.

Similarly, the convex cone generated by the functions in $\mathcal{U}_{\gamma, \beta}$ is given by the functions satisfying

$$\gamma\beta_j u'_i(\mathbf{x}) \leq \beta_i u'_j(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \text{ and for all } i, j \in \{1, \dots, N\}.$$

In the following discussion of transfers we will first restrict our attention to the class $\mathcal{U}_{\gamma, 1}$, i.e., the functions that satisfy property (5.2). In contrast to the γ -transfer, we will now allow the decreasing transfer from \mathbf{x}_4 to \mathbf{x}_3 concerning component i to also be compensated by an increasing transfer from \mathbf{x}_1 to \mathbf{x}_2 concerning some other component j .

Definition 5.2. Consider two discrete cumulative distribution functions F and G with respective mass functions f and g . We say that G is obtained from F via a $(\gamma, \mathbf{1})$ -transfer (along dimensions i, j) if there exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, $\varepsilon_1, \varepsilon_2 > 0$ and $\eta_1, \eta_2 > 0$ such that, for some $i, j \in \{1, \dots, N\}$,

$$\mathbf{x}_2 = \mathbf{x}_1 + \varepsilon_1 \mathbf{e}_i, \quad \mathbf{x}_4 = \mathbf{x}_3 + \varepsilon_2 \mathbf{e}_j, \quad \eta_2 \varepsilon_2 = \gamma \eta_1 \varepsilon_1,$$

and

$$\begin{aligned} g(\mathbf{x}_1) &= f(\mathbf{x}_1) - \eta_1, \\ g(\mathbf{x}_2) &= f(\mathbf{x}_2) + \eta_1, \\ g(\mathbf{x}_3) &= f(\mathbf{x}_3) + \eta_2, \\ g(\mathbf{x}_4) &= f(\mathbf{x}_4) - \eta_2, \\ g(\mathbf{z}) &= f(\mathbf{z}) \quad \text{for all other values } \mathbf{z}. \end{aligned}$$

Fig. 3 shows an example of a $(\gamma, \mathbf{1})$ -transfer with $N = 2$, $\varepsilon_1 = 1.5$, $\varepsilon_2 = 1$, $\gamma = 2/3$, $\eta_1 = \eta_2 = \eta$.

With a proof similar to the proof of Theorem 3.4, we get the following result.

Theorem 5.3. *Let the random vectors \mathbf{X} and \mathbf{Y} assume a finite number of values. Then $\mathbf{X} \leq_{\gamma, \mathbf{1}} \mathbf{Y}$ if and only if the distribution of \mathbf{Y} can be obtained from the distribution of \mathbf{X} by a finite number of*

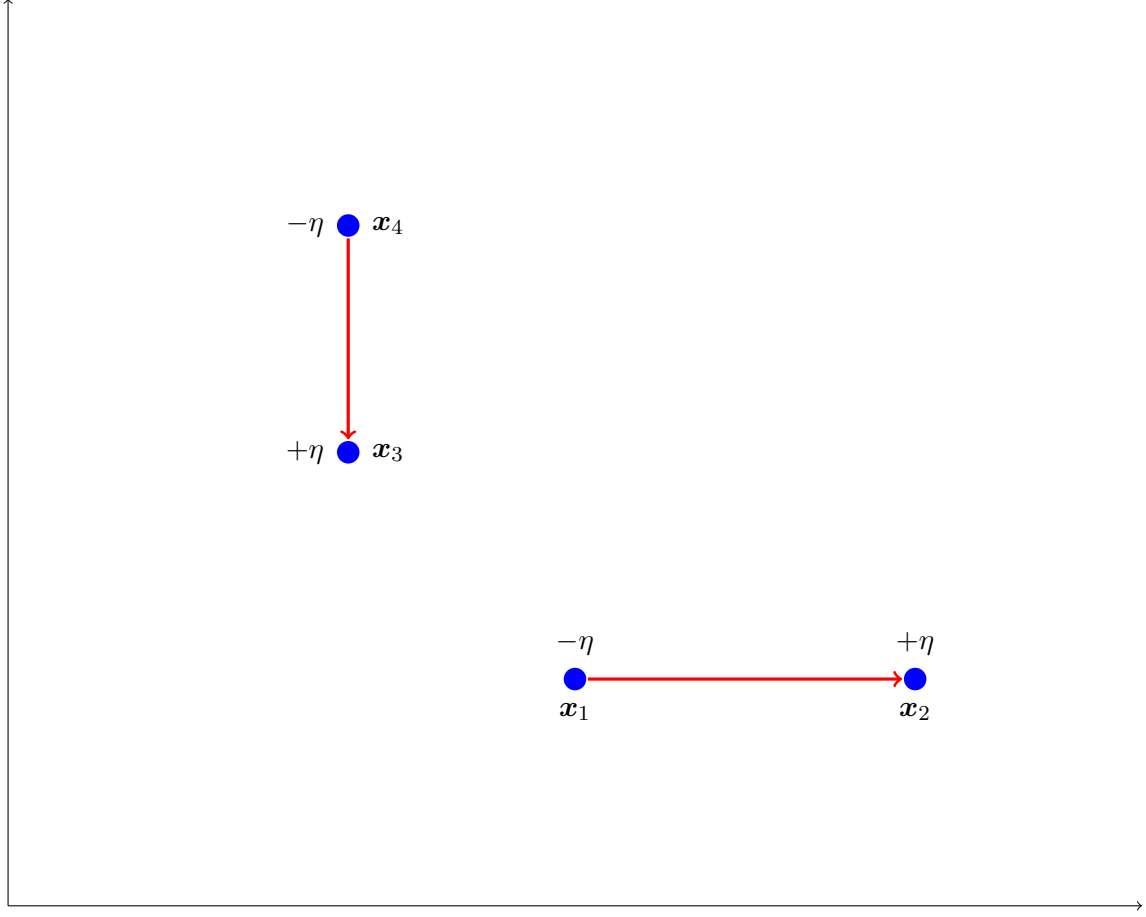


Figure 3: Example of $(\gamma, \mathbf{1})$ -transfer with $\varepsilon_1 = 1.5$, $\varepsilon_2 = 1$, $\gamma = 2/3$, $\eta_1 = \eta_2 = \eta$.

increasing transfers and $(\gamma, \mathbf{1})$ -transfers.

Notice that

$$\mathbb{E}[u(\mathbf{X})] \leq \mathbb{E}[u(\mathbf{Y})] \quad \text{for all } u \in \mathcal{U}_{(\gamma, \beta)}$$

is equivalent to

$$\mathbb{E}[\tilde{u}(\beta_1 X_1, \dots, \beta_N X_N)] \leq \mathbb{E}[\tilde{u}(\beta_1 Y_1, \dots, \beta_N Y_N)] \quad \text{for all } \tilde{u} \in \mathcal{U}_{(\gamma, \mathbf{1})}.$$

From this equivalence we get the general (γ, β) -transfers as follows.

Definition 5.4. Consider two discrete cumulative distribution functions F and G with respective mass functions f and g . We say that G is obtained from F via a (γ, β) -transfer if there are $i, j \in \{1, \dots, N\}$ and exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^N$, $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 > 0$ such that, for some $i, j \in \{1, \dots, N\}$,

$$\mathbf{x}_2 = \mathbf{x}_1 + \varepsilon_1 \mathbf{e}_i, \quad \mathbf{x}_4 = \mathbf{x}_3 + \varepsilon_2 \mathbf{e}_j, \quad \eta_2 \varepsilon_2 \beta_j = \gamma \eta_1 \varepsilon_1 \beta_i,$$

and

$$\begin{aligned}
g(\mathbf{x}_1) &= f(\mathbf{x}_1) - \eta_1, \\
g(\mathbf{x}_2) &= f(\mathbf{x}_2) + \eta_1, \\
g(\mathbf{x}_3) &= f(\mathbf{x}_3) + \eta_2, \\
g(\mathbf{x}_4) &= f(\mathbf{x}_4) - \eta_2, \\
g(\mathbf{z}) &= f(\mathbf{z}) \quad \text{for all other values } \mathbf{z}.
\end{aligned}$$

Theorem 5.5. *Let the random vectors \mathbf{X} and \mathbf{Y} assume a finite number of values. Then $\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$ if and only if the distribution of \mathbf{Y} can be obtained from the distribution of \mathbf{X} by a finite number of increasing transfers and (γ, β) -transfers.*

5.3 Sufficient conditions for (γ, β) -dominance

We can derive sufficient conditions for this version of MASD that are very similar to the conditions described in [Section 4](#).

Theorem 5.6. *Assume that the marginal distributions of the components of \mathbf{X} and \mathbf{Y} are known. Let $\delta_i := \inf\{x: F_i(x) + G_i(x) \geq 1\}$ and let*

$$\gamma := \frac{\sum_{i=1}^N \beta_i (\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+])}{\sum_{i=1}^N \beta_i (\mathbb{E}[(Y_i - \delta_i)_+] + \mathbb{E}[(\delta_i - X_i)_+])}.$$

If

$$\sum_{i=1}^N \beta_i \mu_{X_i} \leq \sum_{i=1}^N \beta_i \mu_{Y_i}, \tag{5.4}$$

then $\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$.

We next address the case where only means and variances are known.

Theorem 5.7. *Let the two random vectors \mathbf{X} and \mathbf{Y} have finite means and variances. Let*

$$\gamma = \frac{\sum_{i=1}^N \beta_i \left(\sqrt{(\sigma_{X_i} + \sigma_{Y_i})^2 + (\mu_{Y_i} - \mu_{X_i})^2} - (\mu_{Y_i} - \mu_{X_i}) \right)}{\sum_{i=1}^N \beta_i \left(\sqrt{(\sigma_{X_i} + \sigma_{Y_i})^2 + (\mu_{Y_i} - \mu_{X_i})^2} + (\mu_{Y_i} - \mu_{X_i}) \right)}.$$

If (5.4) holds, then $\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$.

Example 5.8. Returning to the setting of [Example 3.2](#), suppose the board has to decide whether project \mathbf{Y} (with $N = 2$, Y_1 being NPV and Y_2 being MS) is worth undertaking, thus comparing \mathbf{Y} to the status quo $\mathbf{0}$. It is hard to assess the joint distribution of \mathbf{Y} , but estimates of the means and variances are available:

$$\mu_{Y_1} = -5, \quad \sigma_{Y_1} = 4, \quad \mu_{Y_2} = 2, \quad \sigma_{Y_2} = 1.$$

In expectation, this risky project will decrease NPV by \$5 million and increase MS by 2%. To apply [Theorem 5.7](#) we choose the parameters β_1, β_2 as the sup of the partial derivatives of the utility function:

$$\frac{\beta_2}{\beta_1} = \frac{\sup(u'_2(\mathbf{x}))}{\sup(u'_1(\mathbf{x}))}.$$

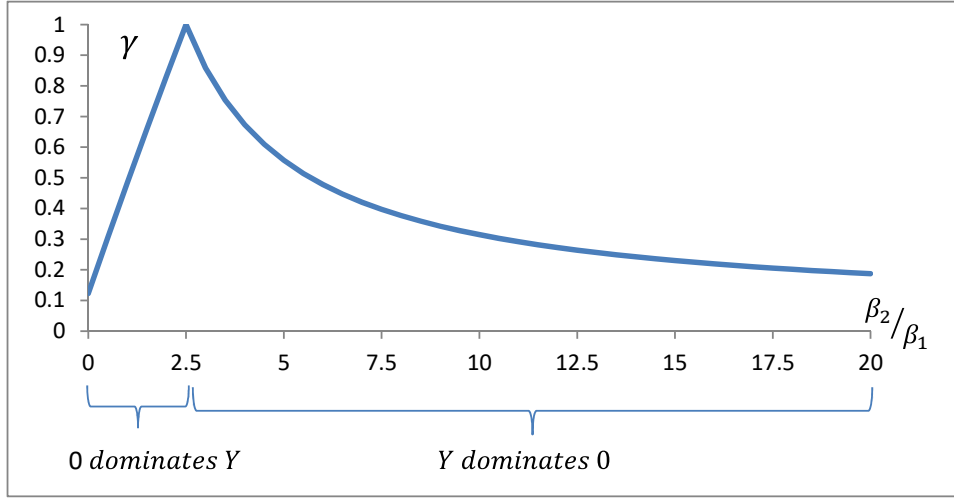


Figure 4: γ as a function of β_2/β_1 .

[Fig. 4](#) plots γ from [Theorem 5.7](#) as a function of β_2/β_1 . As we can see, for $\beta_2/\beta_1 < 2.5$, it is better not to undertake the project. After discussing the issue in a board meeting, all members of the board agree that $\beta_2/\beta_1 \geq 18$. From [Fig. 4](#), if $\beta_2/\beta_1 = 18$, then $\gamma = 0.2$, which implies that \mathbf{Y} dominates $\mathbf{0}$ with respect to the utility class $\mathcal{U}_{0.2, \beta}$ with $\beta_2/\beta_1 \geq 18$. If, as in [Example 3.2](#), all board members agree that $\mathcal{U}_{(0.2, 0.4)}$ includes their preferences, then the support for the project is unanimous, and further information on the joint distribution of the components of \mathbf{Y} is not needed.

Example 5.9. Returning to the data analytics startup from the scenario (b) in the Introduction, the current approach is to maximize $\mathbb{E}\left[\sum_{i=1}^N \beta_i X_i\right]$, as mentioned at the beginning of [Section 5](#). At the same time, some bounds on variances might be available ([Arlotto et al., 2014](#)). To take into account the riskiness of different solutions, the startup team can apply [Theorem 5.7](#). For any two contending solutions that will give γ ; by varying γ , they can identify dominated and dominating sets of the available alternatives, and see how these sets change as β varies. Then these results can be discussed with a

client.

6 A case study on investments in photovoltaic power systems

A company wants to compare the efficiency of a photovoltaic solar power system in two different locations. The productivity of a photovoltaic (PV) solar power system depends on solar irradiance, which varies through the day and depends on latitude and climate. We want to compare the two possible locations of Rome in Italy and Siegen in Germany. Data for solar irradiance are publicly available for all locations in Europe from the Copernicus Atmosphere Monitoring Service CAMS (2019) <http://www.soda-pro.com/web-services/radiation/cams-radiation-service>. From this source we downloaded the hourly data for the so-called global horizontal irradiation (GHI) for the year 2020. For each location we get a sample of 365 vectors of daily GHI data. These are displayed for the two cities of Rome and Siegen in Fig. 5 together with the hourly means.

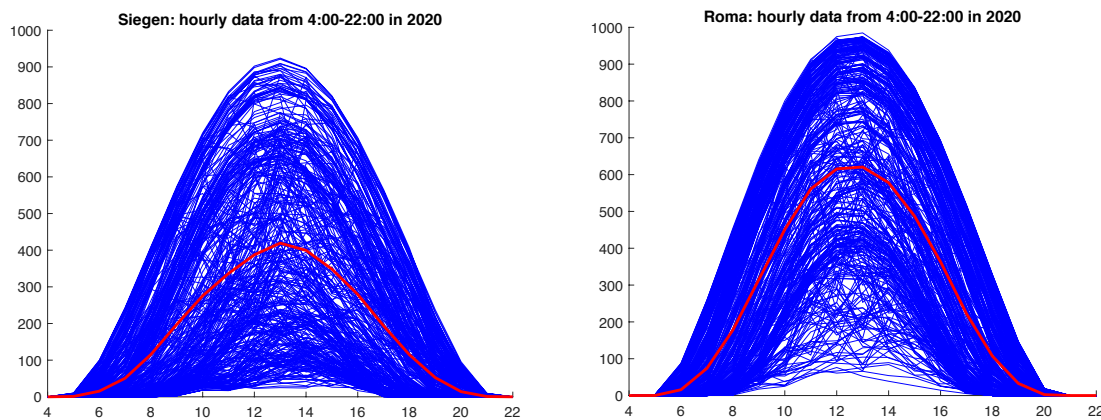


Figure 5: Global horizontal irradiation in Siegen and Rome.

Positive values are only possible between 5 a.m. and 10 p.m. so that only 17 hours of the day are relevant. Therefore we can describe the possible productivity of the PV systems by a random vector $\mathbf{X} = (X_1, \dots, X_{17})$ for Siegen and by a similar random vector \mathbf{Y} for Rome, whose distributions we estimate by the empirical distributions of the data. It is not surprising that the values for Rome are typically larger than the ones for Siegen as Rome is more than 1000 km south of Siegen and less rainy. However, we do not have multivariate FSD between the two distributions as not even all hourly means are larger. This can easily be explained by the fact that in the summer days are longer in the north and therefore very early in the morning and late in the evening Siegen has a higher (though small) solar irradiance on average, whereas in the rest of the day Rome has much higher irradiance, as shown in Fig. 6. It is quite clear, however, that an investment in Rome should be more profitable as there is some kind of almost FSD.

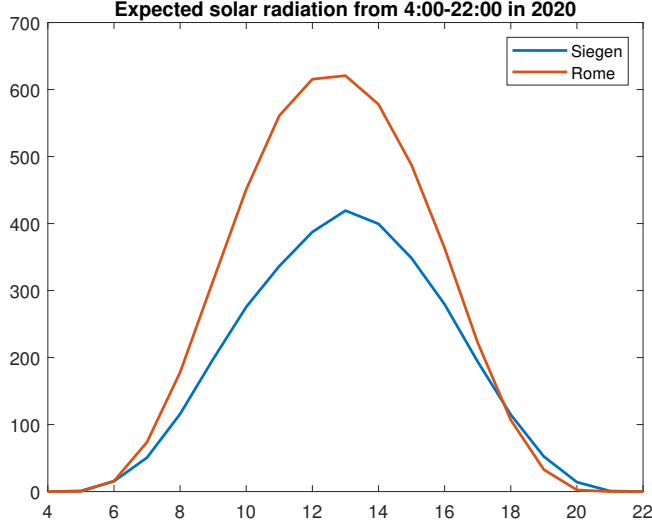


Figure 6: Expected hourly GHI for Siegen and Rome.

We will show now that it is reasonable to assume that the expected reward of the decision maker can be written in form of an expected utility $\mathbb{E}[u(\mathbf{X})]$ that fulfills the assumptions of [Definition 5.1](#) for appropriate parameters and that we can show with the results of [Section 5](#) that $\mathbf{X} \leq_{\gamma, \beta} \mathbf{Y}$ holds for appropriate parameters. For this illustrative example we make a few simplifying assumptions as the real world problem is very complex. We assume that the output of the PV system is exactly proportional to the GHI. The decision maker is assumed to be a so called prosumer, who at the same time is a producer as well as a consumer of electricity. For simplicity we assume that the consumption can be described by a random vector \mathbf{Z} that is independent of the production of the PV system. Notice, however, that with the methods described in this paper, we can also handle situations with dependent background risk; see [Theorem 4.15](#).

Unfortunately, the multivariate distribution of the consumption vector \mathbf{Z} of a company typically also has a multivariate distribution that is difficult to assess, see, e.g., [Berk et al. \(2018\)](#) for an attempt to describe electricity demand patterns of companies by a stochastic model. It is reasonable to assume that there always is the possibility of buying electricity for a price of β per unit and selling electricity for a lower price of $\gamma\beta$ with $0 < \gamma < 1$. In practice, these prices may also vary with the hour of the day and therefore there would be a price vector $\boldsymbol{\beta}$, but we assume here for simplicity that β is constant. Prosumers have the incentive to consume the produced electricity themselves as far as possible to avoid the higher cost of β per unit for buying electricity. Electricity could be sold for the lower price $\gamma\beta$ if production exceeds consumption. If this simple strategy is applied, then, for a given output vector \mathbf{x} of the PV system and a consumption vector \mathbf{z} , the payoff is

$$v(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{17} \beta \min\{x_i, z_i\} + \gamma\beta(x_i - z_i)_+. \quad (6.1)$$

For a random consumption vector \mathbf{Z} , the expected reward given output vector \mathbf{x} is

$$u(\mathbf{x}) := \mathbb{E}[v(\mathbf{x}, \mathbf{Z})]. \quad (6.2)$$

In the case of this simple separable utility function the strong positive dependence between the production in different hours is irrelevant. However, strategic behavior of the prosumer may lead to a higher payoff. For instance, battery storage could be employed to store the produced electricity, so that the prosumer could adopt a policy π that allows electricity to be used later instead of being sold for a cheap price. Therefore the real value that one gets as expected payoff is much more complicated and not separable any more. Thus the dependence structure of the multivariate distribution of \mathbf{X} will also be relevant. When we have a random electricity consumption and in addition the possibility of using battery storage, we will not be able to give a simple explicit expression for the value of the expected payoff of an operating policy π . However, it is still true that the marginal utilities are bounded by $\gamma\beta$ and β , so that we have $u_\pi \in \mathcal{U}_{\gamma,\beta}$. Therefore the decision maker with operating policy π will prefer the investment with a production vector \mathbf{Y} to the one with production vector \mathbf{X} if $\mathbf{X} \leq_{\gamma,\beta} \mathbf{Y}$. If the prosumer solves an optimization problem to find an optimal operating policy among all possible operating policies, the expected value of the PV system will have the form $V(\mathbf{X}) = \sup_\pi \mathbb{E}[u_\pi(\mathbf{X})]$, which may not have the form of an expected utility anymore as the optimal policy may depend on the random vector \mathbf{X} . Nevertheless, it is still true that $\mathbf{X} \leq_{\gamma,\beta} \mathbf{Y}$ implies $V(\mathbf{X}) \leq V(\mathbf{Y})$ as the ordering property is preserved by taking a supremum.

It is also very difficult to determine the complicated dependence structure of the random vector of GHI data; see, e.g., Müller and Reuber (2022) for an attempt to describe the whole multivariate distribution of this time series by a stochastic model using time dependent beta distributions and copula models. Therefore the results of Section 5 are useful to obtain bounds for the parameter γ that ensures $\mathbf{X} \leq_{\gamma,\beta} \mathbf{Y}$. Approximating the marginal distributions with their empirical counterparts, and ignoring the dependence structure, from Theorem 5.6 we can derive the value $\gamma = 0.525$. Using only means and variances of the marginals, from Theorem 5.7 we get the value $\gamma = 0.576$. The difference between the two values is comparable to the difference for the values of γ_i that one obtains for normally distributed marginals in Section 4, as described in Fig. 6.

In the context of this example the transfers corresponding to the stochastic dominance rule also have a simple and intuitive interpretation. If the marginal utility of the produced electricity is bounded by the buying price β and the selling price $\gamma\beta$, then we prefer a scenario where we produce more in hour i and less in hour j as long as the lower production in hour j is bounded by a fraction γ of the higher production in hour i .

Table 1 shows the means, standard deviations, and the corresponding univariate γ for the different time slots. The numbers in boldface refer to the times when Siegen dominates Rome. One can see that the value of $\gamma = 0.576$ obtained for the multivariate version of the bounds discussed in Section 5 is about the same size as the bounds that one gets when considering the univariate problem for a single hour in the most relevant hours in the middle of the day.

Table 1: Expected values, standard deviations and γ 's for different time slots

time	Expected value		Standard deviation		γ
	Siegen	Roma	Siegen	Roma	
04:00	0.000	0.000	0.000	0.000	-
05:00	1.003	0.199	2.393	0.542	0.582
06:00	15.401	15.744	26.398	25.754	0.987
07:00	50.707	73.894	70.451	87.724	0.747
08:00	115.811	177.978	128.355	157.311	0.649
09:00	197.805	314.928	185.353	204.742	0.553
10:00	275.557	450.991	224.584	232.883	0.473
11:00	336.451	561.013	249.567	248.803	0.418
12:00	387.412	615.468	262.666	260.380	0.429
13:00	419.324	620.697	266.839	263.684	0.476
14:00	399.560	577.937	257.419	252.466	0.504
15:00	348.290	487.155	235.694	234.401	0.558
16:00	279.321	363.207	209.142	212.732	0.674
17:00	193.808	222.166	171.812	176.166	0.850
18:00	114.787	106.833	122.830	113.958	0.935
19:00	52.316	32.758	68.294	47.429	0.714
20:00	14.082	2.432	24.152	5.123	0.460
21:00	0.821	0.000	2.020	0.000	0.453
22:00	0.000	0.000	0.000	0.000	-

7 Conclusions

SD is a useful concept, especially in a multivariate context, where assessing multiattribute utility is challenging and different stakeholders might have divergent views. However, applying multivariate SD is difficult for two reasons: First, often distributions cannot be ranked (e.g., by FSD); this can be overcome by using γ -MASD. Second, integral conditions for multivariate SD do not exist; to overcome this challenge, we develop sufficient conditions for γ -MASD that are based on marginal distributions of the compared alternatives or just on their means and variances. This makes our conditions very practical, as full assessments of joint multivariate distributions are usually difficult. In the framework of portfolio analysis, Arvanitis et al. (2021) study stochastic bounding of a portfolio by another, i.e., they look at conditions under which a set of portfolios contains one portfolio that stochastically dominates all portfolios in another set. When these conditions are not satisfied, they look for approximate bounds, in the spirit of ASD.

Another distinction of the multivariate case, compared to the univariate case, is that a real coordinate space is not completely ordered. To attain a path to a complete order, we need to constrain maximal marginal utilities for different attributes. Section 5 presents the corresponding definition of (γ, β) -MASD, its characterization via transfers, and sufficient conditions for comparing two risky alternatives.

Within the expected utility framework, γ -MASD and (γ, β) -MASD translate into bounds on marginal utilities (Definitions 3.1 and 5.1). Alternatively, these preferences can be characterized via transfers (Definitions 3.3 and 5.4 and Theorems 3.4 and 5.5). Such transfers might be easier to explain to decision makers and use for elicitation of γ and β .

Examples 3.2, 4.14 and 5.8 illustrate our approach in a classical decision-making setting, where one needs to choose between two alternatives. We also discuss broader potential applications (Example 5.9) for screening of the most promising solutions in (potentially large-scale) optimization problems such as risk-averse revenue management (Gönsch, 2017). There is always a tension between a careful comparison and evaluation of available alternatives and a search for new solutions. With multiple attributes, the former is difficult and laborious. Our results provide tools for “fast and frugal” screening and evaluation, while properly accounting for tradeoffs and riskiness. As the world moves toward decisions with multiple objectives (e.g., many environmental, social and governance (ESG) criteria in addition to the financial performance of a company), such tools, consistent with normative decision analysis, should become even more in demand.

A Note on general distribution functions

The following theorem shows how the previous results on transfers can be adapted to the case of random variables that are not finite.

Theorem A.1. *Let $B \subset \mathbb{R}^N$ be bounded and let \mathcal{U} be a class of continuous increasing functions $u : B \rightarrow \mathbb{R}$. Let the random vectors \mathbf{X}, \mathbf{Y} take values in B . Then $\mathbf{X} \leq_{\mathcal{U}} \mathbf{Y}$ if and only if there exist*

two sequences $(\mathbf{X}_n)_{n \in \mathbb{N}}$ and $(\mathbf{Y}_n)_{n \in \mathbb{N}}$ such that

$$\mathbf{X}_n \rightarrow \mathbf{X} \quad \text{a.s.}, \quad \mathbf{Y}_n \rightarrow \mathbf{Y} \quad \text{a.s.}, \quad \text{and} \quad \mathbf{X}_n \leq_{\mathcal{U}} \mathbf{Y}_n \quad \text{for all } n \in \mathbb{N}.$$

Proof. For bounded univariate random variables X, Y we can construct sequences $(X_n), (Y_n)$ with $X_n \leq X$, $Y_n \geq Y$, and $X_n \rightarrow X, Y_n \rightarrow Y$ a.s.. A concrete construction is given in the proof of Theorem 2.8 in Müller et al. (2017). We can apply this procedure componentwise to bounded random vectors \mathbf{X} and \mathbf{Y} . Thus we get sequences such that, almost surely, $\mathbf{X}_n \leq \mathbf{X}$, $\mathbf{Y}_n \geq \mathbf{Y}$, $\mathbf{X}_n \rightarrow \mathbf{X}$, and $\mathbf{Y}_n \rightarrow \mathbf{Y}$. Since $\mathbf{X} \leq \mathbf{Y}$ a.s. implies $\mathbf{X} \leq_{\mathcal{U}} \mathbf{Y}$ we thus get sequences with

$$\mathbf{X}_n \leq_{\mathcal{U}} \mathbf{X} \leq_{\mathcal{U}} \mathbf{Y} \leq_{\mathcal{U}} \mathbf{Y}_n.$$

This shows the only-if-part. The if-part follows from the fact that any SD relation $\leq_{\mathcal{U}}$ is closed under convergence in distribution if \mathcal{U} consists only of bounded continuous functions, see, e.g., Müller (1997, Theorem 4.2). \square

B Proofs

Proofs of Section 3

The proof of Theorem 3.4 requires the following lemma.

Lemma B.1. *Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be continuously differentiable. Then $u \in \mathcal{U}_{\gamma}$ if and only if*

$$\eta_2(u(\mathbf{x}_4) - u(\mathbf{x}_3)) \leq \eta_1(u(\mathbf{x}_2) - u(\mathbf{x}_1)) \tag{B.1}$$

for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ satisfying (3.3) for some i and γ_i .

Proof. If part: Assume that u fulfills (B.1) for some i and γ_i . Then

$$\eta_2(\mathbf{x}_4 - \mathbf{x}_3) = \gamma_i \eta_1(\mathbf{x}_2 - \mathbf{x}_1) \implies \mathbf{x}_3 = \mathbf{x}_4 - \gamma_i \eta_1 \mathbf{e}_i$$

and so (B.1) implies

$$\gamma_i \frac{\partial}{\partial x_i} u(\mathbf{x}_4) = \gamma_i \lim_{\eta_1 \rightarrow 0} \frac{u(\mathbf{x}_4) - u(\mathbf{x}_3)}{\gamma_i \eta_1} \leq \lim_{\eta_2 \rightarrow 0} \frac{u(\mathbf{x}_2) - u(\mathbf{x}_1)}{\eta_2} = \frac{\partial}{\partial x_i} u(\mathbf{x}_1).$$

As this holds for arbitrary $\mathbf{x}_1, \mathbf{x}_4$ and the derivatives are assumed to be continuous, by (3.1), we get $u \in \mathcal{U}_{\gamma}$.

Only if part: Now assume that $u \in \mathcal{U}_{\gamma}$ is continuously differentiable. Let $\mathbf{h} := \mathbf{x}_2 - \mathbf{x}_1$. For $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ satisfying (3.3) for some i and γ_i , from $\eta_2(\mathbf{x}_4 - \mathbf{x}_3) = \gamma_i \eta_1(\mathbf{x}_2 - \mathbf{x}_1)$, we get that

$$\mathbf{x}_4 - \mathbf{x}_3 = \frac{\gamma_i \eta_1}{\eta_2} (\mathbf{x}_2 - \mathbf{x}_1).$$

Thus, from (B.1) we can deduce

$$\begin{aligned}
\eta_1(u(\mathbf{x}_2) - u(\mathbf{x}_1)) &= \int_0^1 \frac{\partial}{\partial x_i} u(\mathbf{x}_1 + t\mathbf{h}) dt \\
&\geq \eta_1 \gamma_i \int_0^1 \frac{\partial}{\partial x_i} u\left(\mathbf{x}_3 + t \frac{\gamma_i \eta_1}{\eta_2} \mathbf{h}\right) dt \\
&= \eta_2 \frac{\gamma_i \eta_1}{\eta_2} \int_0^1 \frac{\partial}{\partial x_i} u\left(\mathbf{x}_3 + t \frac{\gamma_i \eta_1}{\eta_2} \mathbf{h}\right) dt \\
&= \eta_2(u(\mathbf{x}_4) - u(\mathbf{x}_3)). \quad \square
\end{aligned}$$

Proof of Theorem 3.4. The proof is based on the duality theory for transfers that as described in Müller (2013) Lemma B.1 shows that \mathcal{U}_γ can be described by a set of inequalities. Therefore it is induced by the corresponding set of transfers as described in Müller (2013, Definition 2.2.1). The proof thus follows from Müller (2013, Theorem 2.4.1). \square

Proof of Theorem 3.5. Note that $u'_i(\mathbf{x}) \leq \sup(u'_i(\mathbf{x})) = b_i$ and that by inequality (3.2) we have $u'_i(\mathbf{x}) \geq \gamma_i b_i$. By a multivariate first-order Taylor expansion, $u(\mathbf{x}) - u(\mathbf{z}) = \sum_{i=1}^N u'_i(\mathbf{y})(x_i - z_i)$, where y_i is between x_i and z_i . Then, using $u'_i(\mathbf{y}) \leq b_i$ if $x_i > z_i$ and $u'_i(\mathbf{y}) \geq \gamma_i b_i$ if $x_i < z_i$ provides an upper bound, while using $u'_i(\mathbf{y}) \geq \gamma_i b_i$ if $x_i > z_i$ and $u'_i(\mathbf{y}) \leq b_i$ if $x_i < z_i$ provides a lower bound. \square

Proofs of Section 4

Proof of Proposition 4.1. We will prove (b). The proof for (a) is similar. Let $u \in \mathcal{U}_\gamma$ and let

$$b_i := \sup_{\mathbf{x} \in \mathbb{R}^N} u'_i(\mathbf{x}). \quad (\text{B.2})$$

Without any loss of generality, assume $u(\mathbf{c}) = 0$. By Theorem 3.5 we have

$$u(\mathbf{x}) \leq \sum_{i=1}^N b_i v_U(x_i - c_i; \gamma_i), \quad (\text{B.3})$$

where $v_U(x_i - c_i; \gamma_i) = -\gamma_i(c_i - x_i)_+ + (x_i - c_i)_+$. This implies

$$\mathbb{E}[u(\mathbf{X})] \leq \sum_{i=1}^N b_i (-\gamma_i \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+]) \quad (\text{B.4})$$

Therefore $\mathbb{E}[u(\mathbf{X})] \leq 0$ if $-\gamma_i \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ for all $i = 1, \dots, N$.

Notice that $-\gamma_i \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ is equivalent to $X_i \leq_{\gamma_i} c_i$. This proves the if part.

Now we prove the only if part. Consider a sequence of utility functions

$$u_n(\mathbf{x}) = \sum_{i=1}^N b_{i,n} v_U(x_i - c_i; \gamma_i)_+ \in \mathcal{U}_\gamma \quad (\text{B.5})$$

such that $\lim_{n \rightarrow \infty} b_{j,n} = 0$ for $j \neq i$ and $b_{i,n} \equiv 1$ for all n .

If $\mathbf{X} \leq_{\gamma} \mathbf{c}$, then $\mathbb{E}[u_n(\mathbf{X})] \leq u_n(\mathbf{c}) = 0$. This implies $-\gamma_i \mathbb{E}[(c_i - X_i)_+] + \mathbb{E}[(X_i - c_i)_+] \leq 0$ for all $i = 1, \dots, N$, i.e., $X_i \leq_{\gamma_i} c_i$, for all $i = 1, \dots, N$. \square

Proof of Theorem 4.3. Given $u \in \mathcal{U}_{\gamma}$, let $b_i = \sup(u'_i(\mathbf{x}))$, and without loss of generality, assume $u(\boldsymbol{\delta}) = 0$. By Theorem 3.5 we have

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \leq u(\mathbf{x}) \leq \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i).$$

First, we show that, for $i = 1, \dots, N$, for any δ_i we have

$$\mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] = \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)]$$

for γ_i defined as in Eq. (4.4). This follows from

$$\begin{aligned} \mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] &= -\mathbb{E}[(\delta_i - Y_i)_+] + \gamma_i \mathbb{E}[(Y_i - \delta_i)_+], \\ \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)] &= -\gamma_i \mathbb{E}[(\delta_i - X_i)_+] + \mathbb{E}[(X_i - \delta_i)_+], \end{aligned}$$

and the definition of γ_i .

Therefore, from inequality (3.4) it follows that

$$\mathbb{E}[u(\mathbf{Y})] \geq \sum_{i=1}^N b_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] = \sum_{i=1}^N b_i \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)] \geq \mathbb{E}[u(\mathbf{X})]$$

holds for arbitrary δ_i . We want to choose δ_i such that γ_i is as small as possible. As

$$\gamma_i = \frac{\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]}{\mathbb{E}[(Y_i - \delta_i)_+] + \mathbb{E}[(\delta_i - X_i)_+]} = \frac{\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]}{\mu_{Y_i} - \delta_i + \mathbb{E}[(\delta_i - Y_i)_+] + \delta_i - \mu_{X_i} + \mathbb{E}[(X_i - \delta_i)_+]},$$

we have to minimize $\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]$ with respect to δ_i . The right derivative is

$$\frac{\partial^+}{\partial \delta_i} (\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]) = \mathbb{E}[\mathbf{1}_{[\delta_i - Y_i \geq 0]}] - \mathbb{E}[\mathbf{1}_{[X_i - \delta_i \geq 0]}] = G_i(\delta_i) - 1 + F_i(\delta_i).$$

Therefore, δ_i is minimized for $\delta_i = \inf\{x: F_i(x) + G_i(x) \geq 1\}$. \square

Proof of Proposition 4.5. In this case we can solve for δ_i from Theorem 4.3:

$$\begin{aligned}
F_i(\delta_i) + G_i(\delta_i) = 1 &\iff H\left(\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}}\right) + H\left(\frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}}\right) = 1 \\
&\iff H\left(\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}}\right) = H\left(\frac{\mu_{Y_i} - \delta_i}{\sigma_{Y_i}}\right) \\
&\iff \frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}} = \frac{\mu_{Y_i} - \delta_i}{\sigma_{Y_i}} \\
&\iff \delta_i = \frac{\mu_{X_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}.
\end{aligned}$$

Hence

$$\gamma_i = \frac{\mathbb{E}[(Y_i - \delta_i)_+] + \mathbb{E}[(\delta_i - X_i)_+]}{\mathbb{E}[(\delta_i - Y_i)_+] + \mathbb{E}[(X_i - \delta_i)_+]} = \frac{\sigma_{Y_i} \mathbb{E}[(Z - \tau_i)_+] + \sigma_{X_i} \mathbb{E}[(Z - \tau_i)_+]}{\sigma_{Y_i} \mathbb{E}[(\tau_i - Z)_+] + \sigma_{X_i} \mathbb{E}[(\tau_i - Z)_+]} = \eta(\tau_i). \quad \square$$

The proof of Proposition 4.7 is along the lines of Müller et al. (2017, example 2.11).

Proof of Proposition 4.7. The following condition for γ_i^M -dominance in location-scale models can be found in Müller et al. (2017, bottom of page 2940):

$$\gamma_i^M = \frac{\int_{-\infty}^{\infty} (G_i(x) - F_i(x))_+ dx}{\int_{-\infty}^{\infty} (F_i(x) - G_i(x))_+ dx} = \frac{\int_{-\infty}^{\infty} \left(H\left(\frac{x - \mu_{Y_i}}{\sigma_{Y_i}}\right) - H\left(\frac{x - \mu_{X_i}}{\sigma_{X_i}}\right) \right)_+ dx}{\int_{-\infty}^{\infty} \left(H\left(\frac{x - \mu_{X_i}}{\sigma_{X_i}}\right) - H\left(\frac{x - \mu_{Y_i}}{\sigma_{Y_i}}\right) \right)_+ dx}. \quad (\text{B.6})$$

The two distribution functions F_i and G_i single-cross at a point δ_i such that

$$\frac{\delta_i - \mu_{X_i}}{\sigma_{X_i}} = \frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}}, \quad (\text{B.7})$$

which implies

$$\delta_i = \frac{\mu_{Y_i}\sigma_{X_i} - \mu_{X_i}\sigma_{Y_i}}{\sigma_{X_i} - \sigma_{Y_i}}. \quad (\text{B.8})$$

Notice that, for $x < \delta_i$, the distribution with a larger variance takes larger values than the other one. Moreover, integrating by parts, we get the well-known equalities:

$$\int_{-\infty}^{\delta_i} F_i(x) dx = \mathbb{E}[(\delta_i - X_i)_+], \quad \int_{\delta_i}^{\infty} F_i(x) dx = \mathbb{E}[(X_i - \delta_i)_+]. \quad (\text{B.9})$$

Therefore, when $\sigma_{Y_i} > \sigma_{X_i}$, Eq. (B.6) becomes

$$\gamma_i^M = \frac{\int_{-\infty}^{\delta_i} \left(H\left(\frac{x - \mu_{Y_i}}{\sigma_{Y_i}}\right) - H\left(\frac{x - \mu_{X_i}}{\sigma_{X_i}}\right) \right) dx}{\int_{\delta_i}^{\infty} \left(H\left(\frac{x - \mu_{X_i}}{\sigma_{X_i}}\right) - H\left(\frac{x - \mu_{Y_i}}{\sigma_{Y_i}}\right) \right) dx} = \frac{\mathbb{E}[(\delta_i - Y_i)_+] - \mathbb{E}[(\delta_i - X_i)_+]}{\mathbb{E}[(Y_i - \delta_i)_+] - \mathbb{E}[(X_i - \delta_i)_+]}. \quad (\text{B.10})$$

Since

$$\mathbb{E}[(\delta_i - Y_i)_+] = \mathbb{E}[(\delta_i - \mu_{Y_i} - \sigma_{Y_i}Z)_+] = \sigma_{Y_i} \mathbb{E}\left[\left(\frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}} - Z\right)_+\right], \quad (\text{B.11})$$

we have

$$\begin{aligned} \frac{\delta_i - \mu_{Y_i}}{\sigma_{Y_i}} &= \frac{1}{\sigma_{Y_i}} \left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{X_i}}{\sigma_{Y_i} - \sigma_{X_i}} - \mu_{Y_i} \right) \\ &= \frac{1}{\sigma_{Y_i}} \left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{X_i} - \mu_{Y_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{Y_i} - \sigma_{X_i}} \right) \\ &= \frac{1}{\sigma_{Y_i}} \left(\frac{\mu_{X_i}\sigma_{Y_i} - \mu_{Y_i}\sigma_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}} \right) \\ &= \frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}}. \end{aligned} \quad (\text{B.12})$$

This implies that

$$\mathbb{E}[(\delta_i - Y_i)_+] = \sigma_{Y_i} \mathbb{E}\left[\left(\frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}} - Z\right)_+\right]. \quad (\text{B.13})$$

Applying a similar argument to the other components in Eq. (B.10), we obtain

$$\gamma_i^M = \frac{\mathbb{E}\left[\left(\frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}} - Z\right)_+\right]}{\mathbb{E}\left[Z - \left(\frac{\mu_{X_i} - \mu_{Y_i}}{\sigma_{Y_i} - \sigma_{X_i}}\right)_+\right]}. \quad (\text{B.14})$$

A similar derivation holds for $\sigma_{Y_i} > \sigma_{X_i}$. □

Proof of Theorem 4.10. The proof uses similar ideas as the proof of Theorem 3 in Müller et al. (2021). Fix arbitrary $\boldsymbol{\delta}$, consider $u \in \mathcal{U}_\gamma$, and let $b_i = \sup(u'_i(\mathbf{x}))$. Without loss of generality assume $u(\boldsymbol{\delta}) = 0$. By Theorem 3.5,

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \leq u(\mathbf{x}) \leq \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i).$$

We need to show that, for some appropriate δ_i and γ_i , $\mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] \geq \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)]$ for $i = 1, \dots, N$. With the same tedious but straightforward calculation as in the proof of Theorem 3 in Müller et al. (2021), we can establish that the smallest possible choice for γ_i is obtained by choosing

$$\delta_i = \frac{\mu_{X_i}\sigma_{Y_i} + \mu_{Y_i}\sigma_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}$$

and

$$\gamma_i = \frac{1}{1 + 2t(t + \sqrt{t^2 + 1})}$$

for

$$t = \frac{\mu_{Y_i} - \mu_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}. \quad \square$$

Proof of Theorem 4.15. The proof is similar to the one of Theorem 4.3. We get

$$\sum_{i=1}^N b_i v_L(x_i - \delta_i; \gamma_i) \leq u(\mathbf{x}, \mathbf{z}) - u(\boldsymbol{\delta}, \mathbf{z}) \leq \sum_{i=1}^N b_i v_U(x_i - \delta_i; \gamma_i),$$

and thus

$$\begin{aligned} \mathbb{E}[u(\mathbf{Y}, \mathbf{Z})] &\geq \sum_{i=1}^N b_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma_i)] + \mathbb{E}[u(\boldsymbol{\delta}, \mathbf{Z})] \\ &= \sum_{i=1}^N b_i \mathbb{E}[v_U(X_i - \delta_i; \gamma_i)] + \mathbb{E}[u(\boldsymbol{\delta}, \mathbf{Z})] \\ &\geq \mathbb{E}[u(\mathbf{X}, \mathbf{Z})]. \end{aligned}$$

□

Proofs of Section 5

Proof of Theorem 5.6. As in Theorem 3.5, we get for $\mathcal{U}_{\gamma, \beta}$

$$\sum_{i=1}^N \beta_i v_L(x_i - \delta_i; \gamma) \leq u(\mathbf{x}) - u(\boldsymbol{\delta}) \leq \sum_{i=1}^N \beta_i v_U(x_i - \delta_i; \gamma).$$

Therefore we can derive as in Theorem 4.3 that a sufficient condition for $\mathbb{E}[u(\mathbf{Y})] \geq \mathbb{E}[u(\mathbf{X})]$ is

$$\sum_{i=1}^N \beta_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \geq \sum_{i=1}^N \beta_i \mathbb{E}[v_U(X_i - \delta_i; \gamma)],$$

which is equivalent to

$$\gamma \geq \frac{\sum_{i=1}^N \beta_i (\mathbb{E}[(X_i - \delta_i)_+] + \mathbb{E}[(\delta_i - Y_i)_+])}{\sum_{i=1}^N \beta_i (\mathbb{E}[(\delta_i - X_i)_+] + \mathbb{E}[(Y_i - \delta_i)_+])}.$$

□

Proof of Theorem 5.7. Assume that (5.4) holds. Fix arbitrary $\boldsymbol{\delta}$, consider $u \in \mathcal{U}_{\gamma, \beta}$, and without loss of generality set $u(\boldsymbol{\delta}) = 0$. As in Theorem 3.5, it follows that

$$\sum_{i=1}^N \beta_i v_L(x_i - \delta_i; \gamma) \leq u(\mathbf{x}) \leq \sum_{i=1}^N \beta_i v_U(x_i - \delta_i; \gamma).$$

It is sufficient to show that for some $\boldsymbol{\delta}$ we have

$$\sum_{i=1}^N \beta_i \mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \geq \sum_{i=1}^N \beta_i \mathbb{E}[v_U(X_i - \delta_i; \gamma)]$$

for any \mathbf{X} and \mathbf{Y} such that (4.1) holds. As in the proof of Theorem 3 in Müller et al. (2021), we get

$$\mathbb{E}[v_L(Y_i - \delta_i; \gamma)] \geq \gamma(\mu_{Y_i} - \delta_i) - (1 - \gamma) \frac{1}{2} \left(\delta_i - \mu_{Y_i} + \sqrt{\sigma_{Y_i}^2 + (\mu_{Y_i} - \delta_i)^2} \right)$$

and

$$\mathbb{E}[v_U(X_i - \delta_i; \gamma)] \leq \gamma(\mu_{X_i} - \delta_i) + (1 - \gamma) \frac{1}{2} \left(\mu_{X_i} - \delta_i + \sqrt{\sigma_{X_i}^2 + (\mu_{X_i} - \delta_i)^2} \right).$$

Thus, we need to find some γ such that

$$\begin{aligned} \sum_{i=1}^N \beta_i \left(\gamma(\mu_{Y_i} - \delta_i) - (1 - \gamma) \frac{1}{2} \left(\delta_i - \mu_{Y_i} + \sqrt{\sigma_{Y_i}^2 + (\mu_{Y_i} - \delta_i)^2} \right) \right) \\ \geq \sum_{i=1}^N \beta_i \left(\gamma(\mu_{X_i} - \delta_i) + (1 - \gamma) \frac{1}{2} \left(\mu_{X_i} - \delta_i + \sqrt{\sigma_{X_i}^2 + (\mu_{X_i} - \delta_i)^2} \right) \right) \end{aligned}$$

for some δ . Following Müller et al. (2021, Theorem 3), we choose

$$\delta_i = \frac{\mu_{X_i} \sigma_{Y_i} + \mu_{Y_i} \sigma_{X_i}}{\sigma_{Y_i} + \sigma_{X_i}},$$

so that

$$\frac{\mu_{Y_i} - \delta_i}{\sigma_{Y_i}} = t_i \quad \text{and} \quad \frac{\mu_{X_i} - \delta_i}{\sigma_{X_i}} = -t_i, \quad \text{where} \quad t_i = \frac{\mu_{Y_i} - \mu_{X_i}}{\sigma_{X_i} + \sigma_{Y_i}}.$$

Then the equation for γ becomes

$$\begin{aligned} \sum_{i=1}^N \beta_i \left(\gamma \sigma_{Y_i} t_i - (1 - \gamma) \frac{1}{2} \left(-\sigma_{Y_i} t_i + \sigma_{Y_i} \sqrt{1 + t_i^2} \right) \right) \\ = \sum_{i=1}^N \beta_i \left(\gamma (-\sigma_{X_i} t_i) + (1 - \gamma) \frac{1}{2} \left(-\sigma_{X_i} t_i + \sigma_{X_i} \sqrt{1 + t_i^2} \right) \right), \end{aligned}$$

which is equivalent to

$$\gamma \sum_{i=1}^N \beta_i t_i (\sigma_{Y_i} + \sigma_{X_i}) = (1 - \gamma) \frac{1}{2} \sum_{i=1}^N \beta_i \left(-\sigma_{X_i} t_i - \sigma_{Y_i} t_i + (\sigma_{X_i} + \sigma_{Y_i}) \sqrt{1 + t_i^2} \right).$$

Define

$$\Delta = \sum_{i=1}^N \beta_i t_i (\sigma_{Y_i} + \sigma_{X_i}) = \sum_{i=1}^N \beta_i (\mu_{Y_i} - \mu_{X_i}).$$

Then

$$\left(\gamma + (1 - \gamma) \frac{1}{2} \right) \Delta = (1 - \gamma) \frac{1}{2} \sum_{i=1}^N \beta_i (\sigma_{X_i} + \sigma_{Y_i}) \sqrt{1 + t_i^2},$$

or equivalently,

$$(1 + \gamma) \Delta = (1 - \gamma) \sum_{i=1}^N \beta_i (\sigma_{X_i} + \sigma_{Y_i}) \sqrt{1 + t_i^2}.$$

This yields

$$\gamma = \frac{\sum_{i=1}^N \beta_i (\sigma_{X_i} + \sigma_{Y_i}) \sqrt{1 + t_i^2} - \Delta}{\Delta + \sum_{i=1}^N \beta_i (\sigma_{X_i} + \sigma_{Y_i}) \sqrt{1 + t_i^2}}.$$

Alternatively, we can express γ as

$$\gamma = \frac{\sum_{i=1}^N \beta_i \left(\sqrt{(\sigma_{X_i} + \sigma_{Y_i})^2 + (\mu_{Y_i} - \mu_{X_i})^2} - (\mu_{Y_i} - \mu_{X_i}) \right)}{\sum_{i=1}^N \beta_i \left(\sqrt{(\sigma_{X_i} + \sigma_{Y_i})^2 + (\mu_{Y_i} - \mu_{X_i})^2} + (\mu_{Y_i} - \mu_{X_i}) \right)}.$$

□

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